Fourier Series as an Inner Product Space

- **Space** : $|f\rangle \equiv$ **τ-periodic functions** $f(t)$ that are periodic over $t = [-\tau/2 \to \tau/2]$, with $\omega = 2\pi/\tau$

  - **Basis #1** : $|n\rangle \equiv \begin{cases} \sin(n\omega t) & n = 1,\ldots,\infty \\ 1/\sqrt{2} & n = 0 \\ \cos(n\omega t) & n = -1,\ldots,-\infty \end{cases}$

  - **Inner Product #1** : $\langle g|f\rangle \equiv \dfrac{2}{\tau}\displaystyle\int_{-\tau/2}^{\tau/2} g(t)f(t)\,dt$

  - **Basis #2** : $|n\rangle = e^{in\omega t}$

  - **Inner Product #2** : $\langle \tilde{g}|\tilde{f}\rangle \equiv \dfrac{1}{\tau}\displaystyle\int_{-\tau/2}^{\tau/2} g^*(t)f(t)\,dt$

➔ **Basis is Orthonormal** : $\langle n|m\rangle = \delta_{nm}$

➔ **Completeness** : any $|f\rangle = \displaystyle\sum_{n=-\infty}^{+\infty} |n\rangle\langle n|f\rangle$