Phys 326 Discussion 14 – GR: Reduced Circumference and Speed of Light

Here is the Schwarzschild metric describing the spacetime curvature around a spherically-symmetric non-rotating mass M:

$$d\tau^2 = dt^2\left(1-\frac{2M}{r}\right) - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad \text{where } M = \frac{G}{c^2} M_{\text{kg}} \text{ and } t = c\, t_{\text{sec}}.$$

Problem 1 : Circles around a Black Hole

Checkpoints [1]

Let's explore the spatial part of the Schwarzschild metric on its own: if we always make measurements at the *same times* (as we do when measuring purely spatial distances like the radii and circumferences of circles), the dt term in the metric is zero, leaving only the dr and $d\phi$ terms. Since all such same-time measurements are spacelike, we switch to from proper time to **proper distance** as our metric:

$$d\sigma^2 \equiv -d\tau^2 = \frac{dr^2}{(1-2M/r)} + r^2 d\phi^2$$

The *only* reason for this sign change is to avoid getting an imaginary result, which you do when you calculate $d\tau$ for spacelike separated events. With this change, $d\sigma$ exactly plays the role of dl = ruler distance. Let's check out what circles look like near black holes! Note: we are just using "black hole" here as cool-sounding shorthand for "something that produces a gravitational field strong enough to require General Relativity".)

(a) Let's measure a circle of coordinate-radius r_0 around the origin $(r,\phi) = (0, \text{irrelevant})$, which is the center of the mass M. First, calculate/"measure" the circumference C by integrating $\int d\sigma$ around the circle. Your result will depend on r_0 of course.

(b) Set up — but don't integrate! — the $\int d\sigma$ integral you would need to measure the radius R of the circle.

(c) Uh oh, we have a problem: you undoubtedly used 0 and r_0 as your lower and upper bounds of integration … but look at what happens to the integrand at $r = 0$: it becomes *imaginary!* From lecture, we know that $2M$ is called the **Schwarzschild radius**, and when r goes down to $2M$, the $dr^2/(1-2M/r)$ term *blows up*, indicating that we have reached the limits of applicability of this metric. We can't use the Schwarzschild metric to explore radii *inside* the Schwarzschild radius. (This is exactly analogous to a familiar situation in Special Relativity: the collapse of the Lorentz transformation at speeds greater than c; for boost speeds $\beta > 1$ the gamma factors become imaginary, so the equations have officially broken down.) Oh boy … we *cannot* measure the radius of

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[1] **(a)** $C = 2\pi r_0$ **(c)** $\dfrac{dk}{dr_0} = 2\pi\left(\dfrac{1}{\sqrt{1-2M/r_0}} - 1\right)$ **(d)** You found that, for circles around a black hole, $2\pi dR \gg dC$ when you get close to the event horizon at $r_0 = 2M$. In words: near the event horizon, it takes a very large change in radius to produce a very small change in the circumference of a circle at that radius … and the discrepancy grows as you get closer to $r = 2M$. Yow! Such a phenomenon can never be drawn on a flat piece of paper … but it *can* be drawn on a flexible sheet curved into the shape of a funnel ... which is the famous picture. **(e)** Hint: Schwarzschild-r is sometimes called the **reduced circumference** … why? recall part (a) … possible measurement technique: Maintain a constant distance from the planet by e.g. bouncing a laser beam off its nearest surface and timing how long it takes to get back to you. Travel around the planet in a circle by keeping that laser-bounce-time constant, and measure the distance you travel as you go along. Once you've returned to your starting point, divide your measured circumference by $2\pi \rightarrow$ that's your r coordinate. **(f)** Maintain a constant distance from the planet's surface using the same laser-bounce technique, but make sure your circular path takes you directly over the ice cap this time. Measure the distance you travel as you proceed, and stop measuring when you're over the ice cap. If you measured a distance d, the polar angle θ at your starting point was $2\pi\,(d\,/\,C) = 2\pi$ times (the fraction of a full circle you had to travel).

our circles, so we cannot investigate the curvature $k = (2\pi R - C)$, how disappointing! But wait: we can do something else, we can investigate how this curvature *varies* with coordinate-radius r_0 ➔ calculate the "differential" curvature $\frac{dk}{dr_0} = 2\pi \frac{dR}{dr_0} - \frac{dC}{dr_0}$ and see if it is positive, negative, or zero. NOTE: at no point do you have to *do* the integral in part (b), you only need to remember how to take a derivative with respect to the *upper bound* of an integral.

(d) **What did we just learn?** ➔ The calculation you performed provides the explanation for one of the most famous plots you see "explaining" General Relativity: the "funnel picture". I've included the version from Taylor & Wheeler's book "Exploring Black Holes" on the last page, as Edwin Taylor has been kind enough to provide a free copy of Chapter 2 on his website at http://www.eftaylor.com/download.html#general_relativity . Many people have seen this funnel picture, but very few know what it actually represents. Examine the figures intently and make sure you understand them completely ... it takes some thought, but it is nothing more than a graphical depiction of the exact calculation you just performed. ☺ Check your thinking against the checkpoint, but if you are not 100% sure of your understanding of this famous picture, talk to your TA!

(e) If you were in a spaceship sitting somewhere near a massive spherical planet, what exact measurement(s) could you perform to figure out your location in the Schwarzschild coordinate r?

(f) Same spaceship, different question. You can see the polar ice cap marking the planet's North Pole, so you call that direction the $+z$-axis. What measurement(s) could you perform to figure out your location in Schwarzschild coordinate θ? (The polar angle, the one we keep dropping from the metric.)

☞ We now have a thorough understanding of the Schwarzschild coordinates (t, r, θ, ϕ) in terms of which the Schwarzschild metric is written. Summary:

- The angles θ and ϕ are what they always are, since there are no mass-dependent factors in the $d\theta$ or $d\phi$ terms of the metric.
- t is the time recorded by imaginary clocks that are placed throughout space and are rate-adjusted to compensate for gravitational time dilation. These clocks thus tick at the same rate as a clock placed at infinity (and they tick faster than local clocks sitting next to them that have not been tampered with).
- r is $C/2\pi$, where C is the circumference of a circle drawn around M and measured by actual people laying down actual rulers all the way around the circle.

Problem 2 : Curvature of a Mystery Space *Checkpoints* [2]

(a) Here is the metric for a 2D space: $dl^2 = \frac{16}{3}\rho^2\, d\rho^2 + \rho^4\, d\phi^2$. This metric describes a rather simple surface, but it is disguised via the use of an unusual polar coordinate system: ϕ is still the azimuthal angle and ρ is the radial coordinate, but you can immediately see that ρ doesn't have units of distance. Treat this mysterious metric as an experimental tool: it allows you to *measure distance* in this mystery space, via path integrals, and thereby figure out the nature of the space. So let's measure! Calculate the quantity $k = (2\pi R - C)$ for a circle of coordinate-radius ρ_0 around the origin $(\rho, \phi) = (0, \text{irrelevant})$. Is it positive or negative? If it's positive, it is "sphere-like": it has some "bowl-like" shape around the origin that is making circles smaller than $2\pi \times$ the

[2] **(a)** $k = 2\pi\rho_0^{\,2}\left(\frac{2}{\sqrt{3}} - 1\right)$ **(b)** after substitution: $dl^2 = dr^2 + \frac{3}{4}r^2 d\phi^2$ … Hint 2: Compare this to the metric for 3D spherical coord's (r,θ,ϕ) … which is $dl^2 = dr^2 + r^2 d\theta^2 + (r\sin\theta)^2 d\phi^2$ … what constraint can you impose to turn this familiar metric into the strange one with the ¾ factor … Hint 3: fix θ to a constant value … the object is a cone of 60° opening angle. ☺

distance it takes to *get* to the circle. If it's negative, the reverse is true.

(b-OPTIONAL) If you would like a challenge, or if you are dying of curiosity, you might like to try figuring out what surface this is. ☺ Here's your first hint: replace the radial coordinate with $r = 2\rho^2 / \sqrt{3}$, change the metric accordingly, and see if you can figure out what surface it describes. Further hints are in the footnote.

Problem 3 : The Path of Light *Checkpoints* [3]

We discussed in class how how a stationary observer standing at Schwarzschild radius *r* from the center of a mass *M* will measure events near their location using wristwatches that measure proper time and rulers that measure proper distance. We call this person a **local observer** because they are *at* (or near) the events they are observing. In symbols:

$$dt_{\text{local}} = d\tau\Big|_{\substack{\text{const}\\ r,\theta,\phi}} = dt\sqrt{1-\frac{2M}{r}} \qquad\qquad dr_{\text{local}} = d\sigma\Big|_{\substack{\text{const}\\ t,\theta,\phi}} = dr \Big/ \sqrt{1-\frac{2M}{r}}$$

(a) Light follows a very special geodesic: the **null geodesic** $d\tau = 0$. In words, proper time is slowed down to a complete stop for light, compared to any observer that is not moving at the speed of light. See if you can remind yourself how to show this using special relativity, i.e. in free space, when there is no mass *M* around.

(b) What about when there is a mass *M* present? One way to show that $d\tau = 0$ for a light ray is using one of the defining properties of light: it has zero rest mass. On this week's homework, you did / will show that the Schwarzschild metric allows you to identify this quantity as the dimensionless energy $e \equiv E_{\text{TOTAL}} / E_{\text{at REST}}$ of a particle moving radially :

$$e = \frac{dt}{d\tau}\left(1-\frac{2M}{r}\right)$$

(The expression is only slightly changed when angular motion is added, don't worry, you just get another term in the brackets.) Do you still agree that $d\tau = 0$ for light?

(c) Given that $d\tau = 0$ for light, calculate the speed of light as a function of *r* for a photon falling radially toward the center of a black hole. Calculate two versions of this speed: the "Bookkeeper" speed $|dr/dt|$ and the local speed seen by stationary local observers measuring local radial distance with their rulers and local time with their wristwatches as the photon passes right by them. Plot these two speeds vs r/M, being careful to label which is which, and to label the one important value on the horizontal axis.

[3] (a) Lorentz time dilation … Derivation thereof: For a particle moving with speed u as measured in an inertial frame with coordinates (t, x, y, z), proper time along the particle's path is $d\tau = (dt^2 - dl^2 / c^2)^{½} = dt\,(1 - u^2/c^2)^{½} = dt / \gamma_u$ = Lorentz time dilation formula ... If the particle is light, moving at $u = c$, then $d\tau = 0 \cdot dt = 0$ always. (b) $E_{\text{at REST}} = m\,c^2$ where m = rest mass. For a photon, $m = 0$, so $d\tau / dt = 0$ (note that the photon is never at rest so $E_{\text{REST}} = 0$ is not a problem.) (c) Bookkeeper speed = $1 - 2M/r$, local speed = 1 in natural units

Figure from Taylor & Wheeler, "Exploring Black Holes" illustrating result of Problem 2

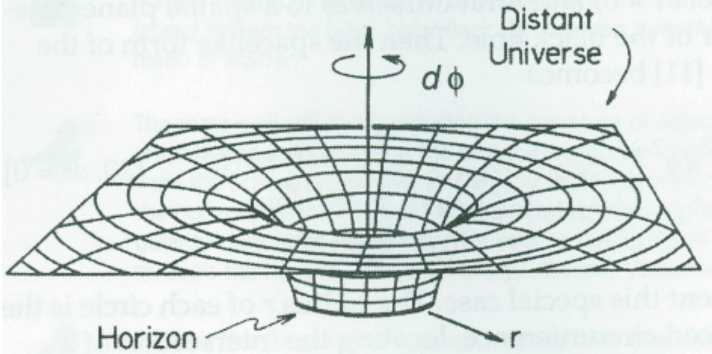

Figure 6 Space geometry for a plane sliced through the center of a black hole, the result "embedded" in a three-dimensional Euclidean perspective. All of the curvature of empty space (space free of any mass–energy whatsoever) derives from the mass of the black hole. Circles are the intersections of the spherical shells with the slicing plane. WE add the vertical dimension to show that dσ is greater than dr in the spatial part of the Schwarzschild metric, as shown more clearly in Figure 7.

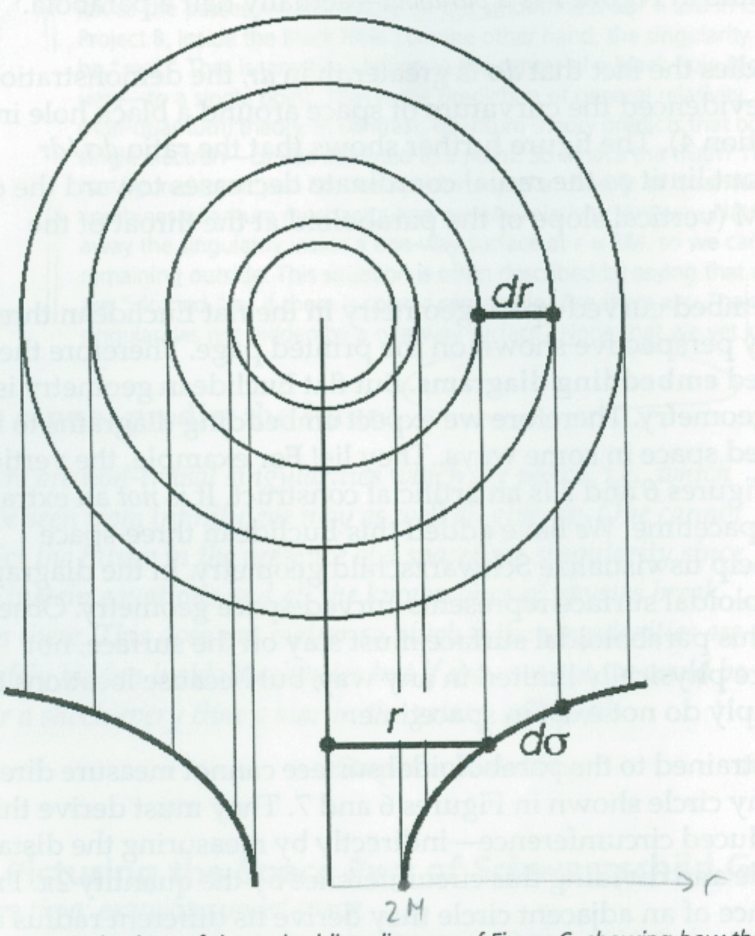

Figure 7 Projections of the embedding diagram of Figure 6, showing how the directly measured radial distance dσ between two adjacent spherical shells is greater than the difference dr in r-coordinates. Real observers exist only on the paraboloidal surface (shown edge-on as the heavy curved line). They can measure dσ directly but not r or dr. They derive the r-coordinate (the reduced circumference) of a given circle by measuring its circumference and dividing by 2π. Then dr is the computed difference between the reduced circumferences of adjacent circles.