Phys 326 Discussion 8 – Captured Paths and Cross-Sections; the Virial Theorem

- **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\mu = \frac{m_1 m_2}{M}$

- **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective $U^* = U + U_{\rm cf}$

- **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\rm cf}(r)$, $E = T + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\rm cf}(r) + U(r)$

- **Path Equation** : $u(\phi) \equiv 1/r(\phi) \;\rightarrow\; u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu\dot{r}}{L}$

- **Ellipse / Hyperbola** with (r,ϕ) centered on a focal point : $\frac{1}{r} = \frac{a}{b^2}(1 + e\cos\phi)$, $e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a}$

- **Kepler Orbits** for $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{1 + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu\gamma} = \frac{b^2}{a}$, $E = \mp\frac{\gamma}{2a} = \frac{\gamma(e^2 - 1)}{2r_0}$,

$$\tau^2 = \frac{4\pi^2\mu}{\gamma}a^3 , \quad r_0 = \frac{2r_{\rm min}r_{\rm max}}{r_{\rm min} + r_{\rm max}} , \quad e = \frac{r_{\rm max} - r_{\rm min}}{r_{\rm max} + r_{\rm min}}$$

Problem 1 : A Captured Path *Checkpoints* [1] *and* [2]

In lecture we introduced the capture cross-section, $\sigma = \pi b_{\rm max}^2$: particles fired with a given initial velocity $\vec{v}_0$ at a potential well are "captured" when their impact parameter b is below some value $b_{\rm max}$. "Captured" means the particle cannot return to $r = \infty$ (cannot escape the potential well). What exactly happens to such a captured particle? Let's find out, using an attractive force, $F = -k/r^3$, which corresponds to the potential $U = -k/2r^2$. Let's say this mysterious force is emitted by the Death Star, with some huge mass M located at the origin. We will send in particles with initial velocity v_0, impact parameter b, and mass $m << M$.

(a) Plot $U^*(r)$. You will see that it can take two different shapes, depending on the relative size of the force constant k and L^2/m for the incoming particle. Using the fact that L is related to b, find the capture condition on b, and from that get $\sigma = \pi b_{\rm max}^2$.

(b) Use the path equation to determine the path $r(\phi)$ for captured particles of mass m. You will quickly find it convenient to define the symbol $\alpha \equiv \sqrt{(mk/L^2) - 1}$.

(c) Sketch the path! Important: $\dot{\phi} = L/\mu r^2$ tells us that the angle is constantly increasing with time; it will keep growing until r reaches some limit like $r = \infty$ or $r = 0$. What shape is the path? Are the particles actually "captured" by the Death Star?

(d) Calculate the speed components $v_r = \dot{r}$ and $v_\phi = r\dot{\phi}$ as functions of ϕ. What happens to them in the limit $\phi \to \infty$? (We are using ϕ as a substitute for time; we could solve for time, but it is really messy.) Think of one or two reasons why the limit you obtained will not occur in practice, despite your calculation.

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[1] (a) $\sigma = \pi(k/mv_0^2)$ (b) $r(\phi) = \alpha b/\sinh(\alpha\phi)$ (c) path = inward spiral (d) $v_r = -v_0\cosh(\alpha\phi)$, $v_\phi = (v_0/\alpha)\sinh(\alpha\phi)$

[2] $\Delta\Omega_{\rm moon} \approx 6.45 \times 10^{-5}$ sr; $\Delta\Omega_{\rm sun} \approx 6.76 \times 10^{-5}$ sr $\rightarrow$ solar eclipse with visible corona

Problem 2 : Solid Angle Quickie *Checkpoints on prev page*

Solid angle is the 2D version of angle; as area is to length, solid angle is to angle. As we wrote in class, the differential element of solid angle can be defined by this wordy relation: $d\Omega = \dfrac{dA \text{ on spherical surface}}{r \text{ of spherical surface}}$.

Solid angle is measured in **steradians** ≡ radians^2 and, honestly, it's best explained with a familiar example:

Calculate the solid angles $\Delta\Omega_{\text{moon}}$ and $\Delta\Omega_{\text{sun}}$ subtended by the moon and by the sun respectively, both <u>as seen from the earth</u>. Data: The radii of the moon and sun are $R_m = 1.74 \times 10^6$ m and $R_S = 6.96 \times 10^8$ m. Their distances from earth are $d_m = 3.84 \times 10^8$ m and $d_S = 1.50 \times 10^{11}$ m (one A.U.) What do your answers tell you about the <u>appearance</u> of the moon and sun in the sky as seen from earth? What remarkable phenomenon is visible to earthlings only because of the rather extraordinary comparison between $\Delta\Omega_{\text{moon}}$ and $\Delta\Omega_{\text{sun}}$?

Problem 3 : The Virial Theorem

The **virial theorem** is a very well-known theorem in physics. The most common way it is stated is that a particle held in a bound state by an inverse-square central force field (e.g. gravity, the electric force) has <u>kinetic energy equal to half the magnitude of its potential energy</u>. Since the particle is bound, the force is necessarily attractive ($F \sim -1/r^2$), so the potential energy $U \sim -1/r$ is negative. The total energy is thus $E = T + U = \frac{1}{2}|U| - |U| = -\frac{1}{2}|U| = -T$ → it's negative, as it must be for a bound state. This "rule of thumb" that the kinetic energy fills up half of the potential-energy well works wonderfully for many common bound states (planets in orbit, electrons in atoms); it is a really useful fact to know! Let's examine the virial theorem in greater detail.

(a) A mass m moves in a <u>circular</u> orbit in the field of a stationary, attractive central force with potential energy $U = kr^n$ (with $kn > 0$ to ensure that the force is attractive). Prove the **virial theorem** : $T = nU/2$.
(The most common appearance of the theorem is for the case $n = -1$, i.e. for a gravity-like potential $U \sim -1/r$. This is the case described in the introductory paragraph. As you see, it gives $T = -U/2 = \frac{1}{2}|U|$.)

(b) We can be even more general than that! Let's derive a form of the theorem that applies to <u>any periodic orbit</u> of a particle, not just a circle. First, find the time derivative of the quantity $G = \vec{r} \cdot \vec{p}$ and, by integrating from time 0 to t, show that $\dfrac{G(t) - G(0)}{t} = 2\langle T \rangle + \langle \vec{F} \cdot \vec{r} \rangle$, where $\vec{F}$ is the net force on the particle and $\langle f \rangle$ denotes the average over time of any quantity f.

(c) Explain why, if the particle's orbit is periodic and if we make t sufficiently large, we can make the left-hand side of this equation as small as we please. That is, the left side approaches zero as $t \to \infty$. FYI: Notice that the left-hand side can *also* be made zero if we choose $t = n\tau$ = any integer multiple of the orbital period.

(d) Use this result to prove that if $\vec{F}$ comes from the potential energy $U = kr^n$, then $\langle T \rangle = n\langle U \rangle / 2$, if now $\langle f \rangle$ denotes the time average over a very long time. FYI: Following the FYI from part (c), you can see that the theorem also applies if $\langle f \rangle$ denotes instead the time average over a *cycle* of the periodic orbit. That is the more common interpretation of $\langle T \rangle = n\langle U \rangle / 2$. Taylor's point is that if you average a periodic quantity over a very long time, you get the cycle average anyway without having to stop your integral at t = exactly $n\tau$.