Phys 326 Discussion 1 – Weakly Coupled Oscillator Demo

In this discussion, we will analyze the behavior of the demo we viewed in lecture: two equal masses on identical springs were suspended from either side of a horizontal bar that was attached to a flexible support. Since the support could flex a bit, it **coupled** the left and right springs together: if one mass moved, the support moved slightly, and this eventually caused the other mass to start oscillating too. In Problem 1 we will find the general solution for this system; in Problem 2 we will impose boundary conditions to match what we did in the demo. Next week we will analyze the solution and find the distinctive behavior of weakly coupled oscillators.

Problem 1 : Solve the Demo using Normal Coordinates *Checkpoints*[1]

To analyze this apparatus, we will represent the flexible support as a third spring. Since its motions were small, we can approximate it as a *linear* spring, i.e. obeying Hooke's Law. (Remember, small enough excursions from equilibrium can *always* be approximated with a linear restoring force = first term in the Taylor expansion of the force's position-dependence.) In our demo, the support could only flex slightly, so it had a much stiffer spring constant than the two actual springs from which the masses were suspended. Let k be the force constant of each actual spring, and let K be the force constant of the support; the stiffness of the support tells us that $K \gg k$.

(a) Sketch the demo! Draw a fixed point at the top of your diagram and hang the support-spring K from it. Attach a horizontal cross-bar to the bottom of the support spring, and suspend the two actual springs k from either side of this cross-bar. Place equal masses m at the ends of the two actual springs. Finally, establish a coordinate system: have the x direction pointing downwards, then define a coordinate for the bottom end of each spring that represents its deviation from the equilibrium position. Use x_0 , x_1, and x_2 for the bottom ends of the support spring, the left spring, and the right spring respectively.

(b) First, let's agree to **IGNORE GRAVITY** entirely. (The whole apparatus could be laid on its side and the behavior would be the same, you would just have to build frictionless tracks for the masses to move in. Air resistance is so small that moving the masses vertically through air is essentially frictionless ... much easier than moving them on a table!) Now that we have g out of the way, write down the equations of motion (EOMs) for this system. Since this system is so simple, it is easier to do this with $F=ma$ than with a Lagrangian.

(c) Now *pause* … how many EOMs do you have? I'm sure you have these two : $F=ma$ applied to mass 1 and to mass 2. But do you have a *third* equation? You have three coordinates, so you need three EOMs to solve for them all. The third equation has to do with the **massless coupling** issue that you may recall from 325.
The third equation is determined by applying $F=ma$ to the horizontal crossbar that connects the springs.
The crossbar was a little stick of wood whose mass was completely negligible ➔ a massless coupling. When you apply $F=ma$ to this massless connector, the "m" is zero, so the total force on the massless connector must be zero or you would get infinite acceleration! Follow the technique from class: use the massless-coupling condition to remove one of your coordinates. Specifically, get rid of the coordinate x_0 by writing down what it is in terms of x_1 and x_2.

(d) Write the two remaining EOMs entirely in terms of x_1, x_2, and constants. Also, now is the moment to pick a good representation for those constants; instead of using m, k, and K, switch to these variables:

$$\omega_0^2 \equiv \frac{k}{m} \qquad \text{and} \qquad \eta \equiv \frac{k}{K} \ll 1$$

We've gone from three constants to two, and our η ("eta") is both dimensionless and tiny ➔ perfect parameter!

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[1] (b) patience ☺ (c) $x_0 = (x_1 + x_2)k/(K+2k)$ (d) $\ddot{x}_1 = \left[-x_1(1+\eta) + x_2\eta\right]\omega_0^2/(1+2\eta)$ and $\ddot{x}_2 = \left[x_1\eta - x_2(1+\eta)\right]\omega_0^2/(1+2\eta)$
(e) patience ☺ (f) $\omega_S = \omega_0/\sqrt{1+2\eta} \approx \omega_0(1-\eta)$ and $\omega_F = \omega_0$
(g) $\xi_+(t) = a_S\cos\left[\omega_S t - \delta_S\right]$ and $\xi_-(t) = a_F\cos\left[\omega_F t - \delta_F\right]$ (h) two (i) $\vec{x}(t) = \begin{pmatrix}1\\1\end{pmatrix}A_S\cos(\omega_S t - \delta_S) + \begin{pmatrix}1\\-1\end{pmatrix}A_F\cos(\omega_F t - \delta_F)$

Now we must find the general solution for $x_1(t)$ and $x_2(t)$. In class we presented the standard method where we postulate a normal-mode solution, switch to matrix notation, etc. For this particular problem, however, we have the rare situation when a faster method is available: we can actually **decouple** our two EOMs quite easily! To do so, we must make a change of variables from the generalized coordinates x_i to the so-called **normal coordinates** ξ_i. Normal coordinates are, by definition, those that decouple the EOMs. They are usually impossible to guess … *unless* your system is *highly symmetric* ...

(e) Did you notice that your EOMs are symmetric under the coordinate exchange $x_1 \leftrightarrow x_2$? This is the one instance when the normal coordinates ξ_i can be readily guessed! Define $\xi_+ \equiv x_1 + x_2$ and $\xi_- \equiv x_1 - x_2$, and rewrite your EOMs in terms of them. The equations should **decouple**, with one involving only ξ_+ and the other involving only ξ_-. (If the EOMs *don't* decouple, you haven't found the correct normal coordinates.)

(f) The equations you obtained are so simple that you can solve them by inspection. What you should see from your inspection is that each normal coordinate oscillates at a single frequency. These are the system's **eigenfrequencies**; what are they for this system? For convenience, label them ω_S for the slow mode and and ω_F for the fast mode. Note: since our parameter η is very small, we will only keep terms that are linear in η; you will need to make such an approximation to get the solution form in the checkpoint.

(g) Write down the general solutions for $\xi_+(t)$ and $\xi_-(t)$. Each normal coordinate is associated with a single mode and it's important to keep track of which is which. It's a good idea to write down the association and circle it. For this problem, write "+ is the SLOW mode". And circle it. (Seriously!)

(h) How many free parameters (constants of integration) are there in each normal-coordinate solution?

(i) Finish up the problem by writing down the general solution for $x_1(t)$ and $x_2(t)$.
Note: Each x-coordinate solution has *four* free parameters. Does it make sense that each one has so many? The problems in the last section will help to clarify that although $x_1(t)$ and $x_2(t)$ each have four parameters, they are the *same* parameters, so the total for the full system is still four, as it must be.

Problem 2 : Imposing Initial Conditions *Checkpoints*[2]

Now we will impose some initial conditions and use them to fix the free parameters of your general solution. You can determine the free parameters by working with the $x_1(t)$ and $x_2(t)$ solutions *or* the normal-coordinate solutions $\xi_+(t)$ and $\xi_-(t)$. The best choice depends on what initial conditions you are given; practice will help you figure that out. For each set of initial conditions below, obtain the specific solutions for $x_1(t)$ and $x_2(t)$:

(a) At time $t = 0$, the masses are placed at positions $(x_1, x_2) = (b, b)$ and released from rest.

(b) The masses are placed at rest in their equilibrium positions $(x_1, x_2) = (0,0)$. Then, at time $t = 0$, an impulse is delivered to mass #1 that gives it a downward velocity of $\dot{x}_1 = v_0$. (Mass #2 is left at rest.)

Problem 3 : The Standard Method *Reproduce 1(i) general solution*

The normal-coordinate trick you used to solve problem 1 is great, but if your system doesn't have 1 <–> 2 exchange symmetry, you must use the standard "determinant" method. It's new, so let's practice. The steps are: (1) put EOMs in matrix form (2) postulate normal-mode solution form (3) find eigenfrequencies via determinant (4) find eigenvectors. Start with these EOMs: $\ddot{x}_1 = \omega_0^2\left[-x_1(1-\eta) + x_2\eta\right]$ & $\ddot{x}_2 = \omega_0^2\left[x_1\eta - x_2(1-\eta)\right]$. They are your 1(d) EOMs approximated for $\eta \ll 1$; keep only linear terms in η throughout your work, as you did before.

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[2] (a) Normal coordinates are easier to work with here because *only one mode* (slow) is excited $\rightarrow$ $x_1(t) = x_2(t) = b\cos(\omega_S t)$

(b) Coord x_1, x_2 are easier here as the initial conditions excite *only one mass* $\rightarrow$ $\vec{x}(t) = \dfrac{v_0}{2\omega_0}\left[\begin{pmatrix} 1+\eta \\ 1+\eta \end{pmatrix}\sin(\omega_S t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(\omega_F t)\right]$