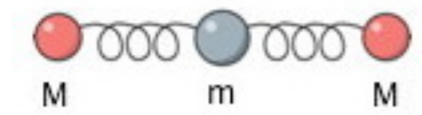
Phys 326 Discussion 4 – The Inner Product ; Degenerate Eigenstates

Problem 1 : The Inner Product of Normal Mode Space *Checkpoints*[1]

Recall the CO_2 molecule you analyzed last week, shown at right. Here are the system's M and K matrices and the three resulting eigenmodes (eigenfrequencies and associated eigenvectors) :



$$\mathbf{M} = \begin{pmatrix} M & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & M \end{pmatrix}, \quad \mathbf{K} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \rightarrow \quad \omega^2_{z,s,f} = 0, \frac{k}{M}, \frac{k}{M}\left(1 + \frac{2M}{m}\right) \quad \text{with } \vec{a}_{z,s,f} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} m \\ -2M \\ m \end{pmatrix}$$

As our first task in the next lecture, we will prove that the eigenvectors of any linear coupled oscillator form are always **orthogonal** to each other if we introduce an appropriate definition of the **inner product**.

(a) Two elements $|a\rangle$ and $|b\rangle$ of an inner product space are **orthogonal** if their inner product $\langle a|b\rangle$ is zero. Vectors in 3D space are the most familiar example of an IPS; in that space, the inner product is the familiar dot-product: $\langle \vec{a}|\vec{b}\rangle \equiv \vec{a}\cdot\vec{b} = a_i b_i$. Show that the eigenvectors of the CO_2 molecule are *not* all orthogonal to each other using this definition of the inner product.

(b) Here is the normal dot-product rewritten in index notation and matrix notation:

$$\langle \vec{a}|\vec{b}\rangle \equiv a_i b_i = \vec{a}^T\vec{b}$$

Here it is again, this time with the **Kronecker delta** δ_{ij} inserted between the two elements:

$$\langle \vec{a}|\vec{b}\rangle \equiv a_i\,\delta_{ij}\,b_j = \vec{a}^T\mathbf{D}\vec{b} \qquad \text{where } \mathbf{D} \text{ is … ?}$$

First, make sure that it is 100% clear to you that $a_i\delta_{ij}b_j$ is the same thing as $a_i b_i$. To demonstrate your understanding, what is the matrix "**D**" in the right-hand expression? i.e. what is the matrix representation of δ_{ij}?

(c) We are now just one slight modification away from the inner-product definition we need for normal-mode space. Let $\vec{q}_1(t)$ and $\vec{q}_2(t)$ be two solutions of a linear oscillator written in terms of generalized coordinates $\vec{q}$, and consider this definition of the inner product:

$$\boxed{\langle \vec{q}_1|\vec{q}_2\rangle \equiv \vec{q}_1^{\,T}\mathbf{M}\vec{q}_2} \qquad \text{where } \mathbf{M} \text{ is the system's mass matrix.}$$

Check all three inner products for the normal modes of the CO_2 molecule — $\langle \vec{a}_z|\vec{a}_s\rangle$, $\langle \vec{a}_z|\vec{a}_s\rangle$, and $\langle \vec{a}_z|\vec{a}_s\rangle$ — and show that this definition *does* make the three normal modes orthogonal to each other.

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[1] (a) $\vec{a}_z\cdot\vec{a}_S = 0$ and $\vec{a}_f\cdot\vec{a}_S = 0$ but $\vec{a}_z\cdot\vec{a}_f = 2(m-M) \neq 0$ (b) **D** is the identity matrix **1** (a.k.a the unit matrix)

(d) $\hat{a}_z = \frac{1}{\sqrt{2M+m}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \hat{a}_s = \frac{1}{\sqrt{2M}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \hat{a}_f = \frac{1}{\sqrt{2mM(m-2M)}}\begin{pmatrix} m \\ -2M \\ m \end{pmatrix}$

We haven't proved it yet (that will be our first task next lecture), but this definition of the inner product always makes the eigenvectors of a coupled linear oscillator orthogonal to each other. Thus, the normal modes — the eigenmodes — of such a system provide a suitable **basis** for describing all of its solutions as an inner product space. Except for one last item …

(d) To make everything simple, we would like our basis elements to be not just orthogonal, but **orthonormal**: we would like $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$. That is easily accomplished: use the new inner product to find the **magnitude** of your eigenvectors $\vec{a}_m$, then turn them into "unit eigenvectors" $\hat{a}_m$ by dividing out this magnitude. The magnitude of an IPS element is the square-root of its inner product with itself:

$$\text{magnitude } |\vec{a}_m| \equiv \sqrt{\langle \vec{a}_m | \vec{a}_m \rangle} \quad \rightarrow \quad \text{normalized eigenvectors } \hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$$

Normalize the eigenvectors of the CO_2 molecule to produce a set of unit eigenvectors. Your unit eigenvectors $\hat{a}_m$ provide a perfect orthonormal basis for all motions of the molecule.

So : here is a summary of our description of normal mode solutions as an inner product space:

- **Space** : $|\vec{q}(t)\rangle \equiv$ all solutions of a linear oscillator system in terms of generalized coordinates $\vec{q}$
- **Inner Product** : $\boxed{\langle \vec{q}_1 | \vec{q}_2 \rangle \equiv \vec{q}_1^{\,T} \mathbf{M} \vec{q}_2}$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q} | \vec{q} \rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M} \vec{a}_m$ and normalized by $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$
- **Basis is Orthonormal** : $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$
- **Completeness** for $\vec{q}(t)$: $|\vec{q}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle \langle \hat{a}_m | \vec{q}(t) \rangle \equiv \sum_{\text{modes } m} \hat{a}_m \; \tilde{A}_m \, e^{i\omega_m t}$

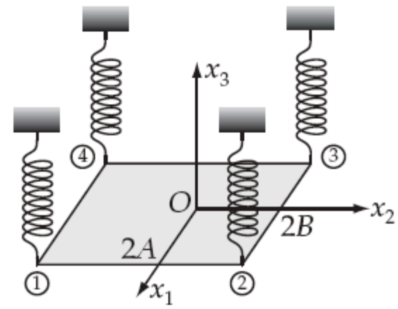
We have one more element to add to this (normal coordinates), but just to remind you: the point of introducing all this math is that it appears *everywhere* in physics. You will find this IPS structure *all over the place*: a set of mathematical objects described as a linear combination of a set of orthonormal basis elements, with an inner-product operation that tells you how to project any member of the set onto those basis elements.

If you're lost, always hang on to the simplest example you can find. In this case, it is 3D vectors:

- **Space** : $|\vec{v}\rangle \equiv$ all vectors in 3D space
- **Inner Product** : $\boxed{\langle \vec{u} | \vec{v} \rangle \equiv \vec{u}^{\,T} \vec{v} = \vec{u}^{\,T} \mathbf{1} \vec{v}}$ and associated **magnitude** : $|\vec{v}|^2 \equiv \langle \vec{v} | \vec{v} \rangle$
- **Basis** : $|\hat{x}_i\rangle$ of Cartesian unit vectors $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\} \equiv \{\hat{x}, \hat{y}, \hat{z}\}$
- **Basis is Orthonormal** : $\langle \hat{x}_i | \hat{x}_j \rangle = \delta_{ij}$
- **Completeness** for $\vec{v}$: $|\vec{v}\rangle = \sum_{i=x,y,z} |\hat{x}_i\rangle \langle \hat{x}_i | \vec{v} \rangle \equiv \sum_{i=x,y,z} \hat{x}_i \, (\vec{v} \cdot \hat{x}_i)$

Problem 2 : The Degenerate Modes of a Suspended Plate *Checkpoints*[2]

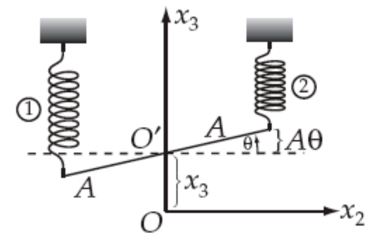
A thin, flat, homogeneous plate has mass M and lies in the x_1-x_2 plane with its center at the origin. The plate's sides have length $2A$ in the x_2 direction and $2B$ in the x_1 direction. The plate is suspended from a fixed support by four springs of equal spring-constant k at the four corners of the plate. The top figure shows the equilibrium configuration of this system.



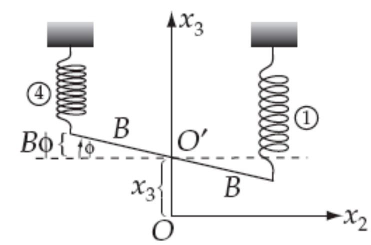
We have three degrees of freedom:

1. vertical motion, with the center of the plate moving along $\pm\hat{x}_3$
2. tipping motion around the $\hat{x}_1$ axis, described by the angle θ
3. tipping motion around the $\hat{x}_2$ axis, described by the angle ϕ

Our generalized coordinates for this problem are thus $\vec{q} = \{x_3, \theta, \phi\}$;
the angles remain small to ensure a linear system. The plate's moments of inertia are $I_1 = ⅓\, MA^2$ around $\hat{x}_1$ and $I_2 = ⅓\, MB^2$ around $\hat{x}_2$.



(a) Find the eigenfrequencies of this system. You should find that two of this system's normal modes are **degenerate**, which means that two of the eigenfrequencies are *the same*.



(b) To find out how this plate likes to move, you must find the eigenvector $\vec{a}$ for each mode. Because of the degeneracy, you must make some arbitrary choice when building $\vec{a}$ for the two modes with the same frequency. Use this common tactic: set one of the free components of $\vec{a}$ to zero for one of the degenerate modes, then figure out the other $\vec{a}$ using orthogonality. Careful … the inner product that defines "orthogonal" in eigenvector space is not the dot-product … but happily, you will quickly find that it is *proportional* to the dot-product for this particular system.

(c) Normalize your eigenvectors to form an orthonormal set.

(d) Now add to the plate a thin bar of mass m and length $2A$ situated (at equilibrium) along the x_2-axis. The bar's moment of inertia around $\hat{x}_1$ is $I_1 = ⅓\, mA^2$ and it is thin enough to have zero moment of inertia around $\hat{x}_2$. Find the new eigenfrequencies of the system and show that the degeneracy of the system is gone.

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[2] (a) $\mathbf{M} = \frac{M}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & B^2 \end{pmatrix}$, $\mathbf{K} = 4k\begin{pmatrix} 1 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & B^2 \end{pmatrix}$, $\rightarrow \omega_{\mathrm{S,F1,F2}} = 2\sqrt{\frac{k}{M}}, 2\sqrt{\frac{3k}{M}}, 2\sqrt{\frac{3k}{M}}$ (b) $\vec{a}_{\mathrm{S,F1,F2}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $\hat{a}_{\mathrm{S,F1,F2}} = \frac{1}{\sqrt{M}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \sqrt{\frac{3}{MA^2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \sqrt{\frac{3}{MB^2}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (d) $\omega_{\mathrm{F1}} = 2\sqrt{\frac{3k}{M+m}}$, others unchanged