Phys 326 Discussion 13 – GR: Curvature and Reduced Circumference

Discussion 12 summarized the essential principles of General Relativity, including the (non-axiomatic) **Schwarzschild metric** that we are using as our one example of curved spacetime. It is the metric describing the spacetime curvature around a spherically-symmetric non-rotating mass M.

$$\boxed{d\tau^2 = dt^2\left(1-\frac{2M}{r}\right) - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2}$$

where $M = \frac{G}{c^2}M_{\text{kg}}$ and $t = c\, t_{\text{sec.}}$

One of the two main challenges of GR is figuring out how to use the metric to calculate interesting quantities. (The other challenge is how to find the metric for different mass distributions from Einstein's field equations … but that is an advanced topic treated in graduate-level courses. Einstein himself found it challenging: Schwarzschild, not Einstein, was the first one to come up with such a solution. ☺)

Problem 1 : Gaussian Curvature of a Spherical Surface

Checkpoint on next page

To get any feeling at all for what the bizarre notion of "curved spacetime" means, we must first explore some *simple* curved geometries that we *understand*. The geometries of curved surfaces are perfect: we can picture them and check our calculations against intuitive sketches. Every feature of a space's geometry flows from the **metric**, as the metric is the relation that translates changes in our chosen coordinates into physical distances. In spatial geometries (no time coordinate), physical distance is just spatial distance, dl. (I'll avoid the more common symbol "ds" as it conflicts with our use of "s" as the cylindrical-radial coordinate.)

The surface of a sphere of radius a is the most familiar example of all of a curved geometry since we *live* on such a surface ☺. For our coordinates, let's pick the spherical-polar angles θ and ϕ; in terms of these coordinates, the metric is $dl^2 = (a\,d\theta)^2 + (a\sin\theta\, d\phi)^2$. One way to characterize the curvature of a space is to evaluate its **Gaussian curvature** at one or more points. This quantity compares the circumference of a circle with its radius to see if they satisfy the flat-space relation $C = 2\pi R$. The exact definition of Gaussian curvature is $K = \lim_{R\to 0}(2\pi R - C)\cdot\frac{3}{\pi R^3}$. Your task is to evaluate the Gaussian curvature at the "North Pole", i.e. at $(\theta, \phi) = (0, \text{irrelevant})$. Your task requires two path integrals. Here are the steps:

- Parametrize the path of your circle. A circle around the north pole is defined by a constant value of θ and a full sweep of ϕ, so the path for our circle will be $\{\theta = \theta_0,\ \phi: 0 \to 2\pi\}$ for some arbitrary constant θ_0 describing the size of the circle.
- Calculate the circumference C of the circle: $C = \int dl$ along the path we just described.
- Calculate the radius R of the circle: $R = \int dl$ along a geodesic path – i.e. a path of shortest distance – from the center-point of your circle to any point on the circle. REMEMBER REMEMBER: the most common problem people have with path integrals is *not parametrizing the PATH of integration!* You *have* to specify a path before you can do a path integral! We did it above for the circumference-path; you provide the simple path for the radial integral. Note that this path, just like the other one, will necessarily depend on the constant θ_0 we introduced to describe the size of our circle in coordinate space.

OK off you go! Calculate the Gaussian curvature $K = \lim_{R\to 0}(2\pi R - C)\cdot\frac{3}{\pi R^3}$ at the North Pole of a sphere using the metric $dl^2 = (a\,d\theta)^2 + (a\sin\theta\, d\phi)^2$.

☞ The purpose of taking the limit $R \to 0$ is to determine the curvature right *at* the North Pole. To take this small-circle limit, please note that you cannot change the parameter a: it is the radius of the sphere and so an unchangeable property of the geometry.

Problem 2 : Circles around a Black Hole *Checkpoints* [1,2]

On to spacetime! Let's explore only the spatial part of the Schwarzschild metric: if we always make measurements at the *same times* (as we do when measuring purely spatial distances like the radii and circumferences of circles), the dt term in the metric is zero, leaving only the dr and $d\phi$ terms. Since all such same-time measurements are spacelike, we switch to from proper time to **proper distance** as our metric:

$$d\sigma^2 \equiv -d\tau^2 = \frac{dr^2}{(1-2M/r)} + r^2 d\phi^2$$

The *only* reason for this sign change is to avoid getting an imaginary result, which you do when you calculate $d\tau$ for spacelike separated events. With this change, $d\sigma$ exactly plays the role of dl = ruler distance. Let's check out what circles look like near black holes! Note: we are just using "black hole" here as cool-sounding shorthand for "something that produces a gravitational field strong enough to require General Relativity".)

(a) Let's measure a circle of coordinate-radius r_0 around the origin $(r,\phi) = (0,\text{irrelevant})$, which is the center of the mass M. First, calculate/"measure" the circumference C by integrating $\int d\sigma$ around the circle. Your result will depend on r_0 of course.

(b) Set up — but don't integrate! — the $\int d\sigma$ integral you would need to measure the radius R of the circle.

(c) Uh oh, we have a problem: you undoubtedly used 0 and r_0 as your lower and upper bounds of integration … but look at what happens to the integrand at $r = 0$: it becomes *imaginary!* From lecture, we know that $2M$ is called the **Schwarzschild radius**, and when r goes down to $2M$, the $dr^2/(1-2M/r)$ term *blows up*, indicating that we have reached the limits of applicability of this metric. We can't use the Schwarzschild metric to explore radii *inside* the Schwarzschild radius. (This is exactly analogous to a familiar situation in Special Relativity: the collapse of the Lorentz transformation at speeds greater than c; for boost speeds $\beta > 1$ the gamma factors become imaginary, so the equations have officially broken down.) Oh boy … we *cannot* measure the radius of our circles, so we cannot investigate the curvature $k = (2\pi R - C)$, how disappointing! But wait: we can do something else, we can investigate how this curvature *varies* with coordinate-radius r_0 ➔ calculate the "differential" curvature $\frac{dk}{dr_0} = 2\pi\frac{dR}{dr_0} - \frac{dC}{dr_0}$ and see if it is positive, negative, or zero. NOTE: at no point do you

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[1] **Q1** $K = 1/a^2$ **Q2 (a)** $C = 2\pi r_0$ **(c)** $\frac{dk}{dr_0} = 2\pi\left(\frac{1}{\sqrt{1-2M/r_0}} - 1\right)$ **(d)** You found that, for circles around a black hole, $2\pi dR \gg dC$ when you get close to the event horizon at $r_0 = 2M$. In words: near the event horizon, it takes a very large change in radius to produce a very small change in the circumference of a circle at that radius … and the discrepancy grows as you get closer to $r = 2M$. Yow! Such a phenomenon can never be drawn on a flat piece of paper … but it *can* be drawn on a flexible sheet curved into the shape of a funnel ... which is the famous picture. **(e)** Hint: Schwarzschild-r is sometimes called the **reduced circumference** … why? recall part (a) … possible measurement technique: Maintain a constant distance from the planet by e.g. bouncing a laser beam off its nearest surface and timing how long it takes to get back to you. Travel around the planet in a circle by keeping that laser-bounce-time constant, and measure the distance you travel as you go along. Once you've returned to your starting point, divide your measured circumference by $2\pi \rightarrow$ that's your r coordinate. **(f)** Maintain a constant distance from the planet's surface using the same laser-bounce technique, but make sure your circular path takes you directly over the ice cap this time. Measure the distance you travel as you proceed, and stop measuring when you're over the ice cap. If you measured a distance d, the polar angle θ at your starting point was $2\pi\,(d/C) = 2\pi$ times (the fraction of a full circle you had to travel).

[2] **Q3 (a)** $k = 2\pi\rho_0^2\left(\frac{2}{\sqrt{3}} - 1\right)$ **(b)** after substitution: $dl^2 = dr^2 + \frac{3}{4}r^2 d\phi^2$ … Hint 2: Compare this to the metric for 3D spherical coord's (r,θ,ϕ) … which is $dl^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2$ … what constraint can you impose to turn this familiar metric into the strange one with the ¾ factor … Hint 3: fix θ to a constant value … the object is a cone of 60° opening angle. ☺

have to *do* the integral in part (b), you only need to remember how to take a derivative with respect to the *upper bound* of an integral.

(d) **What did we just learn? ➔** The calculation you performed provides the explanation for one of the most famous plots you see "explaining" General Relativity: the "funnel picture". I've included the version from Taylor & Wheeler's book "Exploring Black Holes" on the last page, as Edwin Taylor has been kind enough to provide a free copy of Chapter 2 on his website at http://www.eftaylor.com/download.html#general_relativity . Many people have seen this funnel picture, but very few know what it actually represents. Examine the figures intently and make sure you understand them completely ... it takes some thought, but it is nothing more than a graphical depiction of the exact calculation you just performed. ☺ Check your thinking against the checkpoint, but if you are not 100% sure of your understanding of this famous picture, talk to your TA!

(e) If you were in a spaceship sitting somewhere near a massive spherical planet, what exact measurement(s) could you perform to figure out your location in the Schwarzschild coordinate r?

(f) Same spaceship, different question. You can see the polar ice cap marking the planet's North Pole, so you call that direction the $+z$-axis. What measurement(s) could you perform to figure out your location in Schwarzschild coordinate θ? (The polar angle, the one we keep dropping from the metric.)

☞ We now have a thorough understanding of the Schwarzschild coordinates (t, r, θ, ϕ) in terms of which the Schwarzschild metric is written. Summary:

- The angles θ and ϕ are what they always are, since there are no mass-dependent factors in the $d\theta$ or $d\phi$ terms of the metric.
- t is the time recorded by imaginary clocks that are placed throughout space and are rate-adjusted to compensate for gravitational time dilation. These clocks thus tick at the same rate as a clock placed at infinity (and they tick faster than local clocks sitting next to them that have not been tampered with).
- r is $C/2\pi$, where C is the circumference of a circle drawn around M and measured by actual people laying down actual rulers all the way around the circle.

Problem 3 : Curvature of a Mystery Space *Checkpoints on previous page*

(a) Here is the metric for a 2D space: $dl^2 = \frac{16}{3}\rho^2\, d\rho^2 + \rho^4\, d\phi^2$. This metric describes a rather simple surface, but it is disguised via the use of an unusual polar coordinate system: ϕ is still the azimuthal angle and ρ is the radial coordinate, but you can immediately see that ρ doesn't have units of distance. Treat this mysterious metric as an experimental tool: it allows you to *measure distance* in this mystery space, via path integrals, and thereby figure out the nature of the space. So let's measure! Calculate the quantity $k = (2\pi R - C)$ for a circle of coordinate-radius ρ_0 around the origin $(\rho,\phi) = (0,\text{irrelevant})$. Is it positive or negative? If it's positive, it is "sphere-like": it has some "bowl-like" shape around the origin that is making circles smaller than $2\pi \times$ the distance it takes to *get* to the circle. If it's negative, the reverse is true.

(b-OPTIONAL) If you would like a challenge, or if you are dying of curiosity, you might like to try figuring out what surface this is. ☺ Here's your first hint: replace the radial coordinate with $r = 2\rho^2/\sqrt{3}$, change the metric accordingly, and see if you can figure out what surface it describes. Further hints are in the footnote.

Figure from Taylor & Wheeler, "Exploring Black Holes" illustrating result of Problem 2

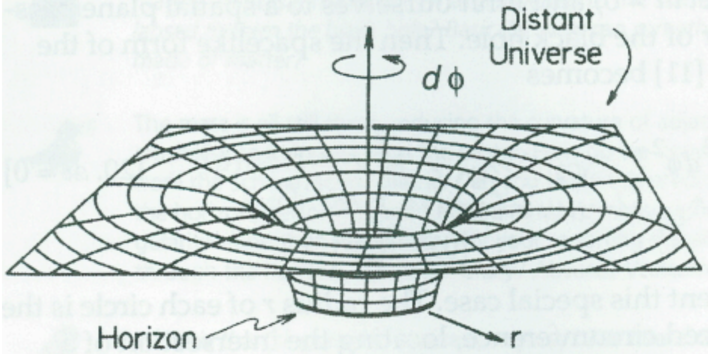

***Figure 6** Space geometry for a plane sliced through the center of a black hole, the result "embedded" in a three-dimensional Euclidean perspective. All of the curvature of empty space (space free of any mass–energy whatsoever) derives from the mass of the black hole. Circles are the intersections of the spherical shells with the slicing plane. WE add the vertical dimension to show that dσ is greater than dr in the spatial part of the Schwarzschild metric, as shown more clearly in Figure 7.*

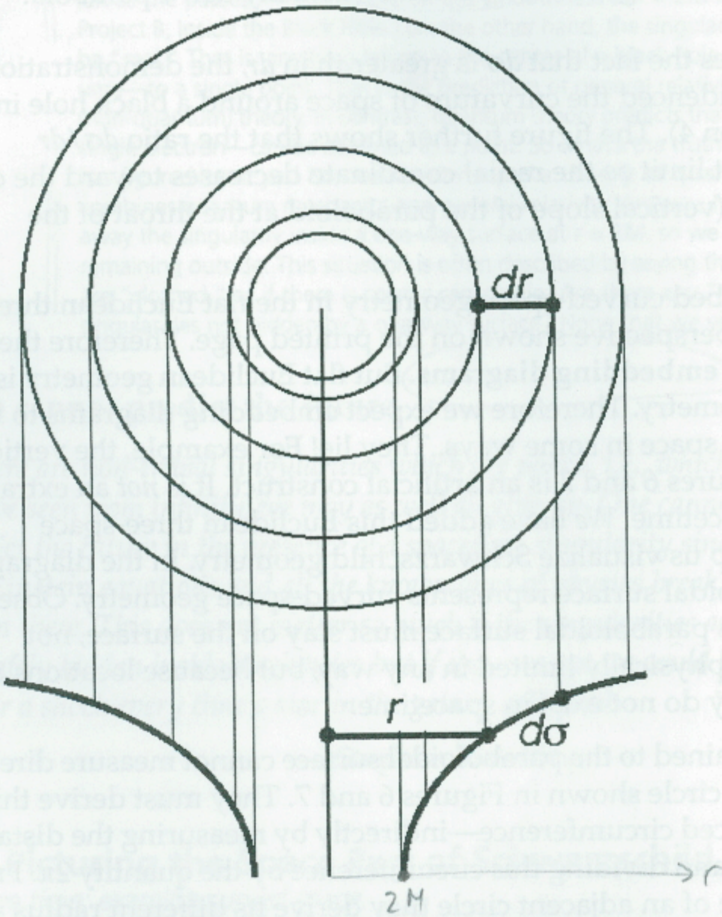

Figure 7** Projections of the embedding diagram of Figure 6, showing how the directly measured radial distance dσ between two adjacent spherical shells is greater than the difference dr in r-coordinates. Real observers exist only on the paraboloidal surface (shown edge-on as the heavy curved line). They can measure dσ directly but not r or dr. They derive the r-coordinate (the* **reduced circumference) of a given circle by measuring its circumference and dividing by 2π. Then dr is the computed difference between the reduced circumferences of adjacent circles.*