Phys 326 Discussion 5 – Two-Body Central Force Problems

New subject! We will study the dynamics of two bodies that form an isolated system (no external forces) but interact with each other via a **central force**. As we learned in 325, a central force is defined by these properties:

- It acts between exactly two bodies.
- Its direction lies along the line connecting the two bodies.
- Its magnitude depends only on the distance between the two bodies.

The gravitational and electrostatic forces are the prime examples. So: consider an isolated system composed of masses m_1 and m_2 at positions $\vec{r}_1$ and $\vec{r}_2$ respectively. Define $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ as the relative position vector pointing from m_2 to m_1. A central force $\vec{F}_{12} = -\vec{F}_{21}$ acting between these two bodies points along $\pm\hat{r}$ and its magnitude depends only on $r \equiv |\vec{r}|$. Clearly the **potential energy** from such a force also depends only on r : $U(r)$.

This system can be enormously simplified by doing two things:

(1) Make a sensible **change of variables** from the individual position vectors $\vec{r}_1$, $\vec{r}_2$ to the relative position vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ and the CM position vector $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/M$.

(2) Switch to the **CM frame** where $\vec{R} = 0$ and $\dot{\vec{R}} = 0$ (i.e. where the CM is at rest at the origin).

As there is no external force on this system, the CM frame is inertial, so we can switch to it without concern. In a moment you will determine the Lagrangian for this system and you will obtain

$$\mathcal{L}(\vec{r},\dot{\vec{r}}) = \tfrac{1}{2}\mu\left|\dot{\vec{r}}^{\,2}\right| - U(r)$$ where $\mu \equiv m_1 m_2/(m_1+m_2)$ is called the **reduced mass** of the pair.

Key Point: This is the Lagrangian of a **single particle** ! It's the Lagrangian of a mass μ moving in a central potential $U(r)$. We can thus solve the entire 2-body problem by treating it as a 1-body system !

- **Coordinates** : $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2$ …. $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$

Problem 1 : The Reduced Mass → 6 Degrees of Freedom down to 1 *Hints*[1]

(a) As advertised, your first task is to derive that Lagrangian: follow the simplification steps (1) and (2) and write down the Lagrangian for a 2-body system in its CM frame with a potential energy $U(r)$. You *need* to derive this. The transition from a { mass 1 + mass 2 } system to an equivalent { mass μ + fixed central force field } system is so magical it's oddly dangerous: it will be "obvious" while in short term memory, then you'll forget the derivation and wonder what the reduced mass was ever for … Here's what it's for: replacing an actual 2-body system with a fake but equivalent 1-body system. You *need* to know how the 2-body to 1-body transition happens. So go for it → derive the Lagrangian and the expression $\mu = m_1 m_2/(m_1+m_2)$ from scratch.

(b) So the system's Lagrangian is $\mathcal{L} = \frac{1}{2}\mu\left|\dot{\vec{r}}^{\,2}\right| - U(r)$. We now re-imagine the system as some fake fixed source of the potential $U(r)$ sitting at the origin with our fake mass μ whizzing around it. The motion of this mass will be confined to a plane (we'll discuss that in lecture if it's not obvious), so it can be described by two degrees of freedom. The polar coordinates r and ϕ are perfect, giving $\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$. We now need two equations of motion. We can get two second-order EOMs from the Lagrange equations … or we can get two first-order EOMs if we can think of two conserved quantities. Figure out what they are, and write an expression for each in terms of the reduced mass μ, the coordinates $r, \dot{r}, \phi, \dot{\phi}$, and the central potential $U(r)$.

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[1] (a) Hints: Write the system's 2-particle Lagrangian $\mathcal{L}(\vec{r}_1, \dot{\vec{r}}, \vec{r}_2, \dot{\vec{r}}_2)$ for a central potential $U(r)$ between the two masses … switch coordinates to $\vec{R}$ and $\vec{r}$ … convince yourself you can set $\vec{R} = 0$ & $\dot{\vec{R}} = 0$ … rewrite the Lagrangian and μ will be revealed. ☺

(b) Hints: Conserved quantities are energy E and angular momentum L … get L from the fact that ϕ is cyclic (or build $\vec{r} \times \vec{p}$) … get E from T+U (or build the Hamiltonian, which is just T+U) … Answers: $L = \mu r^2\dot{\phi}$ and $E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U$.

(c) $\mu\ddot{r} = -U'(r) + \mu r\dot{\phi}^2 = F(r) + F_{CF}$ = sum of the central force and the **centrifugal** pseudo-force (F_{CF} is there because you have $\ddot{r}$ on the left-hand side, and that is NOT the radial component of acceleration in polar coordinates. F_{CF} is the missing acceleration term from $a_r = \ddot{r} - r\dot{\phi}^2$ moved over to the force side of the equation.) (d) see next page

(c) We have a complete set of EOMs, both first-order. We can call them the "E-equation" and the "L-equation". As it happens, it is also useful to have one second-order equation around: the radial force equation. Obtain this via $\partial\mathcal{L}/\partial r = d(\partial\mathcal{L}/\partial\dot{r})/dt$. (You could also take the time-derivative of the E-equation.) You will get the form: $\mu\ddot{r} =$ <term1> + <term2> … do you recognize both terms on the right-hand side?

(d) Use the L-equation to get rid of $\dot{\phi}$ in the E- and Force equations. This separates the problem into

- <u>radial motion</u> $r(t) \rightarrow$ determined by E-equation or Force equation, for a given value of L
- <u>angular motion</u> $\phi(t) \rightarrow$ determined by L-equation once radial motion is known

All done! Here are all the equations we will use to study to behavior of 2-body central force systems:

- **L-Equation** : $\dot{\phi} = \dfrac{L}{\mu r^2}$
- **E-Equation** : $E = \dfrac{1}{2}\mu\dot{r}^2 + \left(U + \dfrac{L^2}{2\mu r^2}\right) = \dfrac{1}{2}\mu\dot{r}^2 + U^*$
- **Reduced Mass** : $\mu = \dfrac{m_1 m_2}{M}$
- **Force Equation** : $\mu\ddot{r} = F(r) + \dfrac{L^2}{\mu r^3} = F + F_{\text{cf}}$
- ○ Effective Potential : $U^* \equiv U + U_{\text{cf}}$
- ○ Centrifugal force & PE : $\vec{F}_{\text{cf}} = \dfrac{L^2}{\mu r^3}\hat{r}\,,\quad U_{\text{cf}} = \dfrac{L^2}{2\mu r^2}$

Problem 2 : Asteroid incoming! *Checkpoints* [2]

An asteroid of mass m heads toward the Sun, which has mass $M_\odot$. When it is very far away, the asteroid's speed squared is $v_0^2 = 4GM_\odot/3b$ where b is the **impact parameter** of its trajectory toward the Sun. Recall the definition of an impact parameter: if you have one object (call it the beam particle) approaching another (call it the target particle) in a straight line, the impact parameter is the minimum distance of approach that the beam will come to the target. Makes sense? Impact parameters are used all the time in scattering problems … which this problem is, in fact. ☺ So: when our asteroid is far away, it has the speed v_0 given above and it is on a straight-line course that, *if left undeflected*, would have it pass by the Sun at a minimum distance of approach of b. But the sun *is* there, of course, and it affects the asteroid's trajectory when it gets close enough.

(a) Throughout this problem, you can assume that the asteroid is so much lighter than the sun ($m \ll M_\odot$) that you can approximate μ as simply equal to one of the two masses. Which one?

(b) The E-equation can be written $E = \frac{1}{2}\mu\dot{r}^2 + U^*(r)$ where the **effective potential energy** U^* has two terms: $U(r)$ from the central force plus the term $L^2/2\mu r^2$, which can be termed "centrifugal potential". Make a quick sketch of the effective potential $U^* = U + U_{\text{cf}}$ seen by the asteroid as a function of r. This plot is absolutely crucial to understanding what the asteroid will do!

As we will see, the r coordinate will always be either trapped between some minimum value r_{min} and some maximum value r_{max} (**bounded orbit**), or else it will come in from infinity, reach some value r_{min}, then head back out to infinity (**unbounded orbit**). The type of orbit you get depends on the system's energy E. The radial "turning points" at r_{min} and/or r_{max} are such key features of orbits that they have special names:

- **apse** ≡ point on orbit where $r = r_{\text{min}}$ or r_{max}
- **apsidal distances** ≡ r_{min} and/or r_{max}
- **apsidal angle** ≡ angle between successive apses
- **peri-** ≡ means "near", refers to the r_{min} apse
- **ap(o)-** ≡ means "far", refers to the r_{max} apse
- **perigee / apogee** ≡ apses around <u>Earth</u>
- **perihelion, aphelion** ≡ apses around <u>Sun</u>

(c) Calculate the **perihelion distance** reached by the asteroid, i.e. the closest distance of approach to the Sun.

(d) Calculate the asteroid's speed at perihelion. (e) Sketch the asteroid's path as it passes the Sun.

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[2] (a) the asteroid (c) $b/2$ (d) $2v_0$. Did you get zero? You're correct … for one of the *components* of velocity. There's another one!