Physics Formulary

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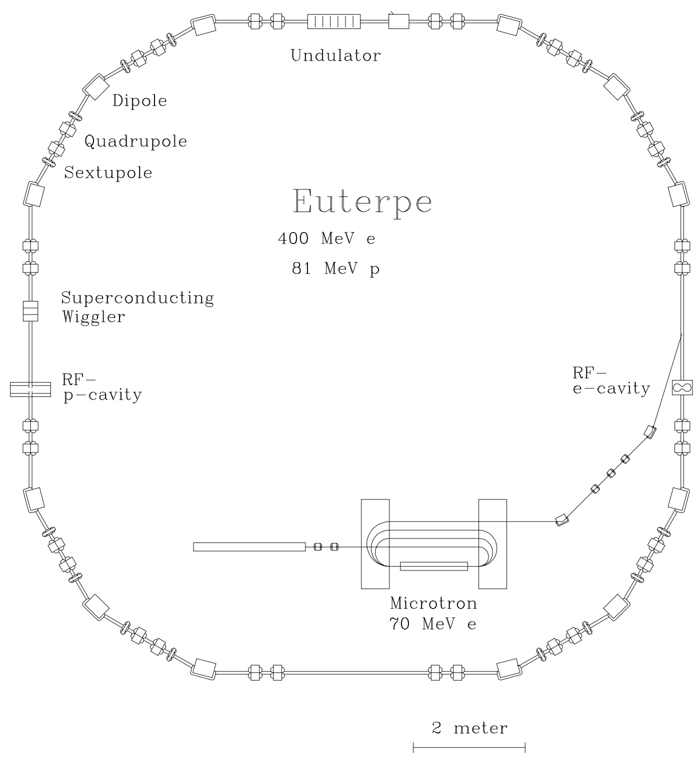

Dear reader,

This document contains a 108 page LaTeX file which contains a lot equations in physics. It is written at advanced undergraduate/postgraduate level. It is intended to be a short reference for anyone who works with physics and often needs to look up equations.

This, and a Dutch version of this file, can be obtained from the author, Johan Wevers (`johanw@vulcan.xs4all.nl`).

It can also be obtained on the WWW. See `http://www.xs4all.nl/~johanw/index.html`, where also a Postscript version is available.

If you find any errors or have any comments, please let me know. I am always open for suggestions and possible corrections to the physics formulary.

The Physics Formulary is made with teTEX. This pdf file is made with pdfTEX.

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Physical Constants

Name	Symbol	Value	Unit
Number π	π	3.14159265358979323846	
Number e	e	2.71828182845904523536	
Euler's constant	$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \ln(n)\right) = 0.5772156649$		
Elementary charge	e	$1.60217733 \cdot 10^{-19}$	C
Gravitational constant	G, κ	$6.67259 \cdot 10^{-11}$	$m^3kg^{-1}s^{-2}$
Fine-structure constant	$\alpha = e^2/2hc\varepsilon_0$	$\approx 1/137$	
Speed of light in vacuum	c	$2.99792458 \cdot 10^8$	m/s (def)
Permittivity of the vacuum	ε_0	$8.854187 \cdot 10^{-12}$	F/m
Permeability of the vacuum	μ_0	$4\pi \cdot 10^{-7}$	H/m
$(4\pi\varepsilon_0)^{-1}$		$8.9876 \cdot 10^9$	Nm^2C^{-2}
Planck's constant	h	$6.6260755 \cdot 10^{-34}$	Js
Dirac's constant	$\hbar = h/2\pi$	$1.0545727 \cdot 10^{-34}$	Js
Bohr magneton	$\mu_B = e\hbar/2m_e$	$9.2741 \cdot 10^{-24}$	Am^2
Bohr radius	a_0	0.52918	Å
Rydberg's constant	Ry	13.595	eV
Electron Compton wavelength	$\lambda_{Ce} = h/m_e c$	$2.2463 \cdot 10^{-12}$	m
Proton Compton wavelength	$\lambda_{Cp} = h/m_p c$	$1.3214 \cdot 10^{-15}$	m
Reduced mass of the H-atom	μ_H	$9.1045755 \cdot 10^{-31}$	kg
Stefan-Boltzmann's constant	σ	$5.67032 \cdot 10^{-8}$	$Wm^{-2}K^{-4}$
Wien's constant	k_W	$2.8978 \cdot 10^{-3}$	mK
Molar gasconstant	R	8.31441	$J\cdot mol^{-1}\cdot K^{-1}$
Avogadro's constant	N_A	$6.0221367 \cdot 10^{23}$	mol^{-1}
Boltzmann's constant	$k = R/N_A$	$1.380658 \cdot 10^{-23}$	J/K
Electron mass	m_e	$9.1093897 \cdot 10^{-31}$	kg
Proton mass	m_p	$1.6726231 \cdot 10^{-27}$	kg
Neutron mass	m_n	$1.674954 \cdot 10^{-27}$	kg
Elementary mass unit	$m_u = \frac{1}{12}m(^{12}_{6}C)$	$1.6605656 \cdot 10^{-27}$	kg
Nuclear magneton	μ_N	$5.0508 \cdot 10^{-27}$	J/T
Diameter of the Sun	$D_\odot$	$1392 \cdot 10^6$	m
Mass of the Sun	$M_\odot$	$1.989 \cdot 10^{30}$	kg
Rotational period of the Sun	$T_\odot$	25.38	days
Radius of Earth	R_A	$6.378 \cdot 10^6$	m
Mass of Earth	M_A	$5.976 \cdot 10^{24}$	kg
Rotational period of Earth	T_A	23.96	hours
Earth orbital period	Tropical year	365.24219879	days
Astronomical unit	AU	$1.4959787066 \cdot 10^{11}$	m
Light year	lj	$9.4605 \cdot 10^{15}$	m
Parsec	pc	$3.0857 \cdot 10^{16}$	m
Hubble constant	H	$\approx (75 \pm 25)$	$km\cdot s^{-1}\cdot Mpc^{-1}$

Chapter 1

Mechanics

1.1 Point-kinetics in a fixed coordinate system

1.1.1 Definitions

The position $\vec{r}$, the velocity $\vec{v}$ and the acceleration $\vec{a}$ are defined by: $\vec{r} = (x, y, z)$, $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$, $\vec{a} = (\ddot{x}, \ddot{y}, \ddot{z})$. The following holds:

$$s(t) = s_0 + \int |\vec{v}(t)| dt \ ; \quad \vec{r}(t) = \vec{r}_0 + \int \vec{v}(t) dt \ ; \quad \vec{v}(t) = \vec{v}_0 + \int \vec{a}(t) dt$$

When the acceleration is constant this gives: $v(t) = v_0 + at$ and $s(t) = s_0 + v_0 t + \frac{1}{2} a t^2$.
For the unit vectors in a direction $\perp$ to the orbit $\vec{e}_{\rm t}$ and parallel to it $\vec{e}_{\rm n}$ holds:

$$\vec{e}_{\rm t} = \frac{\vec{v}}{|\vec{v}|} = \frac{d\vec{r}}{ds} \quad \dot{\vec{e}}_{\rm t} = \frac{v}{\rho} \vec{e}_{\rm n} \ ; \quad \vec{e}_{\rm n} = \frac{\dot{\vec{e}}_{\rm t}}{|\dot{\vec{e}}_{\rm t}|}$$

For the *curvature* k and the *radius of curvature* ρ holds:

$$\vec{k} = \frac{d\vec{e}_{\rm t}}{ds} = \frac{d^2\vec{r}}{ds^2} = \left| \frac{d\varphi}{ds} \right| \ ; \quad \rho = \frac{1}{|k|}$$

1.1.2 Polar coordinates

Polar coordinates are defined by: $x = r\cos(\theta)$, $y = r\sin(\theta)$. So, for the unit coordinate vectors holds: $\dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta$, $\dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r$

The velocity and the acceleration are derived from: $\vec{r} = r\vec{e}_r$, $\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$, $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$.

1.2 Relative motion

For the motion of a point D w.r.t. a point Q holds: $\vec{r}_{\rm D} = \vec{r}_{\rm Q} + \dfrac{\vec{\omega} \times \vec{v}_{\rm Q}}{\omega^2}$ with $\vec{\rm QD} = \vec{r}_{\rm D} - \vec{r}_{\rm Q}$ and $\omega = \dot{\theta}$.

Further holds: $\alpha = \ddot{\theta}$. $'$ means that the quantity is defined in a moving system of coordinates. In a moving system holds:
$\vec{v} = \vec{v}_{\rm Q} + \vec{v}\,' + \vec{\omega} \times \vec{r}\,'$ and $\vec{a} = \vec{a}_{\rm Q} + \vec{a}\,' + \vec{\alpha} \times \vec{r}\,' + 2\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$
with $\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = -\omega^2 \vec{r}\,'_n$

1.3 Point-dynamics in a fixed coordinate system

1.3.1 Force, (angular)momentum and energy

Newton's 2nd law connects the force on an object and the resulting acceleration of the object where the *momentum* is given by $\vec{p} = m\vec{v}$:

$$\vec{F}(\vec{r}, \vec{v}, t) = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v}\,)}{dt} = m\frac{d\vec{v}}{dt} + \vec{v}\,\frac{dm}{dt} \overset{m={\rm const}}{=} m\vec{a}$$

Newton's 3rd law is given by: $\vec{F}_{\rm action} = -\vec{F}_{\rm reaction}$.

For the power P holds: $P = \dot{W} = \vec{F} \cdot \vec{v}$. For the total energy W, the kinetic energy T and the potential energy U holds: $W = T + U$; $\dot{T} = -\dot{U}$ with $T = \frac{1}{2}mv^2$.

The *kick* $\vec{S}$ is given by: $\vec{S} = \Delta\vec{p} = \int \vec{F} dt$

The work A, delivered by a force, is $A = \int\limits_1^2 \vec{F} \cdot d\vec{s} = \int\limits_1^2 F\cos(\alpha) ds$

The torque $\vec{\tau}$ is related to the angular momentum $\vec{L}$: $\vec{\tau} = \dot{\vec{L}} = \vec{r} \times \vec{F}$; and
$\vec{L} = \vec{r} \times \vec{p} = m\vec{v} \times \vec{r}$, $|\vec{L}| = mr^2\omega$. The following equation is valid:

$$\tau = -\frac{\partial U}{\partial \theta}$$

Hence, the conditions for a mechanical equilibrium are: $\sum \vec{F}_i = 0$ and $\sum \vec{\tau}_i = 0$.

The *force of friction* is usually proportional to the force perpendicular to the surface, except when the motion starts, when a threshold has to be overcome: $F_{\rm fric} = f \cdot F_{\rm norm} \cdot \vec{e}_{\rm t}$.

1.3.2 Conservative force fields

A conservative force can be written as the gradient of a potential: $\vec{F}_{\rm cons} = -\vec{\nabla} U$. From this follows that $\nabla \times \vec{F} = \vec{0}$. For such a force field also holds:

$$\oint \vec{F} \cdot d\vec{s} = 0 \Rightarrow U = U_0 - \int\limits_{r_0}^{r_1} \vec{F} \cdot d\vec{s}$$

So the work delivered by a conservative force field depends not on the trajectory covered but only on the starting and ending points of the motion.

1.3.3 Gravitation

The Newtonian law of gravitation is (in GRT one also uses κ instead of G):

$$\vec{F}_{\rm g} = -G\frac{m_1 m_2}{r^2}\vec{e}_r$$

The gravitational potential is then given by $V = -Gm/r$. From Gauss law it then follows: $\nabla^2 V = 4\pi G\varrho$.

1.3.4 Orbital equations

If $V = V(r)$ one can derive from the equations of Lagrange for ϕ the conservation of angular momentum:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial V}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}(mr^2\phi) = 0 \Rightarrow L_z = mr^2\phi = \text{constant}$$

For the radial position as a function of time can be found that:

$$\left(\frac{dr}{dt}\right)^2 = \frac{2(W-V)}{m} - \frac{L^2}{m^2r^2}$$

The angular equation is then:

$$\phi - \phi_0 = \int\limits_0^r \left[\frac{mr^2}{L}\sqrt{\frac{2(W-V)}{m} - \frac{L^2}{m^2r^2}}\right]^{-1} dr \stackrel{r^{-2}\text{field}}{=} \arccos\left(1 + \frac{\frac{1}{r} - \frac{1}{r_0}}{\frac{1}{r_0} + km/L_z^2}\right)$$

If $F = F(r)$: L =constant, if F is conservative: W =constant, if $\vec{F} \perp \vec{v}$ then $\Delta T = 0$ and $U = 0$.

Kepler's orbital equations

In a force field $F = kr^{-2}$, the orbits are conic sections with the origin of the force in one of the foci (Kepler's 1st law). The equation of the orbit is:

$$r(\theta) = \frac{\ell}{1 + \varepsilon\cos(\theta - \theta_0)}\ , \ \text{ or: } x^2 + y^2 = (\ell - \varepsilon x)^2$$

with

$$\ell = \frac{L^2}{G\mu^2 M_{\rm tot}}\ ; \quad \varepsilon^2 = 1 + \frac{2WL^2}{G^2\mu^3 M_{\rm tot}^2} = 1 - \frac{\ell}{a}\ ; \quad a = \frac{\ell}{1-\varepsilon^2} = \frac{k}{2W}$$

a is half the length of the long axis of the elliptical orbit in case the orbit is closed. Half the length of the short axis is $b = \sqrt{a\ell}$. ε is the *excentricity* of the orbit. Orbits with an equal ε are of equal shape. Now, 5 types of orbits are possible:

1. $k < 0$ and $\varepsilon = 0$: a circle.
2. $k < 0$ and $0 < \varepsilon < 1$: an ellipse.
3. $k < 0$ and $\varepsilon = 1$: a parabole.
4. $k < 0$ and $\varepsilon > 1$: a hyperbole, curved towards the centre of force.
5. $k > 0$ and $\varepsilon > 1$: a hyperbole, curved away from the centre of force.

Other combinations are not possible: the total energy in a repulsive force field is always positive so $\varepsilon > 1$.

If the surface between the orbit covered between t_1 and t_2 and the focus C around which the planet moves is $A(t_1, t_2)$, Kepler's 2nd law is

$$A(t_1, t_2) = \frac{L_{\rm C}}{2m}(t_2 - t_1)$$

Kepler's 3rd law is, with T the period and $M_{\rm tot}$ the total mass of the system:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM_{\rm tot}}$$

1.3.5 The virial theorem

The virial theorem for one particle is:

$$\langle m\vec{v}\cdot\vec{r}\rangle = 0 \Rightarrow \langle T\rangle = -\tfrac{1}{2}\left\langle \vec{F}\cdot\vec{r}\right\rangle = \tfrac{1}{2}\left\langle r\frac{dU}{dr}\right\rangle = \tfrac{1}{2}n\langle U\rangle \ \text{ if } U = -\frac{k}{r^n}$$

The virial theorem for a collection of particles is:

$$\langle T\rangle = -\tfrac{1}{2}\left\langle \sum_{\rm particles} \vec{F}_i\cdot\vec{r}_i + \sum_{\rm pairs} \vec{F}_{ij}\cdot\vec{r}_{ij}\right\rangle$$

These propositions can also be written as: $2E_{\rm kin} + E_{\rm pot} = 0$.

1.4 Point dynamics in a moving coordinate system

1.4.1 Apparent forces

The total force in a moving coordinate system can be found by subtracting the apparent forces from the forces working in the reference frame: $\vec{F}\,' = \vec{F} - \vec{F}_{\rm app}$. The different apparent forces are given by:

1. Transformation of the origin: $F_{\rm or} = -m\vec{a}_a$
2. Rotation: $\vec{F}_\alpha = -m\vec{\alpha}\times\vec{r}\,'$
3. Coriolis force: $F_{\rm cor} = -2m\vec{\omega}\times\vec{v}$
4. Centrifugal force: $\vec{F}_{\rm cf} = m\omega^2\vec{r}_n\,' = -\vec{F}_{\rm cp}\ ;\ \vec{F}_{\rm cp} = -\dfrac{mv^2}{r}\vec{e}_r$

1.4.2 Tensor notation

Transformation of the Newtonian equations of motion to $x^\alpha = x^\alpha(x)$ gives:

$$\frac{dx^\alpha}{dt} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dt};$$

The chain rule gives:

$$\frac{d}{dt}\frac{dx^\alpha}{dt} = \frac{d^2x^\alpha}{dt^2} = \frac{d}{dt}\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\beta}{dt}\right) = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d^2\bar{x}^\beta}{dt^2} + \frac{d\bar{x}^\beta}{dt}\frac{d}{dt}\left(\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\right)$$

so:

$$\frac{d}{dt}\frac{\partial x^\alpha}{\partial \bar{x}^\beta} = \frac{\partial}{\partial \bar{x}^\gamma}\frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d\bar{x}^\gamma}{dt} = \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{d\bar{x}^\gamma}{dt}$$

This leads to:

$$\frac{d^2x^\alpha}{dt^2} = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}\frac{d^2\bar{x}^\beta}{dt^2} + \frac{\partial^2 x^\alpha}{\partial \bar{x}^\beta \partial \bar{x}^\gamma}\frac{d\bar{x}^\gamma}{dt}\left(\frac{d\bar{x}^\beta}{dt}\right)$$

Hence the Newtonian equation of motion

$$m\frac{d^2x^\alpha}{dt^2} = F^\alpha$$

will be transformed into:

$$m\left\{\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}\right\} = F^\alpha$$

The apparent forces are taken from he origin to the effect side in the way $\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}$.

1.5 Dynamics of masspoint collections

1.5.1 The centre of mass

The velocity w.r.t. the centre of mass $\vec{R}$ is given by $\vec{v} - \dot{\vec{R}}$. The coordinates of the centre of mass are given by:

$$\vec{r}_{\rm m} = \frac{\sum m_i\vec{r}_i}{\sum m_i}$$

In a 2-particle system, the coordinates of the centre of mass are given by:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

With $\vec{r} = \vec{r}_1 - \vec{r}_2$, the kinetic energy becomes: $T = \frac{1}{2}M_{\rm tot}\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$, with the *reduced mass* μ given by: $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

The motion within and outside the centre of mass can be separated:

$$\dot{\vec{L}}_{\rm outside} = \vec{\tau}_{\rm outside}\ ;\quad \dot{\vec{L}}_{\rm inside} = \vec{\tau}_{\rm inside}$$

$$\vec{p} = m\vec{v}_{\rm m}\ ;\quad \vec{F}_{\rm ext} = m\vec{a}_{\rm m}\ ;\quad \vec{F}_{12} = \mu\vec{u}$$

1.5.2 Collisions

With collisions, where B are the coordinates of the collision and C an arbitrary other position, holds: $\vec{p} = m\vec{v}_{\rm m}$ is constant, and $T = \frac{1}{2}m\vec{v}_{\rm m}^{\,2}$ is constant. The changes in the *relative velocities* can be derived from: $\vec{S} = \Delta\vec{p} = \mu(\vec{v}_{\rm aft} - \vec{v}_{\rm before})$. Further holds $\Delta\vec{L}_{\rm C} = \vec{\rm CB} \times \vec{S}$, $\vec{p}\ \parallel\ \vec{S}$ =constant and $\vec{L}$ w.r.t. B is constant.

1.6 Dynamics of rigid bodies

1.6.1 Moment of Inertia

The angular momentum in a moving coordinate system is given by:

$$\vec{L}' = I\vec{\omega} + \vec{L}'_n$$

where I is the *moment of inertia* with respect to a central axis, which is given by:

$$I = \sum_i m_i \vec{r}_i^{\ 2} \ ; \quad T' = W_{\rm rot} = \tfrac{1}{2}\omega I_{ij}\vec{e}_i\vec{e}_j = \tfrac{1}{2}I\omega^2$$

or, in the continuous case:

$$I = \frac{m}{V}\int r'^2_n dV = \int r'^2_n dm$$

Further holds:

$$L_i = I^{ij}\omega_j \ ; \quad I_{ii} = I_i \ ; \quad I_{ij} = I_{ji} = -\sum_k m_k x'_i x'_j$$

Steiner's theorem is: $I_{\rm w.r.t.D} = I_{\rm w.r.t.C} + m(DM)^2$ if axis C $\parallel$ axis D.

Object	I	**Object**	I
Cavern cylinder	$I = mR^2$	Massive cylinder	$I = \frac{1}{2}mR^2$
Disc, axis in plane disc through m	$I = \frac{1}{4}mR^2$	Halter	$I = \frac{1}{2}\mu R^2$
Cavern sphere	$I = \frac{2}{3}mR^2$	Massive sphere	$I = \frac{2}{5}mR^2$
Bar, axis $\perp$ through c.o.m.	$I = \frac{1}{12}ml^2$	Bar, axis $\perp$ through end	$I = \frac{1}{3}ml^2$
Rectangle, axis $\perp$ plane thr. c.o.m.	$I = \frac{1}{12}m(a^2+b^2)$	Rectangle, axis $\parallel$ b thr. m	$I = ma^2$

1.6.2 Principal axes

Each rigid body has (at least) 3 principal axes which stand $\perp$ to each other. For a principal axis holds:

$$\frac{\partial I}{\partial \omega_x} = \frac{\partial I}{\partial \omega_y} = \frac{\partial I}{\partial \omega_z} = 0 \ \text{ so } \ L'_n = 0$$

The following holds: $\dot{\omega}_k = -a_{ijk}\omega_i\omega_j$ with $a_{ijk} = \dfrac{I_i - I_j}{I_k}$ if $I_1 \leq I_2 \leq I_3$.

1.6.3 Time dependence

For torque of force $\vec{\tau}$ holds:

$$\vec{\tau}\,' = I\ddot{\theta} \ ; \quad \frac{d''\vec{L}'}{dt} = \vec{\tau}\,' - \vec{\omega}\times\vec{L}'$$

The *torque* $\vec{T}$ is defined by: $\vec{T} = \vec{F}\times\vec{d}$.

1.7 Variational Calculus, Hamilton and Lagrange mechanics

1.7.1 Variational Calculus

Starting with:

$$\delta\int_a^b \mathcal{L}(q,\dot{q},t)dt = 0 \ \text{ with } \ \delta(a) = \delta(b) = 0 \ \text{ and } \ \delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u)$$

the equations of Lagrange can be derived:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

When there are additional conditions applying to the variational problem $\delta J(u) = 0$ of the type $K(u) =$constant, the new problem becomes: $\delta J(u) - \lambda\delta K(u) = 0$.

1.7.2 Hamilton mechanics

The *Lagrangian* is given by: $\mathcal{L} = \sum T(\dot{q}_i) - V(q_i)$. The *Hamiltonian* is given by: $H = \sum \dot{q}_i p_i - \mathcal{L}$. In 2 dimensions holds: $\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r,\phi)$.

If the used coordinates are *canonical* the Hamilton equations are the equations of motion for the system:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\ ; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Coordinates are canonical if the following holds: $\{q_i, q_j\} = 0,\ \{p_i, p_j\} = 0,\ \{q_i, p_j\} = \delta_{ij}$ where $\{,\}$ is the *Poisson bracket*:

$$\{A, B\} = \sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$$

The Hamiltonian of a Harmonic oscillator is given by $H(x,p) = p^2/2m + \frac{1}{2}m\omega^2x^2$. With new coordinates (θ, I), obtained by the canonical transformation $x = \sqrt{2I/m\omega}\cos(\theta)$ and $p = -\sqrt{2Im\omega}\sin(\theta)$, with inverse $\theta = \arctan(-p/m\omega x)$ and $I = p^2/2m\omega + \frac{1}{2}m\omega x^2$ it follows: $H(\theta, I) = \omega I$.

The Hamiltonian of a charged particle with charge q in an external electromagnetic field is given by:

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + qV$$

This Hamiltonian can be derived from the Hamiltonian of a free particle $H = p^2/2m$ with the transformations $\vec{p} \to \vec{p} - q\vec{A}$ and $H \to H - qV$. This is elegant from a relativistic point of view: this is equivalent to the transformation of the momentum 4-vector $p^\alpha \to p^\alpha - qA^\alpha$. A gauge transformation on the potentials A^α corresponds with a canonical transformation, which make the Hamilton equations the equations of motion for the system.

1.7.3 Motion around an equilibrium, linearization

For natural systems around equilibrium the following equations are valid:

$$\left(\frac{\partial V}{\partial q_i}\right)_0 = 0\ ; \quad V(q) = V(0) + V_{ik}q_iq_k \ \text{ with } \ V_{ik} = \left(\frac{\partial^2 V}{\partial q_i \partial q_k}\right)_0$$

With $T = \frac{1}{2}(M_{ik}\dot{q}_i\dot{q}_k)$ one receives the set of equations $M\ddot{q} + Vq = 0$. If $q_i(t) = a_i\exp(i\omega t)$ is substituted, this set of equations has solutions if $\det(V - \omega^2 M) = 0$. This leads to the eigenfrequencies of the problem: $\omega_k^2 = \dfrac{a_k^{\rm T} V a_k}{a_k^{\rm T} M a_k}$. If the equilibrium is stable holds: $\forall k$ that $\omega_k^2 > 0$. The general solution is a superposition if eigenvibrations.

1.7.4 Phase space, Liouville's equation

In phase space holds:

$$\nabla = \left(\sum_i \frac{\partial}{\partial q_i}, \sum_i \frac{\partial}{\partial p_i}\right) \ \text{ so } \ \nabla\cdot\vec{v} = \sum_i \left(\frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

If the equation of continuity, $\partial_t\varrho + \nabla\cdot(\varrho\vec{v}) = 0$ holds, this can be written as:

$$\{\varrho, H\} + \frac{\partial \varrho}{\partial t} = 0$$

For an arbitrary quantity A holds:

$$\frac{dA}{dt}=\{A,H\}+\frac{\partial A}{\partial t}$$

Liouville's theorem can than be written as:

$$\frac{d\varrho}{dt}=0\ ;\quad \text{or: } \int pdq=\text{constant}$$

1.7.5 Generating functions

Starting with the coordinate transformation:

$$\left\{\begin{array}{l} Q_i=Q_i(q_i,p_i,t)\\ P_i=P_i(q_i,p_i,t)\end{array}\right.$$

one can derive the following Hamilton equations with the new Hamiltonian K:

$$\frac{dQ_i}{dt}=\frac{\partial K}{\partial P_i}\ ;\quad \frac{dP_i}{dt}=-\frac{\partial K}{\partial Q_i}$$

Now, a distinction between 4 cases can be made:

1. If $p_i\dot{q}_i-H=P_iQ_i-K(P_i,Q_i,t)-\frac{dF_1(q_i,Q_i,t)}{dt}$, the coordinates follow from:

$$p_i=\frac{\partial F_1}{\partial q_i}\ ;\quad P_i=-\frac{\partial F_1}{\partial Q_i}\ ;\quad K=H+\frac{\partial F_1}{\partial t}$$

2. If $p_i\dot{q}_i-H=-\dot{P}_iQ_i-K(P_i,Q_i,t)+\frac{dF_2(q_i,P_i,t)}{dt}$, the coordinates follow from:

$$p_i=\frac{\partial F_2}{\partial q_i}\ ;\quad Q_i=\frac{\partial F_2}{\partial P_i}\ ;\quad K=H+\frac{\partial F_2}{\partial t}$$

3. If $-\dot{p}_iq_i-H=P_i\dot{Q}_i-K(P_i,Q_i,t)+\frac{dF_3(p_i,Q_i,t)}{dt}$, the coordinates follow from:

$$q_i=-\frac{\partial F_3}{\partial p_i}\ ;\quad P_i=-\frac{\partial F_3}{\partial Q_i}\ ;\quad K=H+\frac{\partial F_3}{\partial t}$$

4. If $-\dot{p}_iq_i-H=-P_iQ_i-K(P_i,Q_i,t)+\frac{dF_4(p_i,P_i,t)}{dt}$, the coordinates follow from:

$$q_i=-\frac{\partial F_4}{\partial p_i}\ ;\quad Q_i=\frac{\partial F_4}{\partial p_i}\ ;\quad K=H+\frac{\partial F_4}{\partial t}$$

The functions F_1, F_2, F_3 and F_4 are called *generating functions*.

Chapter 3

Relativity

3.1 Special relativity

3.1.1 The Lorentz transformation

The Lorentz transformation $(\vec{x}\,', t') = (\vec{x}\,'(\vec{x}, t), t'(\vec{x}, t))$ leaves the wave equation invariant if c is invariant:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

This transformation can also be found when $ds^2 = ds'^2$ is demanded. The general form of the Lorentz transformation is given by:

$$\vec{x}\,' = \vec{x} + \frac{(\gamma - 1)(\vec{x} \cdot \vec{v})\vec{v}}{|v|^2} - \gamma\vec{v}t \ , \ t' = \gamma\left(t - \frac{\vec{x} \cdot \vec{v}}{c^2}\right)$$

where

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

The velocity difference $\vec{v}\,'$ between two observers transforms according to:

$$\vec{v}\,' = \left(\gamma\left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2}\right)\right)^{-1}\left(\vec{v}_2 + (\gamma - 1)\frac{\vec{v}_1 \cdot \vec{v}_2}{v_1^2}\vec{v}_1 - \gamma\vec{v}_1\right)$$

If the velocity is parallel to the x-axis, this becomes $y' = y$, $z' = z$ and:

$$x' = \gamma(x - vt) \ , \quad x = \gamma(x' + vt')$$
$$t' = \gamma\left(t - \frac{xv}{c^2}\right) \ , \quad t = \gamma\left(t' + \frac{x'v}{c^2}\right) \ , \quad v' = \frac{v_2 - v_1}{1 - \dfrac{v_1 v_2}{c^2}}$$

If $\vec{v} = v\vec{e}_x$ holds:

$$p'_x = \gamma\left(p_x - \frac{\beta W}{c}\right) \ , \quad W' = \gamma(W - vp_x)$$

With $\beta = v/c$ the electric field of a moving charge is given by:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\frac{(1 - \beta^2)\vec{e}_r}{(1 - \beta^2\sin^2(\theta))^{3/2}}$$

The electromagnetic field transforms according to:

$$\vec{E}' = \gamma(\vec{E} + \vec{v} \times \vec{B}\,) \ , \quad \vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v} \times \vec{E}}{c^2}\right)$$

Length, mass and time transform according to: $\Delta t_{\rm r} = \gamma\Delta t_0$, $m_{\rm r} = \gamma m_0$, $l_{\rm r} = l_0/\gamma$, with $_0$ the quantities in a co-moving reference frame and $_{\rm r}$ the quantities in a frame moving with velocity v w.r.t. it. The proper time τ is defined as: $d\tau^2 = ds^2/c^2$, so $\Delta\tau = \Delta t/\gamma$. For energy and momentum holds: $W = m_{\rm r}c^2 = \gamma W_0$,

$W^2 = m_0^2c^4 + p^2c^2$. $p = m_{\rm r}v = \gamma m_0 v = Wv/c^2$, and $pc = W\beta$ where $\beta = v/c$. The *force* is *defined* by $\vec{F} = d\vec{p}/dt$.

4-vectors have the property that their modulus is independent of the observer: their components *can* change after a coordinate transformation but not their modulus. The difference of two 4-vectors transforms also as a 4-vector. The 4-vector for the velocity is given by $U^\alpha = \dfrac{dx^\alpha}{d\tau}$. The relation with the "common" velocity $u^i := dx^i/dt$ is: $U^\alpha = (\gamma u^i, ic\gamma)$. For particles with nonzero restmass holds: $U^\alpha U_\alpha = -c^2$, for particles with zero restmass (so with $v = c$) holds: $U^\alpha U_\alpha = 0$. The 4-vector for energy and momentum is given by: $p^\alpha = m_0 U^\alpha = (\gamma p^i, iW/c)$. So: $p_\alpha p^\alpha = -m_0^2c^2 = p^2 - W^2/c^2$.

3.1.2 Red and blue shift

There are three causes of red and blue shifts:

1. Motion: with $\vec{e}_v \cdot \vec{e}_r = \cos(\varphi)$ follows: $\dfrac{f'}{f} = \gamma\left(1 - \dfrac{v\cos(\varphi)}{c}\right)$.
   This can give both red- and blueshift, also $\perp$ to the direction of motion.
2. Gravitational redshift: $\dfrac{\Delta f}{f} = \dfrac{\kappa M}{rc^2}$.
3. Redshift because the universe expands, resulting in e.g. the cosmic background radiation:
   $\dfrac{\lambda_0}{\lambda_1} = \dfrac{R_0}{R_1}$.

3.1.3 The stress-energy tensor and the field tensor

The stress-energy tensor is given by:

$$T_{\mu\nu} = (\varrho c^2 + p)u_\mu u_\nu + pg_{\mu\nu} + \frac{1}{c^2}\left(F_{\mu\alpha}F_\nu^\alpha + \tfrac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

The conservation laws can than be written as: $\nabla_\nu T^{\mu\nu} = 0$. The electromagnetic field tensor is given by:

$$\boxed{F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}}$$

with $A_\mu := (\vec{A}, iV/c)$ and $J_\mu := (\vec{J}, ic\rho)$. The Maxwell equations can than be written as:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu\ ,\ \ \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

The equations of motion for a charged particle in an EM field become with the field tensor:

$$\boxed{\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta}$$

3.2 General relativity

3.2.1 Riemannian geometry, the Einstein tensor

The basic principles of general relativity are:

1. The geodesic postulate: free falling particles move along geodesics of space-time with the proper time τ or arc length s as parameter. For particles with zero rest mass (photons), the use of a free parameter is required because for them holds $ds = 0$. From $\delta\int ds = 0$ the equations of motion can be derived:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0$$

2. The *principle of equivalence*: inertial mass $\equiv$ gravitational mass $\Rightarrow$ gravitation is equivalent with a curved space-time were particles move along geodesics.

3. By a proper choice of the coordinate system it is possible to make the metric locally flat in each point x_i: $g_{\alpha\beta}(x_i) = \eta_{\alpha\beta}$:=diag$(-1, 1, 1, 1)$.

The *Riemann tensor* is defined as: $R^\mu_{\nu\alpha\beta}T^\nu := \nabla_\alpha\nabla_\beta T^\mu - \nabla_\beta\nabla_\alpha T^\mu$, where the covariant derivative is given by $\nabla_j a^i = \partial_j a^i + \Gamma^i_{jk}a^k$ and $\nabla_j a_i = \partial_j a_i - \Gamma^k_{ij}a_k$. Here,

$$\Gamma^i_{jk} = \frac{g^{il}}{2}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right), \text{ for Euclidean spaces this reduces to: } \Gamma^i_{jk} = \frac{\partial^2 \bar{x}^l}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial \bar{x}^l},$$

are the *Christoffel symbols*. For a second-order tensor holds: $[\nabla_\alpha, \nabla_\beta]T^\mu_\nu = R^\mu_{\sigma\alpha\beta}T^\sigma_\nu + R^\sigma_{\nu\alpha\beta}T^\mu_\sigma$, $\nabla_k a^i_j = \partial_k a^i_j - \Gamma^l_{kj}a^i_l + \Gamma^i_{kl}a^l_j$, $\nabla_k a_{ij} = \partial_k a_{ij} - \Gamma^l_{ki}a_{lj} - \Gamma^l_{kj}a_{jl}$ and $\nabla_k a^{ij} = \partial_k a^{ij} + \Gamma^i_{kl}a^{lj} + \Gamma^j_{kl}a^{il}$. The following holds: $R^\alpha_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$.

The *Ricci tensor* is a contraction of the Riemann tensor: $R_{\alpha\beta} := R^\mu_{\alpha\mu\beta}$, which is symmetric: $R_{\alpha\beta} = R_{\beta\alpha}$. The *Bianchi identities* are: $\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0$.

The *Einstein tensor* is given by: $G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$, where $R := R^\alpha_\alpha$ is the *Ricci scalar*, for which holds: $\nabla_\beta G_{\alpha\beta} = 0$. With the variational principle $\delta\int(\mathcal{L}(g_{\mu\nu}) - Rc^2/16\pi\kappa)\sqrt{|g|}d^4x = 0$ for variations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ the *Einstein field equations* can be derived:

$$\boxed{G_{\alpha\beta} = \frac{8\pi\kappa}{c^2}T_{\alpha\beta}} \text{ , which can also be written as } R_{\alpha\beta} = \frac{8\pi\kappa}{c^2}(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T^\mu_\mu)$$

For empty space this is equivalent to $R_{\alpha\beta} = 0$. The equation $R_{\alpha\beta\mu\nu} = 0$ has as only solution a flat space.

The Einstein equations are 10 independent equations, which are of second order in $g_{\mu\nu}$. From this, the Laplace equation from Newtonian gravitation can be derived by stating: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h| \ll 1$. In the stationary case, this results in $\nabla^2 h_{00} = 8\pi\kappa\varrho/c^2$.

The most general form of the field equations is: $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \dfrac{8\pi\kappa}{c^2}T_{\alpha\beta}$

where Λ is the *cosmological constant*. This constant plays a role in inflatory models of the universe.

3.2.2 The line element

The *metric tensor* in an Euclidean space is given by: $g_{ij} = \sum\limits_k \dfrac{\partial\bar{x}^k}{\partial x^i}\dfrac{\partial\bar{x}^k}{\partial x^j}$.

In general holds: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In special relativity this becomes $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. This metric, $\eta_{\mu\nu}$:=diag$(-1, 1, 1, 1)$, is called the *Minkowski metric*.

The *external Schwarzschild metric* applies in vacuum outside a spherical mass distribution, and is given by:

$$ds^2 = \left(-1 + \frac{2m}{r}\right)c^2dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\Omega^2$$

Here, $m := M\kappa/c^2$ is the *geometrical mass* of an object with mass M, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. This metric is singular for $r = 2m = 2\kappa M/c^2$. If an object is smaller than its event horizon $2m$, that implies that its escape velocity is $> c$, it is called a *black hole*. The Newtonian limit of this metric is given by:

$$ds^2 = -(1 + 2V)c^2dt^2 + (1 - 2V)(dx^2 + dy^2 + dz^2)$$

where $V = -\kappa M/r$ is the Newtonian gravitation potential. In general relativity, the components of $g_{\mu\nu}$ are associated with the potentials and the derivatives of $g_{\mu\nu}$ with the field strength.

The Kruskal-Szekeres coordinates are used to solve certain problems with the Schwarzschild metric near $r = 2m$. They are defined by:

- $r > 2m$:

$$\left\{\begin{array}{lcl} u &=& \sqrt{\frac{r}{2m}-1}\exp\left(\frac{r}{4m}\right)\cosh\left(\frac{t}{4m}\right)\\ v &=& \sqrt{\frac{r}{2m}-1}\exp\left(\frac{r}{4m}\right)\sinh\left(\frac{t}{4m}\right)\end{array}\right.$$

- $r < 2m$:

$$\left\{\begin{array}{lcl} u &=& \sqrt{1-\frac{r}{2m}}\exp\left(\frac{r}{4m}\right)\sinh\left(\frac{t}{4m}\right)\\ v &=& \sqrt{1-\frac{r}{2m}}\exp\left(\frac{r}{4m}\right)\cosh\left(\frac{t}{4m}\right)\end{array}\right.$$

- $r = 2m$: here, the Kruskal coordinates are singular, which is necessary to eliminate the coordinate singularity there.

The line element in these coordinates is given by:

$$ds^2 = -\frac{32m^3}{r}\mathrm{e}^{-r/2m}(dv^2 - du^2) + r^2 d\Omega^2$$

The line $r = 2m$ corresponds to $u = v = 0$, the limit $x^0 \rightarrow \infty$ with $u = v$ and $x^0 \rightarrow -\infty$ with $u = -v$. The Kruskal coordinates are only singular on the hyperbole $v^2 - u^2 = 1$, this corresponds with $r = 0$. On the line $dv = \pm du$ holds $d\theta = d\varphi = ds = 0$.

For the metric outside a rotating, charged spherical mass the Newman metric applies:

$$\begin{array}{lcl} ds^2 &=& \left(1-\frac{2mr-e^2}{r^2+a^2\cos^2\theta}\right)c^2dt^2 - \left(\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2-e^2}\right)dr^2 - (r^2+a^2\cos^2\theta)d\theta^2 - \\ && \left(r^2+a^2+\frac{(2mr-e^2)a^2\sin^2\theta}{r^2+a^2\cos^2\theta}\right)\sin^2\theta d\varphi^2 + \left(\frac{2a(2mr-e^2)}{r^2+a^2\cos^2\theta}\right)\sin^2\theta(d\varphi)(cdt)\end{array}$$

where $m = \kappa M/c^2$, $a = L/Mc$ and $e = \kappa Q/\varepsilon_0 c^2$.
A rotating charged black hole has an event horizon with $R_{\rm S} = m + \sqrt{m^2 - a^2 - e^2}$.

Near rotating black holes frame dragging occurs because $g_{t\varphi} \neq 0$. For the Kerr metric ($e = 0$, $a \neq 0$) then follows that within the surface $R_{\rm E} = m + \sqrt{m^2 - a^2\cos^2\theta}$ (de ergosphere) no particle can be at rest.

3.2.3 Planetary orbits and the perihelion shift

To find a planetary orbit, the variational problem $\delta\int ds = 0$ has to be solved. This is equivalent to the problem $\delta\int ds^2 = \delta\int g_{ij}dx^i dx^j = 0$. Substituting the external Schwarzschild metric yields for a planetary orbit:

$$\frac{du}{d\varphi}\left(\frac{d^2u}{d\varphi^2}+u\right) = \frac{du}{d\varphi}\left(3mu + \frac{m}{h^2}\right)$$

where $u := 1/r$ and $h = r^2\dot{\varphi}$ =constant. The term $3mu$ is not present in the classical solution. This term can in the classical case also be found from a potential $V(r) = -\frac{\kappa M}{r}\left(1+\frac{h^2}{r^2}\right)$.

The orbital equation gives r =constant as solution, or can, after dividing by $du/d\varphi$, be solved with perturbation theory. In zeroth order, this results in an elliptical orbit: $u_0(\varphi) = A + B\cos(\varphi)$ with $A = m/h^2$ and B an arbitrary constant. In first order, this becomes:

$$u_1(\varphi) = A + B\cos(\varphi - \varepsilon\varphi) + \varepsilon\left(A + \frac{B^2}{2A} - \frac{B^2}{6A}\cos(2\varphi)\right)$$

where $\varepsilon = 3m^2/h^2$ is small. The perihelion of a planet is the point for which r is minimal, or u maximal. This is the case if $\cos(\varphi - \varepsilon\varphi) = 0 \Rightarrow \varphi \approx 2\pi n(1+\varepsilon)$. For the perihelion shift then follows: $\Delta\varphi = 2\pi\varepsilon = 6\pi m^2/h^2$ per orbit.

3.2.4 The trajectory of a photon

For the trajectory of a photon (and for each particle with zero restmass) holds $ds^2 = 0$. Substituting the external Schwarzschild metric results in the following orbital equation:

$$\frac{du}{d\varphi}\left(\frac{d^2u}{d\varphi^2}+u-3mu\right)=0$$

3.2.5 Gravitational waves

Starting with the approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for weak gravitational fields and the definition $h'_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha_\alpha$ it follows that $\square\, h'_{\mu\nu} = 0$ if the gauge condition $\partial h'_{\mu\nu}/\partial x^\nu = 0$ is satisfied. From this, it follows that the loss of energy of a mechanical system, if the occurring velocities are $\ll c$ and for wavelengths $\gg$ the size of the system, is given by:

$$\frac{dE}{dt}=-\frac{G}{5c^5}\sum_{i,j}\left(\frac{d^3Q_{ij}}{dt^3}\right)^2$$

with $Q_{ij} = \int \varrho(x_ix_j - \frac{1}{3}\delta_{ij}r^2)d^3x$ the mass quadrupole moment.

3.2.6 Cosmology

If for the universe as a whole is assumed:

1. There exists a global time coordinate which acts as x^0 of a Gaussian coordinate system,
2. The 3-dimensional spaces are isotrope for a certain value of x^0,
3. Each point is equivalent to each other point for a fixed x^0.

then the *Robertson-Walker metric* can be derived for the line element:

$$ds^2=-c^2dt^2+\frac{R^2(t)}{r_0^2\left(1-\dfrac{kr^2}{4r_0^2}\right)}(dr^2+r^2d\Omega^2)$$

For the *scalefactor* $R(t)$ the following equations can be derived:

$$\frac{2\ddot{R}}{R}+\frac{\dot{R}^2+kc^2}{R^2}=-\frac{8\pi\kappa p}{c^2}+\Lambda \quad\text{and}\quad \frac{\dot{R}^2+kc^2}{R^2}=\frac{8\pi\kappa\varrho}{3}+\frac{\Lambda}{3}$$

where p is the pressure and ϱ the density of the universe. If $\Lambda = 0$ can be derived for the *deceleration parameter* q:

$$q=-\frac{\ddot{R}R}{\dot{R}^2}=\frac{4\pi\kappa\varrho}{3H^2}$$

where $H = \dot{R}/R$ is *Hubble's constant*. This is a measure of the velocity with which galaxies far away are moving away from each other, and has the value $\approx (75\pm25)$ km·s^{-1}·Mpc^{-1}. This gives 3 possible conditions for the universe (here, W is the total amount of energy in the universe):

1. **Parabolical universe**: $k = 0$, $W = 0$, $q = \frac{1}{2}$. The expansion velocity of the universe $\rightarrow 0$ if $t \rightarrow \infty$. The hereto related *critical density* is $\varrho_{\rm c} = 3H^2/8\pi\kappa$.
2. **Hyperbolical universe**: $k = -1$, $W < 0$, $q < \frac{1}{2}$. The expansion velocity of the universe remains positive forever.
3. **Elliptical universe**: $k = 1$, $W > 0$, $q > \frac{1}{2}$. The expansion velocity of the universe becomes negative after some time: the universe starts collapsing.

Chapter 4

Oscillations

4.1 Harmonic oscillations

The general form of a harmonic oscillation is: $\Psi(t) = \hat{\Psi}e^{i(\omega t \pm \varphi)} \equiv \hat{\Psi}\cos(\omega t \pm \varphi)$,

where $\hat{\Psi}$ is the *amplitude*. A superposition of several harmonic oscillations *with the same frequency* results in another harmonic oscillation:

$$\sum_i \hat{\Psi}_i \cos(\alpha_i \pm \omega t) = \hat{\Phi}\cos(\beta \pm \omega t)$$

with:

$$\tan(\beta) = \frac{\sum\limits_i \hat{\Psi}_i \sin(\alpha_i)}{\sum\limits_i \hat{\Psi}_i \cos(\alpha_i)} \quad \text{and} \quad \hat{\Phi}^2 = \sum_i \hat{\Psi}_i^2 + 2\sum_{j>i}\sum_i \hat{\Psi}_i\hat{\Psi}_j \cos(\alpha_i - \alpha_j)$$

For harmonic oscillations holds: $\displaystyle\int x(t)dt = \frac{x(t)}{i\omega}$ and $\displaystyle\frac{d^n x(t)}{dt^n} = (i\omega)^n x(t)$.

4.2 Mechanic oscillations

For a construction with a spring with constant C parallel to a damping k which is connected to a mass M, to which a periodic force $F(t) = \hat{F}\cos(\omega t)$ is applied holds the equation of motion $m\ddot{x} = F(t) - k\dot{x} - Cx$. With complex amplitudes, this becomes $-m\omega^2 x = F - Cx - ik\omega x$. With $\omega_0^2 = C/m$ follows:

$$x = \frac{F}{m(\omega_0^2 - \omega^2) + ik\omega} \text{ ,and for the velocity holds: } \dot{x} = \frac{F}{i\sqrt{Cm}\delta + k}$$

where $\delta = \dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}$. The quantity $Z = F/\dot{x}$ is called the *impedance* of the system. The *quality* of the system is given by $Q = \dfrac{\sqrt{Cm}}{k}$.

The frequency with minimal $|Z|$ is called *velocity resonance frequency*. This is equal to ω_0. In the *resonance curve* $|Z|/\sqrt{Cm}$ is plotted against ω/ω_0. The width of this curve is characterized by the points where $|Z(\omega)| = |Z(\omega_0)|\sqrt{2}$. In these points holds: $R = X$ and $\delta = \pm Q^{-1}$, and the width is $2\Delta\omega_{\rm B} = \omega_0/Q$.

The *stiffness* of an oscillating system is given by F/x. The *amplitude resonance frequency* $\omega_{\rm A}$ is the frequency where $i\omega Z$ is minimal. This is the case for $\omega_{\rm A} = \omega_0\sqrt{1 - \frac{1}{2}Q^2}$.

The *damping frequency* $\omega_{\rm D}$ is a measure for the time in which an oscillating system comes to rest. It is given by $\omega_{\rm D} = \omega_0\sqrt{1 - \dfrac{1}{4Q^2}}$. A weak damped oscillation $(k^2 < 4mC)$ dies out after $T_{\rm D} = 2\pi/\omega_{\rm D}$. For a *critical damped* oscillation $(k^2 = 4mC)$ holds $\omega_{\rm D} = 0$. A strong damped oscillation $(k^2 > 4mC)$ drops like (if $k^2 \gg 4mC$) $x(t) \approx x_0\exp(-t/\tau)$.

4.3 Electric oscillations

The *impedance* is given by: $Z = R + iX$. The phase angle is $\varphi := \arctan(X/R)$. The impedance of a resistor is R, of a capacitor $1/i\omega C$ and of a self inductor $i\omega L$. The quality of a coil is $Q = \omega L/R$. The total impedance in case several elements are positioned is given by:

1. Series connection: $V = IZ$,

$$Z_{\rm tot} = \sum_i Z_i \ , \ \ L_{\rm tot} = \sum_i L_i \ , \ \ \frac{1}{C_{\rm tot}} = \sum_i \frac{1}{C_i} \ , \ \ Q = \frac{Z_0}{R} \ , \ \ Z = R(1 + iQ\delta)$$

2. parallel connection: $V = IZ$,

$$\frac{1}{Z_{\rm tot}} = \sum_i \frac{1}{Z_i} \ , \ \ \frac{1}{L_{\rm tot}} = \sum_i \frac{1}{L_i} \ , \ \ C_{\rm tot} = \sum_i C_i \ , \ \ Q = \frac{R}{Z_0} \ , \ \ Z = \frac{R}{1 + iQ\delta}$$

Here, $Z_0 = \sqrt{\frac{L}{C}}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$.

The power given by a source is given by $P(t) = V(t) \cdot I(t)$, so $\langle P \rangle_t = \hat{V}_{\rm eff} \hat{I}_{\rm eff} \cos(\Delta\phi)$ $= \frac{1}{2}\hat{V}\hat{I}\cos(\phi_v - \phi_i) = \frac{1}{2}\hat{I}^2{\rm Re}(Z) = \frac{1}{2}\hat{V}^2{\rm Re}(1/Z)$, where $\cos(\Delta\phi)$ is the work factor.

4.4 Waves in long conductors

These cables are in use for signal transfer, e.g. coax cable. For them holds: $Z_0 = \sqrt{\frac{dL}{dx}\frac{dx}{dC}}$.

The transmission velocity is given by $v = \sqrt{\frac{dx}{dL}\frac{dx}{dC}}$.

4.5 Coupled conductors and transformers

For two coils enclosing each others flux holds: if Φ_{12} is the part of the flux originating from I_2 through coil 2 which is enclosed by coil 1, than holds $\Phi_{12} = M_{12} I_2$, $\Phi_{21} = M_{21} I_1$. For the coefficients of mutual induction M_{ij} holds:

$$M_{12} = M_{21} := M = k\sqrt{L_1 L_2} = \frac{N_1 \Phi_1}{I_2} = \frac{N_2 \Phi_2}{I_1} \sim N_1 N_2$$

where $0 \leq k \leq 1$ is the *coupling factor*. For a transformer is $k \approx 1$. At full load holds:

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = -\frac{i\omega M}{i\omega L_2 + R_{\rm load}} \approx -\sqrt{\frac{L_1}{L_2}} = -\frac{N_1}{N_2}$$

4.6 Pendulums

The oscillation time $T = 1/f$, and for different types of pendulums is given by:

- Oscillating spring: $T = 2\pi\sqrt{m/C}$ if the spring force is given by $F = C \cdot \Delta l$.
- Physical pendulum: $T = 2\pi\sqrt{I/\tau}$ with τ the moment of force and I the moment of inertia.
- Torsion pendulum: $T = 2\pi\sqrt{I/\kappa}$ with $\kappa = \frac{2lm}{\pi r^4 \Delta\varphi}$ the constant of torsion and I the moment of inertia.
- Mathematical pendulum: $T = 2\pi\sqrt{l/g}$ with g the acceleration of gravity and l the length of the pendulum.

Chapter 5

Waves

5.1 The wave equation

The general form of the wave equation is: $\Box u = 0$, or:

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$$

where u is the disturbance and v the *propagation velocity*. In general holds: $v = f\lambda$. By definition holds: $k\lambda = 2\pi$ and $\omega = 2\pi f$.

In principle, there are two types of waves:

1. Longitudinal waves: for these holds $\vec{k} \parallel \vec{v} \parallel \vec{u}$.
2. Transversal waves: for these holds $\vec{k} \parallel \vec{v} \perp \vec{u}$.

The *phase velocity* is given by $v_{\rm ph} = \omega/k$. The *group velocity* is given by:

$$v_{\rm g} = \frac{d\omega}{dk} = v_{\rm ph} + k\frac{dv_{\rm ph}}{dk} = v_{\rm ph}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

where n is the refractive index of the medium. If $v_{\rm ph}$ does not depend on ω holds: $v_{\rm ph} = v_{\rm g}$. In a dispersive medium it is possible that $v_{\rm g} > v_{\rm ph}$ or $v_{\rm g} < v_{\rm ph}$, and $v_{\rm g} \cdot v_{\rm f} = c^2$. If one wants to transfer information with a wave, e.g. by modulation of an EM wave, the information travels with the velocity at with a change in the electromagnetic field propagates. This velocity is often almost equal to the group velocity.

For some media, the propagation velocity follows from:

- Pressure waves in a liquid or gas: $v = \sqrt{\kappa/\varrho}$, where κ is the modulus of compression.
- For pressure waves in a gas also holds: $v = \sqrt{\gamma p/\varrho} = \sqrt{\gamma RT/M}$.
- Pressure waves in a thin solid bar with diameter $<< \lambda$: $v = \sqrt{E/\varrho}$
- waves in a string: $v = \sqrt{F_{\rm span} l/m}$
- Surface waves on a liquid: $v = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\gamma}{\varrho\lambda}\right)\tanh\left(\frac{2\pi h}{\lambda}\right)}$
  where h is the depth of the liquid and γ the surface tension. If $h \ll \lambda$ holds: $v \approx \sqrt{gh}$.

5.2 Solutions of the wave equation

5.2.1 Plane waves

In n dimensions a harmonic plane wave is defined by:

$$u(\vec{x}, t) = 2^n \hat{u} \cos(\omega t) \sum_{i=1}^{n} \sin(k_i x_i)$$

The equation for a harmonic traveling plane wave is: $u(\vec{x},t)=\hat{u}\cos(\vec{k}\cdot\vec{x}\pm\omega t+\varphi)$

If waves reflect at the end of a spring this will result in a change in phase. A fixed end gives a phase change of $\pi/2$ to the reflected wave, with boundary condition $u(l)=0$. A lose end gives no change in the phase of the reflected wave, with boundary condition $(\partial u/\partial x)_l=0$.

If an observer is moving w.r.t. the wave with a velocity $v_{\rm obs}$, he will observe a change in frequency: the *Doppler effect*. This is given by: $\displaystyle\frac{f}{f_0}=\frac{v_{\rm f}-v_{\rm obs}}{v_{\rm f}}$.

5.2.2 Spherical waves

When the situation is spherical symmetric, the homogeneous wave equation is given by:

$$\frac{1}{v^2}\frac{\partial^2(ru)}{\partial t^2}-\frac{\partial^2(ru)}{\partial r^2}=0$$

with general solution:

$$u(r,t)=C_1\frac{f(r-vt)}{r}+C_2\frac{g(r+vt)}{r}$$

5.2.3 Cylindrical waves

When the situation has a cylindrical symmetry, the homogeneous wave equation becomes:

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)=0$$

This is a Bessel equation, with solutions which can be written as Hankel functions. For sufficient large values of r these are approximated by:

$$u(r,t)=\frac{\hat{u}}{\sqrt{r}}\cos(k(r\pm vt))$$

5.2.4 The general solution in one dimension

Starting point is the equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\sum_{m=0}^{N}\left(b_m\frac{\partial^m}{\partial x^m}\right)u(x,t)$$

where $b_m\in I\!R$. Substituting $u(x,t)=A{\rm e}^{i(kx-\omega t)}$ gives two solutions $\omega_j=\omega_j(k)$ as dispersion relations. The general solution is given by:

$$u(x,t)=\int\limits_{-\infty}^{\infty}\left(a(k){\rm e}^{i(kx-\omega_1(k)t)}+b(k){\rm e}^{i(kx-\omega_2(k)t)}\right)dk$$

Because in general the frequencies ω_j are non-linear in k there is dispersion and the solution cannot be written any more as a sum of functions depending only on $x\pm vt$: the wave front transforms.

5.3 The stationary phase method

Usually the Fourier integrals of the previous section cannot be calculated exactly. If $\omega_j(k)\in I\!R$ the stationary phase method can be applied. Assuming that $a(k)$ is only a slowly varying function of k, one can state that the parts of the k-axis where the phase of $kx-\omega(k)t$ changes rapidly will give no net contribution to the integral because the exponent oscillates rapidly there. The only areas contributing significantly to the integral are areas with a stationary phase, determined by $\displaystyle\frac{d}{dk}(kx-\omega(k)t)=0$. Now the following approximation is possible:

$$\int\limits_{-\infty}^{\infty}a(k){\rm e}^{i(kx-\omega(k)t)}dk\approx\sum_{i=1}^{N}\sqrt{\frac{2\pi}{\frac{d^2\omega(k_i)}{dk_i^2}}}\exp\left[-i\tfrac{1}{4}\pi+i(k_ix-\omega(k_i)t)\right]$$

5.4 Green functions for the initial-value problem

This method is preferable if the solutions deviate much from the stationary solutions, like point-like excitations. Starting with the wave equation in one dimension, with $\nabla^2 = \partial^2/\partial x^2$ holds: if $Q(x, x', t)$ is the solution with initial values $Q(x, x', 0) = \delta(x - x')$ and $\dfrac{\partial Q(x, x', 0)}{\partial t} = 0$, and $P(x, x', t)$ the solution with initial values $P(x, x', 0) = 0$ and $\dfrac{\partial P(x, x', 0)}{\partial t} = \delta(x - x')$, then the solution of the wave equation with arbitrary initial conditions $f(x) = u(x, 0)$ and $g(x) = \dfrac{\partial u(x, 0)}{\partial t}$ is given by:

$$u(x,t) = \int\limits_{-\infty}^{\infty} f(x')Q(x, x', t)dx' + \int\limits_{-\infty}^{\infty} g(x')P(x, x', t)dx'$$

P and Q are called the *propagators*. They are defined by:

$$\begin{aligned} Q(x, x', t) &= \tfrac{1}{2}[\delta(x - x' - vt) + \delta(x - x' + vt)] \\ P(x, x', t) &= \begin{cases} \dfrac{1}{2v} & \text{if } |x - x'| < vt \\ 0 & \text{if } |x - x'| > vt \end{cases} \end{aligned}$$

Further holds the relation: $Q(x, x', t) = \dfrac{\partial P(x, x', t)}{\partial t}$

5.5 Waveguides and resonating cavities

The boundary conditions for a perfect conductor can be derived from the Maxwell equations. If $\vec{n}$ is a unit vector $\perp$ the surface, pointed from 1 to 2, and $\vec{K}$ is a surface current density, than holds:

$$\begin{array}{ll} \vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma & \vec{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \\ \vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0 & \vec{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \end{array}$$

In a waveguide holds because of the cylindrical symmetry: $\vec{E}(\vec{x}, t) = \vec{\mathcal{E}}(x, y)\mathrm{e}^{i(kz - \omega t)}$ and $\vec{B}(\vec{x}, t) = \vec{\mathcal{B}}(x, y)\mathrm{e}^{i(kz - \omega t)}$. From this one can now deduce that, if $\mathcal{B}_z$ and $\mathcal{E}_z$ are not $\equiv 0$:

$$\begin{array}{ll} \mathcal{B}_x = \dfrac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\dfrac{\partial \mathcal{B}_z}{\partial x} - \varepsilon\mu\omega\dfrac{\partial \mathcal{E}_z}{\partial y}\right) & \mathcal{B}_y = \dfrac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\dfrac{\partial \mathcal{B}_z}{\partial y} + \varepsilon\mu\omega\dfrac{\partial \mathcal{E}_z}{\partial x}\right) \\ \mathcal{E}_x = \dfrac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\dfrac{\partial \mathcal{E}_z}{\partial x} + \varepsilon\mu\omega\dfrac{\partial \mathcal{B}_z}{\partial y}\right) & \mathcal{E}_y = \dfrac{i}{\varepsilon\mu\omega^2 - k^2}\left(k\dfrac{\partial \mathcal{E}_z}{\partial y} - \varepsilon\mu\omega\dfrac{\partial \mathcal{B}_z}{\partial x}\right) \end{array}$$

Now one can distinguish between three cases:

1. $B_z \equiv 0$: the Transversal Magnetic modes (TM). Boundary condition: $\mathcal{E}_z|_{\rm surf} = 0$.
2. $E_z \equiv 0$: the Transversal Electric modes (TE). Boundary condition: $\left.\dfrac{\partial \mathcal{B}_z}{\partial n}\right|_{\rm surf} = 0$.

   For the TE and TM modes this gives an eigenvalue problem for $\mathcal{E}_z$ resp. $\mathcal{B}_z$ with boundary conditions:

   $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\gamma^2\psi \ \text{ with eigenvalues } \ \gamma^2 := \varepsilon\mu\omega^2 - k^2$$

   This gives a discrete solution ψ_ℓ with eigenvalue γ_ℓ^2: $k = \sqrt{\varepsilon\mu\omega^2 - \gamma_\ell^2}$. For $\omega < \omega_\ell$, k is imaginary and the wave is damped. Therefore, ω_ℓ is called the *cut-off frequency*. In rectangular conductors the following expression can be found for the cut-off frequency for modes $\text{TE}_{m,n}$ of $\text{TM}_{m,n}$:

   $$\lambda_\ell = \frac{2}{\sqrt{(m/a)^2 + (n/b)^2}}$$

3. E_z and B_z are zero everywhere: the Transversal electromagnetic mode (TEM). Than holds: $k = \pm\omega\sqrt{\varepsilon\mu}$ and $v_{\rm f} = v_{\rm g}$, just as if here were no waveguide. Further $k \in I\!R$, so there exists no cut-off frequency.

In a rectangular, 3 dimensional resonating cavity with edges a, b and c the possible wave numbers are given by: $k_x = \dfrac{n_1\pi}{a}$, $k_y = \dfrac{n_2\pi}{b}$, $k_z = \dfrac{n_3\pi}{c}$ This results in the possible frequencies $f = vk/2\pi$ in the cavity:

$$f = \frac{v}{2}\sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}}$$

For a cubic cavity, with $a = b = c$, the possible number of oscillating modes $N_{\rm L}$ for longitudinal waves is given by:

$$N_{\rm L} = \frac{4\pi a^3 f^3}{3v^3}$$

Because transversal waves have two possible polarizations holds for them: $N_{\rm T} = 2N_{\rm L}$.

5.6 Non-linear wave equations

The *Van der Pol* equation is given by:

$$\frac{d^2x}{dt^2} - \varepsilon\omega_0(1 - \beta x^2)\frac{dx}{dt} + \omega_0^2 x = 0$$

βx^2 can be ignored for very small values of the amplitude. Substitution of $x \sim {\rm e}^{i\omega t}$ gives: $\omega = \frac{1}{2}\omega_0(i\varepsilon \pm 2\sqrt{1 - \frac{1}{2}\varepsilon^2})$. The lowest-order instabilities grow as $\frac{1}{2}\varepsilon\omega_0$. While x is growing, the 2nd term becomes larger and diminishes the growth. Oscillations on a time scale $\sim \omega_0^{-1}$ can exist. If x is expanded as $x = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \cdots$ and this is substituted one obtains, besides periodic, *secular terms* $\sim \varepsilon t$. If it is assumed that there exist timescales τ_n, $0 \leq \tau \leq N$ with $\partial\tau_n/\partial t = \varepsilon^n$ and if the secular terms are put 0 one obtains:

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \tfrac{1}{2}\omega_0^2 x^2\right\} = \varepsilon\omega_0(1 - \beta x^2)\left(\frac{dx}{dt}\right)^2$$

This is an energy equation. Energy is conserved if the left-hand side is 0. If $x^2 > 1/\beta$, the right-hand side changes sign and an increase in energy changes into a decrease of energy. This mechanism limits the growth of oscillations.

The *Korteweg-De Vries* equation is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \underbrace{au\frac{\partial u}{\partial x}}_{\rm non-lin} + \underbrace{b^2\frac{\partial^3 u}{\partial x^3}}_{\rm dispersive} = 0$$

This equation is for example a model for ion-acoustic waves in a plasma. For this equation, soliton solutions of the following form exist:

$$u(x - ct) = \frac{-d}{\cosh^2(e(x - ct))}$$

with $c = 1 + \frac{1}{3}ad$ and $e^2 = ad/(12b^2)$.

The ∇-operator

In cartesian coordinates (x,y,z) holds:

$$\vec{\nabla}=\frac{\partial}{\partial x}\vec{e}_x+\frac{\partial}{\partial y}\vec{e}_y+\frac{\partial}{\partial z}\vec{e}_z\ ,\ \ \text{grad}f=\vec{\nabla}f=\frac{\partial f}{\partial x}\vec{e}_x+\frac{\partial f}{\partial y}\vec{e}_y+\frac{\partial f}{\partial z}\vec{e}_z$$

$$\text{div}\,\vec{a}=\vec{\nabla}\cdot\vec{a}=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}\ ,\ \ \nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$\text{rot}\,\vec{a}=\vec{\nabla}\times\vec{a}=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\vec{e}_x+\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)\vec{e}_y+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

In cylinder coordinates (r,φ,z) holds:

$$\vec{\nabla}=\frac{\partial}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial}{\partial\varphi}\vec{e}_\varphi+\frac{\partial}{\partial z}\vec{e}_z\ ,\ \ \text{grad}f=\frac{\partial f}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial f}{\partial\varphi}\vec{e}_\varphi+\frac{\partial f}{\partial z}\vec{e}_z$$

$$\text{div}\,\vec{a}=\frac{\partial a_r}{\partial r}+\frac{a_r}{r}+\frac{1}{r}\frac{\partial a_\varphi}{\partial\varphi}+\frac{\partial a_z}{\partial z}\ ,\ \ \nabla^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}$$

$$\text{rot}\,\vec{a}=\left(\frac{1}{r}\frac{\partial a_z}{\partial\varphi}-\frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r+\left(\frac{\partial a_r}{\partial z}-\frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi+\left(\frac{\partial a_\varphi}{\partial r}+\frac{a_\varphi}{r}-\frac{1}{r}\frac{\partial a_r}{\partial\varphi}\right)\vec{e}_z$$

In spherical coordinates (r,θ,φ) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\vec{e}_\varphi\\
\text{grad}f &= \frac{\partial f}{\partial r}\vec{e}_r+\frac{1}{r}\frac{\partial f}{\partial\theta}\vec{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\vec{e}_\varphi\\
\text{div}\,\vec{a} &= \frac{\partial a_r}{\partial r}+\frac{2a_r}{r}+\frac{1}{r}\frac{\partial a_\theta}{\partial\theta}+\frac{a_\theta}{r\tan\theta}+\frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial\varphi}\\
\text{rot}\,\vec{a} &= \left(\frac{1}{r}\frac{\partial a_\varphi}{\partial\theta}+\frac{a_\theta}{r\tan\theta}-\frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial\varphi}\right)\vec{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\varphi}-\frac{\partial a_\varphi}{\partial r}-\frac{a_\varphi}{r}\right)\vec{e}_\theta+\\
&\quad\left(\frac{\partial a_\theta}{\partial r}+\frac{a_\theta}{r}-\frac{1}{r}\frac{\partial a_r}{\partial\theta}\right)\vec{e}_\varphi\\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\tan\theta}\frac{\partial f}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}
\end{aligned}$$

General orthonormal curvelinear coordinates (u,v,w) can be obtained from cartesian coordinates by the transformation $\vec{x}=\vec{x}(u,v,w)$. The unit vectors are then given by:

$$\vec{e}_u=\frac{1}{h_1}\frac{\partial\vec{x}}{\partial u}\ ,\ \ \vec{e}_v=\frac{1}{h_2}\frac{\partial\vec{x}}{\partial v}\ ,\ \ \vec{e}_w=\frac{1}{h_3}\frac{\partial\vec{x}}{\partial w}$$

where the factors h_i set the norm to 1. Then holds:

$$\begin{aligned}
\text{grad}f &= \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{e}_u+\frac{1}{h_2}\frac{\partial f}{\partial v}\vec{e}_v+\frac{1}{h_3}\frac{\partial f}{\partial w}\vec{e}_w\\
\text{div}\,\vec{a} &= \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u}(h_2h_3a_u)+\frac{\partial}{\partial v}(h_3h_1a_v)+\frac{\partial}{\partial w}(h_1h_2a_w)\right)\\
\text{rot}\,\vec{a} &= \frac{1}{h_2h_3}\left(\frac{\partial(h_3a_w)}{\partial v}-\frac{\partial(h_2a_v)}{\partial w}\right)\vec{e}_u+\frac{1}{h_3h_1}\left(\frac{\partial(h_1a_u)}{\partial w}-\frac{\partial(h_3a_w)}{\partial u}\right)\vec{e}_v+\\
&\quad\frac{1}{h_1h_2}\left(\frac{\partial(h_2a_v)}{\partial u}-\frac{\partial(h_1a_u)}{\partial v}\right)\vec{e}_w\\
\nabla^2 f &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial v}\right)+\frac{\partial}{\partial w}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]
\end{aligned}$$

The SI units

Basic units

Quantity	Unit	Sym.
Length	metre	m
Mass	kilogram	kg
Time	second	s
Therm. temp.	kelvin	K
Electr. current	ampere	A
Luminous intens.	candela	cd
Amount of subst.	mol	mol

Extra units

Plane angle	radian	rad
solid angle	sterradian	sr

Derived units with special names

Quantity	Unit	Sym.	Derivation
Frequency	hertz	Hz	s^{-1}
Force	newton	N	$kg \cdot m \cdot s^{-2}$
Pressure	pascal	Pa	$N \cdot m^{-2}$
Energy	joule	J	$N \cdot m$
Power	watt	W	$J \cdot s^{-1}$
Charge	coulomb	C	$A \cdot s$
El. Potential	volt	V	$W \cdot A^{-1}$
El. Capacitance	farad	F	$C \cdot V^{-1}$
El. Resistance	ohm	Ω	$V \cdot A^{-1}$
El. Conductance	siemens	S	$A \cdot V^{-1}$
Mag. flux	weber	Wb	$V \cdot s$
Mag. flux density	tesla	T	$Wb \cdot m^{-2}$
Inductance	henry	H	$Wb \cdot A^{-1}$
Luminous flux	lumen	lm	$cd \cdot sr$
Illuminance	lux	lx	$lm \cdot m^{-2}$
Activity	bequerel	Bq	s^{-1}
Absorbed dose	gray	Gy	$J \cdot kg^{-1}$
Dose equivalent	sievert	Sv	$J \cdot kg^{-1}$

Prefixes

yotta	Y	10^{24}	giga	G	10^{9}	deci	d	10^{-1}	pico	p	10^{-12}
zetta	Z	10^{21}	mega	M	10^{6}	centi	c	10^{-2}	femto	f	10^{-15}
exa	E	10^{18}	kilo	k	10^{3}	milli	m	10^{-3}	atto	a	10^{-18}
peta	P	10^{15}	hecto	h	10^{2}	micro	μ	10^{-6}	zepto	z	10^{-21}
tera	T	10^{12}	deca	da	10	nano	n	10^{-9}	yocto	y	10^{-24}