All you need for GR calculations is the Schwarzschild metric. The Hamiltonian = the particle’s total energy (rest + kinetic + potential energy) is also useful because it is conserved. You derived it on the last homework for radial motion, so you can use it without proof this week:

\[ d\tau^2 = dt^2 \left( 1 - \frac{2M}{r} \right) - \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} - r^2 d\phi^2 \]  
and \[ e = \frac{dt}{d\tau} \left( 1 - \frac{2M}{r} \right) \text{ for radial motion} \]

where \( e = \frac{E_{\text{TOTAL}}}{mc^2} \) in SI units

**Problem 1 : Time of Radial Fall**

Alice is in a spaceship of mass \( m \) and constant energy \( e \) that falls under gravity (no engines) toward the center of a spherical mass \( M \) along a radial path. Last week, you used the energy formula and a bit of algebra to obtain the Bookkeeper speed of such a radially-falling mass: \( \dot{r} = -\left( 1 - \frac{2M}{r} \right) \sqrt{1 - \left( \frac{1}{e} \right)^2 \left( 1 - \frac{2M}{r} \right)} \). To simplify your calculations, let’s use \( e = 1 \).

(a) What does \( e = 1 \) tell us about the spaceship’s initial conditions?

(b) How much Bookkeeper time does it take for the spaceship to fall to the black hole’s event horizon? Calculate a formula for \( \tau \equiv \frac{t}{M} \) as a function of \( \bar{r} \equiv \frac{r}{M} \). (Just start calculating \( t \) and a function of \( r \), then as you construct your integral, notice how \( M \) can be completely removed from the problem by rescaling \( t \) and \( r \).) It requires a nasty integral, but one that wolframalpha can do just fine! To interpret your result, let’s set our clock to \( t = 0 \) at somewhere finite, say at \( r = 5M \). Plot \( \tau(r) - \tau(r = 5M) \) versus \( \bar{r} \) over the range \( \bar{r} = 2 \) to 8.

(c) How much proper time does it take for the spaceship to fall to the black hole’s event horizon? Calculate a formula for \( \tau \equiv \tau / M \) as a function of \( \bar{r} \equiv r / M \), then plot \( \tau(r) - \tau(r = 5M) \) versus \( \bar{r} \) again over the range \( \bar{r} = 2 \) to 8. Finally, in words: what exactly does the proper time difference \( \tau(r=2M) - \tau(r=5M) \) represent? i.e. What does it mean physically?

**Problem 2 : Rolling Coin**

We must return to rotating systems in order to complete a classic qual exam problem. Shown at right is a coin, which we treat as a uniform disk of radius \( a \) and negligible thickness whose edge rolls without slipping on a table. The center of the coin moves in a circle of radius \( b \) when the axis of the coin is tilted at an angle \( \theta \) with respect to the vertical. Find the angular velocity \( \Omega \) of the center of mass of the coin (in the frame of the table).

Note: Don’t assume that we left this problem until now because the Euler angles (\( \theta, \phi, \psi \)) are required. They might be helpful but they are not required. You might alternatively find the Euler equations helpful. Or not. Just clarifying that Euler’s angles and Euler’s equations are different. Euler did a lot.

Hint: A common strategy for this one is to go to a rotating frame … but it’s not the only strategy.
Problem 3: The EM Lagrangian → Something You Need to Know

The Lagrangian for GR and SR is \( L = -mc^2 d\tau / dt \) in SI units. This does not have the form \( L = T - U \) → not all Lagrangians do! (What you showed last week is that it reduces to \( T-U \) in the Newtonian limit of slow speed and weak gravity.) The Lagrangian that describes the motion of a non-relativistic particle of mass \( m \) and charge \( e \) in an electromagnetic field also does not have \( T-U \) form: it is

\[
L_{EM} = \frac{1}{2} m \dot{r}^2 - e(V - \dot{r} \cdot \vec{A}).
\]

Here \( \{ V, \vec{A} \} \) are the \{scalar,vector\} potentials of E&M that give the fields via the relations \( \vec{E} = -\nabla V - \partial \vec{A} / \partial t \) and \( \vec{B} = \vec{\nabla} \times \vec{A} \). We are using \( e \) for the particle’s charge since \( q \) could cause confusion with “generalized coordinate”. Obviously we can write this as \( L_{EM} = T - e(V - \dot{r} \cdot \vec{A}) \), which looks like \( T-U \) form, but it really isn’t. Let’s explore.

(●) Obviously you will be profoundly unhappy unless you verify for yourself that this unusual Lagrangian does indeed reproduce the E&M force law, \( \vec{F} = e(\vec{E} + \vec{v} \times \vec{B}) \)! Use Cartesian coordinates \((x,y,z)\) for the particle’s position, then write down and expand the Euler-Lagrange equation for just one coordinate, \( x \). If you can show that this equation reduces to \( m\ddot{x} = F_x = e(E_x + yB_z - zB_y) \), it will be true for \( y \) and \( z \) as well. This derivation is done in Taylor section 7.9, but please do try it yourself.

(a) \( L_{EM} \) certainly looks like a typical Lagrangian of the form \( L = T - U \), with a velocity-dependent potential energy \( e(V - \dot{r} \cdot \vec{A}) \). However, this is not a potential energy and should even not be given the letter \( U \) … we can call it \( U^* \). Why all the caution? The defining property of a potential energy is that this equation reduces to \( \dot{U} = -\nabla U \). Well this one doesn’t! Calculate the gradient of \( U^* = e(V - \dot{r} \cdot \vec{A}) \) and show that \( -\nabla U^* \) is not the Coulomb-Lorentz force \( \vec{F} = e(\vec{E} + \vec{v} \times \vec{B}) \).

Hint 1: Perform your calculation in Cartesian components with index notation: \( -\nabla U^* = -\hat{x_i} \partial U^* / \partial x_i \)

Hint 2: The easiest way to show that \( -\nabla U^* \) is not \( e(\vec{E} + \vec{v} \times \vec{B}) \) is to find one example that disproves it.

Specifically, find a vector-potential \( \vec{A} \) that produces no magnetic field (i.e. an \( \vec{A} \) that has no curl) but does have some derivative: one that is sufficient to make an appearance in \( -\nabla U^* \) and cause it to not match the force \( e\vec{E} \).

☞ Take-away message: \( L_{EM} \) is another example of a Lagrangian that does not follow the form \( L = T - U \).

(b) A particle of mass \( m \) and charge \( e \) moves in a magnetic field produced by a constant current \( I \) flowing in an infinite straight wire that lies along the \( z \)-axis. This \( B \)-field is given by the potentials \( A_x = A_y = V = 0 \) and \( A_z = -(\mu_0 I / 2\pi) \ln(s) \) in the cylindrical coordinate system \((s,\phi,z)\). Find the Lagrangian of the particle, note that \( \phi \) and \( z \) are cyclic coordinates, and find the corresponding conserved quantities \( p_\phi \) and \( p_z \).

☞ Crucial point: The conserved \( p_z \) you just found is the generalized momentum conjugate to \( z \)… and it is NOT the \( z \)-component of mechanical momentum \( (mv)_z \)! → For this important E&M Lagrangian, we must be very careful to distinguish the two DIFFERENT concepts of “momentum”:

- the canonical momentum \( \hat{p} \) obtained from \( p_i = \partial L / \partial \dot{x_i} \), and
- the mechanical momentum \( \vec{mv} \) → Do NOT call this “\( p \)” for the rest of this problem, call it “\( mv \)”!

☞ Jargon check: The terms “canonical momentum”, “generalized momentum”, and “momentum conjugate to \( q_i \)” all mean exactly the same thing: \( p_i = \partial L / \partial \dot{q_i} \).
(c) Interpret the two conserved quantities you found by expressing them in terms of components of the following physical quantities: the particle’s linear momentum \( m\dot{v} \), the particle’s angular momentum \( \vec{l} \), its charge \( e \), and/or \( \vec{A} \) itself. Remember: to avoid massive confusion do NOT USE THE LETTER “p” for the combination “mv”! We are reserving “p” for canonical momentum.

(d), (e)-combined for clarity

To explore these two definitions of momentum further, consider the E&M Lagrangian and Hamiltonian for any electromagnetic potentials \( V \) and \( \vec{A} \) (not just the ones from part b). Go to Cartesian coordinates \( q_i = (x,y,z) \) and show that:

- the relationship between mechanical and canonical momentum is \( m\dot{v} = \vec{p} - e\vec{A} \);
- the Hamiltonian for a charge \( e \) of mass \( m \) in an EM field is \( H = \frac{1}{2m}(\vec{p} - e\vec{A})^2 + eV \).

\( \textbf{The notorious substitution} \): The relations you just found lead to a confusing “trick” that is invoked in other areas of physics, notably quantum mechanics. In various QM texts, you will see the author take a regular Hamiltonian, \( H = T + U = p^2 / 2m + U \), then make this substitution: they replace \( \vec{p} \) with \( \vec{p} - e\vec{A} \) to “add electromagnetism” to their equations of motion. This substitution is often done with little or no explanation, or by saying it is “familiar from classical mechanics”. (eye-roll) Now you know where it comes from … and you also know that the “p” in question is canonical momentum, NOT mechanical momentum.

(f) Finally, rewrite your Hamiltonian in this way: remove the canonical momentum \( \vec{p} \) and replace it with the combination \( mv + e\vec{A} \). Since we know what \( m\dot{v} \) is — it’s the particle’s actual, mechanical momentum — we can now interpret this Hamiltonian! Use your knowledge of the work done by a magnetic field to show that your E&M Hamiltonian is, in fact, the total (kinetic + potential) energy of a charge in an electromagnetic field.

\( \textbf{A Vital Message} \): The Hamiltonian \( H_{EM} = (\vec{p} - e\vec{A})^2 / 2m + eV \) can lead to enormous confusion.

(I have seen it happen, e.g. throughout a 2-week research workshop with numerous theorists in attendance.) The key point: the “p” in this Hamiltonian is NOT the particle’s momentum \( mv \)! People constantly assume that it is, then go on to believe that the \( -e\vec{A} \) term added to \( \vec{p} \) is some sort of “field momentum”. As you saw in (f), our Hamiltonian is nothing more than \( H_{EM} = mv^2 / 2 + eV = T+U \) of the charged particle; it does not contain any trace whatsoever of the momentum or energy of the EM fields. If field energy plays a role, \( V \) and/or \( \vec{A} \) will be time-dependent and lead to \( \partial H / \partial t \neq 0 \), i.e. a non-conserved Hamiltonian \( H \). To restore conservation of energy, a different expression is needed that includes the energy in both the particles and the fields. Chapter 8 of Griffiths’ “Introduction to Electrodynamics” contains an excellent discussion of energy and momentum conservation in E&M ... though not in Hamiltonian or Lagrangian form. To go further, you must study the Lagrangian formulation of field theories, and that is well beyond the scope of our course.