NOTATION for Collective Properties

- ■ unsubscripted **capital** letter ➔ "TOTAL", except for …
- ■ unsubscripted **capital** position, velocity, accel ➔ "OF THE CM"
- ◇ subscript ≠ coordinate index ➔ "OF"
- ◆ no superscript ➔ "RELATIVE TO ORIGIN"
- ◆ superscript () ➔ "RELATIVE TO (POINT)"
- ◆ superscript prime′ ➔ "RELATIVE TO THE CM"

FORMULAE for Particle Collections

CM: $M\vec{R} \equiv \sum_i m_i \vec{r}_i$ $\quad \vec{P} = M\dot{\vec{R}}$ $\quad \vec{L}_{CM} = \vec{R} \times \vec{P}$

EOM in Inertial Frames

$\vec{F}^{\text{EXT}} = \dot{\vec{P}}$

$\vec{\tau}^{\text{EXT},(A)} = \dot{\vec{L}}^{(A)}$ if reference point A

- is the CM, or
- is not accelerating, or
- $\ddot{\vec{r}}_A \parallel (\vec{R} - \vec{r}_A)$

$T + U^{\text{EXT}} + U^{\text{INT}}$ = conserved for conservative forces

Decompositions

$\vec{P} = \vec{P}_{CM}$ $\quad I^{(B)}_{\hat{\omega}} = I^{(B)}_{\text{CM},\hat{\omega}} + I'_{\hat{\omega}}$

$\vec{L} = \vec{L}_{\text{CM}} + \vec{L}'$ $\quad \vec{\tau}^{\text{EXT}}_{CM} = \dot{\vec{L}}_{CM}$

$T = T_{\text{CM}} + T'$ $\quad T = T^{(\text{stationary point})}$

Rotating Body: For any BODY-FIXED vector $\vec{B}$,

$\dot{\vec{B}} = \vec{\omega} \times \vec{B}$

Moment of Inertia: For any BODY-FIXED point B,

$I^{(B)}_{\hat{\omega}} \equiv \sum m_i \left| \vec{r}_i^{\,(B)} \times \hat{\omega} \right|^2$ $\quad L^{(B)}_{\omega} = I^{(B)}_{\hat{\omega}} \omega$ $\quad T^{(B)} = \tfrac{1}{2} I^{(B)}_{\hat{\omega}} \omega^2$

Uniform Gravity: If $\vec{f}^{\text{EXT}} = m\vec{g}$ is the only external force,

$\vec{F}^{\text{EXT}} = M\vec{g}$ $\quad \vec{\tau}^{\text{EXT}} = \vec{R} \times M\vec{g}$ $\quad \vec{\tau}'^{\text{ EXT}} = 0$ $\quad U^{\text{EXT}} = MgH$

Definition of a Particle Collection

All these formulae apply to a collection of particles, labelled with index i, each of which has mass m_i and position $\vec{r}_i$. The formulae also rely on the collection having these properties:

- The mass m_i of each particle is constant.
- The total mass $M \equiv \sum m_i$ of the collection remains constant.
- The system is non-relativistic.

The collection may or may not be a rigid body, i.e. an object where the particles remain in a fixed spatial relationship to each other. The first condition is only needed for the proofs and can be trivially satisfied by mentally subdividing the collection down to the atomic level.

Collective Force Law & CM $\quad \vec{F}^{\text{EXT}} = \dot{\vec{P}}$ & $\vec{P} = M\dot{\vec{R}} = \vec{P}_{CM}$

In general, each particle i experiences a force f_i caused by both **external** forces from sources outside the collection and **internal** forces exerted by other particles j. The total force on the collection is:

$$\vec{F} = \sum_i \vec{f}_i = \sum_i \left(\vec{f}_i^{\text{EXT}} + \sum_{j \neq i} \vec{f}_{ij}^{\text{INT}} \right) = \sum_i \vec{f}_i^{\text{EXT}} + \sum_i \sum_{j<i} \left(\vec{f}_{ij}^{\text{INT}} + \vec{f}_{ji}^{\text{INT}} \right)$$

From **Newton's third law**, the force of particle i on particle j is equal and opposite to that of j on i : $\vec{f}_{ij}^{\text{INT}} = -\vec{f}_{ji}^{\text{INT}}$. The internal forces thus cancel and the total force is external only: $\vec{F} = \sum \vec{f}_i^{\text{EXT}} \equiv \vec{F}^{\text{EXT}}$.

We now apply Newton's second law, $\vec{f} = d\vec{p}/dt$, to each particle:

$$\vec{F} = \sum_i \vec{f}_i = \sum_i \frac{d\vec{p}_i}{dt} = \frac{d}{dt} \sum_i \vec{p}_i = \frac{d\vec{P}}{dt} \quad \rightarrow \quad \boxed{\vec{F}^{\text{EXT}} = \frac{d\vec{P}}{dt}}$$

Finally, since each mass m_i and the total mass M are all constant,

$$\vec{P} \equiv \sum_i \vec{p}_i = \sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt} \sum_i m_i \vec{r}_i = M \frac{d}{dt} \sum_i \frac{m_i \vec{r}_i}{M} \equiv M \frac{d\vec{R}}{dt}$$

where $\vec{R}$ is the familiar **center-of-mass position** of the collection:

$$\boxed{\vec{R} \equiv \frac{1}{M} \sum_i m_i \vec{r}_i} \quad \rightarrow \quad \boxed{\vec{P} = M \frac{d\vec{R}}{dt} = M\vec{V} = \vec{P}_{CM}}$$

Collective Torque Law

$$\vec{\tau}^{\,\text{EXT},(A)} = d\vec{L}^{(A)}/dt$$

The rate of change of the total angular momentum of the collective relative to some reference point A is :

$$\frac{d\vec{L}^{(A)}}{dt} = \frac{d}{dt}\left(\sum_i \vec{r}_i^{\,(A)} \times m_i\,\dot{\vec{r}}_i^{\,(A)}\right) = \sum_i \dot{\vec{r}}_i^{\,(A)} \times m_i \dot{\vec{r}}_i^{\,(A)} + \sum_i \vec{r}_i^{\,(A)} \times m_i \ddot{\vec{r}}_i^{\,(A)}$$

The first term is zero $\because$ the vector $\dot{\vec{r}}_i^{\,(A)}$ is crossed with itself. In the second term, apply the defining relation $\vec{r}_i^{\,(A)} \equiv \vec{r}_i - \vec{r}_A$ and the fact that $m_i \ddot{\vec{r}}_i$ is the force on particle i :

$$\frac{d\vec{L}^{(A)}}{dt} = \sum_i \vec{r}_i^{\,(A)} \times m_i\left(\ddot{\vec{r}}_i - \ddot{\vec{r}}_A\right) = \sum_i \vec{r}_i^{\,(A)} \times \vec{f}_i - \sum_i \vec{r}_i^{\,(A)} \times m_i\,\ddot{\vec{r}}_A = ① - ②$$

Let's analyze these two sums ① and ② separately.

$$① \;=\; \sum_i \vec{r}_i^{\,(A)} \times \vec{f}_i = \sum_i \vec{r}_i^{\,(A)} \times \left(\vec{f}_i^{\,\text{EXT}} + \sum_{j\neq i} \vec{f}_{ij}^{\,\text{INT}}\right)$$

$$= \sum_i \vec{r}_i^{\,(A)} \times \vec{f}_i^{\,\text{EXT}} + \sum_i \sum_{j>i}\left(\vec{r}_i^{\,(A)} \times \vec{f}_{ij}^{\,\text{INT}} + \vec{r}_j^{\,(A)} \times \vec{f}_{ji}^{\,\text{INT}}\right)$$

$$= \sum_i \vec{\tau}_i^{\,\text{EXT},\,(A)} + \sum_i \sum_{j>i}\left(\vec{r}_i^{\,(A)} - \vec{r}_j^{\,(A)}\right) \times \vec{f}_{ij}^{\,\text{INT}}$$

In the last step, we used Newton's 3rd Law, $\vec{f}_{ji} = -\vec{f}_{ij}$. If we further assume that the only internal forces are **central forces**, i.e. where each f_{ij} is parallel to the line pointing from i to j, the internal torques vanish:

$$① \;= \sum_i \vec{\tau}_i^{\,\text{EXT},\,(A)} + \sum_i \sum_{j>i}\left(\vec{r}_i - \vec{r}_j\right) \times \vec{f}_{ij}^{\,\text{INT}} = \vec{\tau}^{\,\text{EXT},\,(A)}$$

The total torque on the collection is thus due to external forces alone. The relation we seek is $d\vec{L}^{(A)}/dt = \vec{\tau}^{\,\text{EXT},\,(A)}$, and that is what we will get ... but only if sum ② is zero. Is it?

$$② \;= \sum_i \vec{r}_i^{\,(A)} \times m_i\,\ddot{\vec{r}}_A = \left(\sum_i m_i\,\vec{r}_i^{\,(A)}\right) \times \ddot{\vec{r}}_A = M\vec{R}^{(A)} \times \ddot{\vec{r}}_A$$

This is zero under any of the following conditions: if $\vec{R}^{(A)} \equiv \vec{R} - \vec{r}_A = 0$, if $\ddot{\vec{r}}_A = 0$, or if the cross-product is zero. Summarizing:

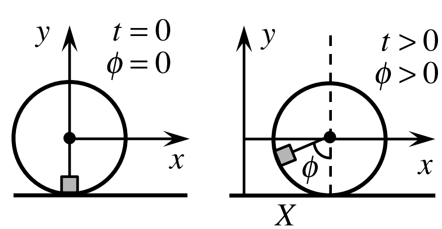
$\vec{\tau}^{\,\text{EXT},(A)} = d\vec{L}^{(A)}/dt$ holds if

- A is not accelerating, or $\quad \rightarrow \ddot{\vec{r}}_A = 0$
- A is the CM, or $\quad \rightarrow \vec{R}^{(A)} = \vec{R}^{(CM)} = \vec{R} - \vec{R} = 0$
- $\ddot{\vec{r}}_A$ is parallel to $\vec{R}^{(A)} \equiv \vec{R} - \vec{r}_A$ $\quad \rightarrow \ddot{\vec{r}}_A \times \vec{R}^{(A)} = 0$

Torque Law at the Contact Point of a Rolling Object

The third condition on the torque law, $\ddot{\vec{r}}_A \parallel \vec{R}^{(A)}$, does not apply to many problems. One important exception occurs for an object that is **rolling without slipping** along a surface. Let's call the rolling object a "wheel" for brevity. Let's also give it a radius b and have its CM move in the $+x$ direction. The **instantaneous contact point** between the wheel and the surface is a valid reference point for the torque law, even though this point is accelerating and is not the CM.

Mentally paint a dot on the wheel's edge and call it point B. (In the figures below, it is indicated by a small grey square.) Point B is only in contact with the surface at a particular moment in time — that's why we call it an "instantaneous" contact point. Without loss of generality, we can choose our coordinates so that B is the contact point at time $t = 0$ and CM position $X = 0$. We will also define the wheel's rotation angle ϕ to be the angle that B makes with the downward vertical axis; contact then occurs at $\phi = 0$. This moment in time is shown in the left-hand figure. A later moment is shown on the right.



The **no-slip rolling condition** between ϕ and X is immediately clear from the right-hand figure: the arc length $b\phi$ is exactly the horizontal distance travelled by the wheel, so $\boldsymbol{X = b\phi}$. This condition is *essential* for the upcoming proof.

Our task is to show that the point B – the small square – satisfies the condition $\ddot{\vec{r}}_B \parallel \vec{R}^{(B)}$ at the moment of contact, i.e. when t, ϕ, X are all 0. At that moment (left-hand figure), the direction of $\vec{R}^{(B)}$ = the vector pointing from B to the CM is upward, which is the $+y$ direction.

We must now show that the acceleration $\ddot{\vec{r}}_B$ of the point B is *also* in the y direction. In Cartesian coordinates, simple geometry gives us

$$\vec{r}_B(t) = \vec{R}(t) + \vec{r}_B'(t) = \left[X\,\hat{x}\right] + b\left[-\sin\phi\,\hat{x} - \cos\phi\,\hat{y}\right].$$

Note: X and ϕ can be arbitrarily complicated functions of time. For example, the surface on which the wheel is rolling may be tilted, causing the wheel to accelerate under gravity. Any manner of external forces may be at work; the only restriction we are imposing on X and ϕ is the no-slip rolling condition $\boldsymbol{X = b\phi}$ that ties them together.
We now calculate the acceleration of the point B:

$$\dot{\vec{r}}_B(t) = \left[\dot{X}\,\hat{x}\right] + b\left[-\dot{\phi}\cos\phi\,\hat{x} + \dot{\phi}\sin\phi\,\hat{y}\right]$$

$$\ddot{\vec{r}}_B(t) = \left[\ddot{X}\,\hat{x}\right] + b\left[\left(-\ddot{\phi}\cos\phi + \dot{\phi}^2\sin\phi\right)\hat{x} + \left(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi\right)\hat{y}\right]$$

We only need this acceleration at the moment when B *is* the contact point, i.e. when $t = 0$ and $\phi = 0$. Since cos(0) = 1 and sin(0) = 0,

$$\ddot{\vec{r}}_B\Big|_{t=0} = \left[\ddot{X}\,\hat{x}\right] + b\left[-\ddot{\phi}\,\hat{x} + \dot{\phi}^2\,\hat{y}\right] = \left(\ddot{X} - b\ddot{\phi}\right)\hat{x} + b\dot{\phi}^2\,\hat{y}$$

This acceleration is indeed in the y direction, just like $\vec{R}^{(B)}$, because <u>the no-slip rolling condition kills the x-component</u> :

$$X = b\phi \quad\rightarrow\quad \left(\ddot{X} - b\ddot{\phi}\right) = 0 \quad\rightarrow\quad \ddot{\vec{r}}_B\Big|_{t=0} = b\dot{\phi}^2\,\hat{y}$$

> We have thus proved that the **contact point** B on a wheel that is **rolling without slipping** is a valid reference point for the torque law, $\vec{\tau}^{(B)} = \dot{\vec{L}}^{(B)}$, even though B *is* accelerating and *is not* the CM.

Remarks : This is an extremely useful result for two reasons.

(1) The contact point is located <u>on the wheel</u> so it is **body-fixed**. That means we can use $L^{(B)} = I^{(B)}\omega$ $\rightarrow$ no need to invoke the spin-orbit decomposition $\vec{L}^{(A)} = \vec{L}_{CM}^{(A)} + \vec{L}'$ that we would need for a reference point located *outside* the wheel.

(2) The torque $\tau^{(B)}$ around the contact point does **not** depend on the **force of friction**, which is always present in no-slip rolling and is never known in advance.

Collective Kinetic Energy – Rotational

An extended object can be both moving and rotating. Let's first calculate its total kinetic energy due **only** to its **rotation**, relative to any **body-fixed point B**. (The object cannot move *relative to itself*, so a body-fixed reference point isolates the rotational motion.)

$$T^{(B)}_{\text{rotational}} = \tfrac{1}{2}\sum_i m_i\left|\dot{\vec{r}}_i^{(B)}\right|^2 = \tfrac{1}{2}\sum_i m_i\left|\vec{\omega}\times\vec{r}_i^{(B)}\right|^2 = \tfrac{1}{2}\sum_i m_i\left(\omega\, r_{i\perp}^{(B)}\right)^2$$

$$= \tfrac{1}{2}\omega^2\sum_i m_i\left(r_{i\perp}^{(B)}\right)^2 = \boxed{\tfrac{1}{2}I^{(B)}\omega^2 = T^{(B)}_{\text{rotational}}}$$

Collective Kinetic Energy – Decomposition

The obtain a kinetic energy formula that includes both the rotational and linear motion of an extended object, let's introduce a **reference point A** other than the origin. Our aim is to find a good choice for A that will split the total KE into two terms: one for rotational motion and one for linear motion.

$$T = \tfrac{1}{2}\sum_i m_i\left(\dot{\vec{r}}_i\bullet\dot{\vec{r}}_i\right) = \tfrac{1}{2}\sum_i m_i\left(\dot{\vec{r}}_i^{(A)} + \dot{\vec{r}}_A\right)\bullet\left(\dot{\vec{r}}_i^{(A)} + \dot{\vec{r}}_A\right)$$

$$= \tfrac{1}{2}\sum_i m_i\left(\dot{\vec{r}}_i^{(A)}\bullet\dot{\vec{r}}_i^{(A)} + 2\,\dot{\vec{r}}_A\bullet\dot{\vec{r}}_i^{(A)} + \dot{\vec{r}}_A\bullet\dot{\vec{r}}_A\right)$$

$$= \tfrac{1}{2}\sum_i m_i v_i^{(A)2} + \vec{v}_A\bullet\sum_i m_i\dot{\vec{r}}_i^{(A)} + \tfrac{1}{2}v_A^2\sum_i m_i$$

$$= T^{(A)} + \vec{v}_A\bullet\vec{V}_{\text{CM}}^{(A)} + \tfrac{1}{2}M\,v_A^2$$

To reduce this to two terms, we can do one of two things:

(1) <u>Choose **A** to be the **CM**</u> : This kills the middle term, since $V_{\text{CM}}^{(A)}$ becomes "velocity of the CM relative to the CM", which is zero. The right-hand term becomes $\tfrac{1}{2}Mv_{CM}^2 = \tfrac{1}{2}MV^2 = T_{CM}$. Thus, $\boxed{T = T_{CM} + T'} = \tfrac{1}{2}MV^2 + \tfrac{1}{2}I'\omega^2$. That's linear KE + rotational KE, just as we wanted, but this lovely split **only** occurs with A = CM.

(2) <u>Choose **A** to be a **stationary point**</u> : If $v_A = 0$, we simply get $\boxed{T = T^{(\text{A stationary})}}$ $\rightarrow$ T relative to the origin is the same as T relative to any other point that isn't moving (which is fairly obvious).