Phys 326 Discussion 7 – Kepler Orbits

• **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\mu = \frac{m_1 m_2}{M}$

• **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective $U^* = U + U_{\text{cf}}$

• **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$, $E = T + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}}(r) + U(r)$

• **Path Equation** : $u(\phi) \equiv 1/r(\phi) \rightarrow u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$

• **Ellipse / Hyperbola** with (r,ϕ) centered on a focal point : $\frac{1}{r} = \frac{a}{b^2}(1 + e\cos\phi)$, $e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a}$

• **Kepler Orbits** for $\vec{F} = -\frac{\gamma}{r^2}\hat{r}$: $r(\phi) = \frac{r_0}{1 + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu\gamma} = \frac{b^2}{a}$, $E = \mp\frac{\gamma}{2a} = \frac{\gamma(e^2-1)}{2r_0}$, $\tau^2 = \frac{4\pi^2\mu}{\gamma}a^3$

Also, the $r(\phi)$ formula in the last line immediately gives $r_0 = \frac{2\, r_{\text{min}} r_{\text{max}}}{r_{\text{min}} + r_{\text{max}}}$ and $e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}$.

Problem 1 : Satellites *Checkpoints for* [1] & [2]

To keep our mess of formulae under control, realize that it takes exactly two parameters to define the shape of an ellipse ... there are just many choices for those shape parameters: $a, b, c, e, r_{\text{min}}, r_{\text{max}}, r_0$. When the ellipse is a Kepler orbit, i.e. the solution of a 2-body system with a $1/r^2$ central force, more choices appear: energy E, angular momentum L, period τ, the velocities at the apses, … but once the reduced mass μ and force constant γ are specified (they *define* the Kepler problem), there are still only two independent parameters for the solution.

EARTH DATA: If you need the radius of the Earth, it is $R_\oplus = 6.4 \times 10^6$ m. If you need the mass of the Earth, it will almost certainly be in the combination $GM_\oplus$ which is equal to $gR_\oplus^2$; use that and $g = 9.8$ m/s^2. Finally, the mass of a satellite can be considered negligible compared to the mass of the Earth.

The height of a satellite above the earth's surface is 300 km at perigee and 3000 km at apogee.

(a) Find the orbit's eccentricity e.

(b) If we take the orbit to define the xy plane and the major axis in the x direction with the earth at the origin, what is the satellite's height above the earth's surface when it crosses the y axis?

Another satellite is observed at perigee to be 250 km above the earth's surface and travelling at about 8500 m/s.

(c) Find the eccentricity of its orbit. (d) Find its height above the earth's surface at apogee.

In Yuri Gagarin's first manned space flight in 1961, the perigee and apogee were 181 km and 327 km above the earth's surface.

(e) Find the period of his orbit. (f) Find his maximum speed in the orbit. (Hints in checkpoint.)

---

[1] (a) 0.17 (b) 1400 km (c) 0.197 (d) 3510 km (e) 90 min (f) Hints: max speed occurs at a particular point on the orbit ... at that point, velocity has only *one component* … get velocity in via T or via L … $v_{\text{max}}^2 = 2gR_\oplus^2 \frac{r_{\text{max}}}{r_{\text{min}}\left(r_{\text{max}} + r_{\text{min}}\right)}$ → $v_{\text{max}} = 7.8$ km/s

[2] (d) Hint: the transfer time is half the period of the transfer orbit. Answer: $T = \pi\sqrt{(R_1 + R_3)^3/(8GM_\odot)} = 2.7$ years.
(e) $(v_1, v_{2L}, v_{2R}, v_3) = (30, 39, 7.4, 13)$ km/sec. (f) It doesn't matter *what* radial speed you use: if you have to *stop* the angular velocity v_1 at L and *create* angular velocity v_3 at R, your total Δv will be at least $v_1 + v_3 = 43$ km/s > Hohmann's 15 km/s !

Problem 2 : Hohmann Transfer to Jupiter *Checkpoints on previous page*

The idea of **Hohmann Transfer Orbits** dates from 1925 when Walter Hohmann found the most cost-effective procedure for transferring a spacecraft from one circular orbit to another circular orbit in the same plane.

(a) Here are the steps involved; draw them as your read so you can visualize the procedure.

1. Draw the Sun, and draw two circular orbits around it. Assign the starting orbit a radius R_1 and assign the ending orbit a radius $R_3 > R_1$. We will imagine that orbit 1 is the Earth's orbit, while orbit 3 is that of Jupiter. Our goal is to get a spaceship from Earth-orbit to Jupiter-orbit (so it can take pretty pictures ☺).
2. Hohmann's technique to get a ship from orbit 1 to orbit 3 is to put the ship into a temporary **transfer orbit**, which we call orbit 2. Draw in this transfer orbit as a dashed line using Hohmann's specification: the transfer orbit is an ellipse that is tangential with both orbit 1 and orbit 3.
3. The ship will briefly fire its rockets at two points: at the point L=Launch where we switch the ship from orbit 1 to orbit 2, and at the point R=Rendezvous where switch from orbit 2 to orbit 3. Label these two points L and R on your drawing. (They are the points where the circular orbits touch the transfer orbit.)

If your drawing is correct you will see that the transfer orbit has perihelion distance R_1 and aphelion distance R_3. You will also see that the two rocket firings apply impulses that are tangential to the ship's current orbit. This avoidance of any radial velocity changes is at the heart of the cost-saving of the Hohmann procedure. It is not the fastest way to get from one orbit to another; rather, it uses the least amount of fuel, and getting *anything* up into space is extremely expensive!

(b) We need very little information to get all the numbers for this problem. We need the distance from the Sun to Jupiter, which is 5.2 A.U., where an A.U. (Astronomical Unit) is the Sun-Earth distance. Almost everything else can be obtained from well-known quantities about the Earth's orbit (well-known because we live here. ☺) For example, the nasty constants G, $M_\odot$, and R_1 = 1 A.U. will appear frequently in our calculations, but in **convenient combinations**. Specifically, $(GM_\odot / 1\,\text{A.U.})$ turns out to be a very pretty quantity: 9×10^8 (m/s)^2. Show that, since the Earth's orbit is nearly circular, the velocity of the Earth around the Sun is the square root of this pretty quantity: $v_1 = \sqrt{GM_\odot / R_1} = 30$ km/s .

(c) Another convenient combination of G, $M_\odot$, and R_1 = 1 A.U. can be obtained from the 1-year period of the Earth around the Sun. Show that $\sqrt{GM_\odot / R_1^3} = 2\pi / \text{year}$. Combined with (b), we also get 30 km/s = 2π AU/yr.

(d) Now calculate the total time T it will take to get from Launch to Rendezvous. A hint is in the footnote.

(e) Calculate the four speeds v_1, v_{2L}, v_{2R}, and v_3 involved in the story. In case the notation is unclear: v_1 is the ship's speed in orbit 1; v_{2L} is the speed right after the ship fires its rockets at point L (which puts it onto orbit 2); v_{2R} is the speed on orbit 2 right *before* the rockets are fired for a second time, at point R; and v_3 is the ship's final speed in orbit 3.

(f) Does Hohmann Transfer really optimize cost by minimizing the amount of fuel burned? As we showed in 325, the amount of fuel burned is proportional to the sum of velocity changes $|\Delta v|$. From part (e), you found $|\Delta v_L| + |\Delta v_R| \approx 15$ km/s for Hohmann, so experiment and see if you can do better! Try a direct radial trajectory: fire your rockets at L=Launch to give the ship zero angular speed and a radial speed that will reach Jupiter's orbit in the same time T = 2.7 years that Hohmann requires. Once you reach Jupiter's orbit, remember that you have another velocity change to make: you must stop your radial velocity and impart an angular velocity matching Jupiter's orbit. See how much "cost" $|\Delta v_L| + |\Delta v_R|$ your plan requires!