If you need them, steps are in the footnote on the next page. Also you will find these trig relations helpful:

\[
\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta), \quad \frac{d}{d\theta}\cot\left(\theta/2\right) = -\frac{1}{2\sin^2(\theta/2)}, \quad \sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)
\]

Our Kepler formula-set is now updated to include: (1) the possibility of repulsive \(1/r^2\) forces with negative force-constants \(\gamma\) (2) relations needed for scattering problems, namely formulae for the scattering angle \(\theta\) and impact parameter \(b\) for unbounded Kepler orbits as well as general cross-section formulae.

- **Coordinates & Reduced Mass**: \(\vec{r}_1 = \vec{R} + \frac{m_1}{M}\vec{r}, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}, \quad \mu = \frac{m_1m_2}{M}\)

- **Centrifugal force & PE**: \(\vec{F}_{cf} = -\frac{L^2}{\mu r^3}\vec{r}, \quad U_{cf} = \frac{L^2}{2\mu r^2}, \quad \text{effective radial } U^* = U + U_{cf}\)

- **Angular EOM**: \(\dot{\phi} = \frac{L}{\mu r^2}\)

- **Radial EOMs**: \(\mu \ddot{r} = F(r) + F_{cf}(r), \quad E = T + U(r) = \frac{1}{2} \mu \dot{r}^2 + U_{cf}(r) + U(r)\)

- **Path Equation**: \(u(\phi) = \frac{1}{r(\phi)} \rightarrow u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}\) and \(u' = -\frac{\mu \dot{r}}{L}\)

- **Conics**: With \((r, \phi)\) centered on a focal point and \(E = \text{Ellipse}, H = \text{Hyperbola}\)

\[
\frac{1}{r} = \frac{a}{b^2}(\pm 1 + e \cos \phi) \quad \begin{cases} +: \text{E or H-near-branch} \quad e = \frac{c}{a} = \sqrt{a^2 + b^2} \quad \begin{cases} +: \text{E} \quad -: \text{H} \\ -: \text{H-far-branch} \quad \begin{cases} -: \text{H} \end{cases} \end{cases} \end{cases}
\]

- **Kepler Orbits**: \(F = -\frac{\gamma}{r^2} : r(\phi) = \frac{r_0}{\sinh[\gamma] + e \cos \phi} \quad \begin{cases} r_o = \frac{L^2}{\mu \gamma} = a \sinh[\gamma], \quad E = \frac{1}{2} \frac{|\gamma|}{2a} |\gamma| \left(\frac{e^2 - 1}{2r_0}\right) \\ \text{Bounded orbits: } \tau^2 = \frac{4\pi^2 \mu}{\gamma} a^3, \quad r_o = \frac{2r_{\min}r_{\max}}{r_{\min} + r_{\max}}, \quad \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}, \quad e = \frac{r_{\max} - r_{\min}}{r_{\min} + r_{\min}} \end{cases}\)

- **Unbounded orbits: scattering angle** \(\theta = \pi - 2\alpha\) with \(\tan \alpha = \frac{b}{a}\), impact parameter \(b = \text{semi-minor axis}\)

- **XSec**: \(d\sigma \equiv dA / r^2 = \left[\frac{\sin \theta d\theta d\phi}{d\Omega}\right], \quad \frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left|\frac{db}{d\theta}\right|\) with \(\theta = \text{scattering angle}\)

- **Lumi**: \(L = n_\lambda N_e, \quad \frac{dN_{ev}}{dt} = L \sigma\)

**Practice Problem 1: The Rutherford Cross Section**

We have all the tools we need to derive the most famous and most commonly-used cross section in the world: the **Rutherford XSec** \(d\sigma / d\Omega\) for the non-relativistic scattering of two charged particles. Ernest Rutherford used this calculation to analyze the 1911 scattering experiment of Geiger and Marsden and deduce that the positive charge in the atom is not smeared uniformly within the atom but concentrated in a tiny volume. This was the discovery of the atomic nucleus. On to our derivation! We give the beam particle a charge \(q\) and the target particle a charge \(Q\); the force between them is then \(F = kQq / r^2\). We also assume that the target particle’s mass is so much greater than the beam particle’s mass \((M \gg m)\) that the target can be treated as fixed. You have all the tools you need to show that this famous cross-section is

\[
\frac{d\sigma}{d\Omega} = \left[\frac{kQq}{4E \sin^2(\theta/2)}\right]^2.
\]

If you need them, steps are in the footnote on the next page. Also you will find these trig relations helpful: