Phys 326 Discussion 15 – Orbits in GR

Using the Schwarzschild metric

$$d\tau^2 = dt^2\left(1-\frac{2M}{r}\right) - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 \qquad \text{where } M = \left(\frac{MG}{c^2}\right)_{\substack{\text{SI}\\\text{units}}} \;\&\; t = (ct)_{\substack{\text{SI}\\\text{units}}}$$

and the Principle of Maximum Aging, we derived the general equations of motion of a particle moving in the curved spacetime of the Schwarzschild metric:

$$\boxed{l = r^2\frac{d\phi}{d\tau}} \quad \text{and} \quad \boxed{\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}} \qquad \text{where } l = \left(\frac{L}{mc}\right)_{\substack{\text{SI}\\\text{units}}} \;\&\; e = \left(\frac{E}{mc^2}\right)_{\substack{\text{SI}\\\text{units}}}$$

The only simplification we made was to set $\theta = 90°$, which we can always arrange for a 2-body central force problem since the motion is always in a plane. These are the GR versions of the Newtonian angular-momentum and energy-conservation equations that we called the "L-equation" and "E-equation". From the E-equation, we can read off the modified gravitational potential per unit mass, $V \equiv U/m$, that we obtain from GR :

$$\boxed{V_{GR} = -\frac{M}{r} - \frac{Ml^2}{r^3}} \qquad \text{or in SI units, } \boxed{V_{GR} = -\frac{GM}{r}\left(1+\frac{L^2}{m^2c^2r^2}\right)} \qquad \text{where } V \equiv \frac{U}{m}$$

Newtonian gravity is the first term; the second term is the correction from GR.

Problem 1 : Advance of the Perihelion of Mercury

The potential in the box above is *not* a pure $1/r$ potential like Newtonian gravity, it has an additional $1/r^3$ term … and that means that it is not *guaranteed* to produce closed orbits by Bertrand's theorem. It can produce circular orbits, which are closed; but if an orbit not *quite* a circle, such as the orbit of the planet Mercury, its perihelion point will not land at the same point with every revolution. The perihelion point will precess a bit. In this problem we will compute another famous result of Einstein's: the angle by which the perihelion point advances with every Mercurial revolution.

(a) To determine the shape of orbits in our GR potential, we will apply the **path equation** from our 2-body central force studies:

$$u'' + u = -\frac{mF(1/u)}{L^2u^2} \quad \text{with} \quad u \equiv \frac{1}{r}\,.$$

(The derivation of the path equation is clearly unaffected by the fact that the EOMs now use proper time, τ, as an independent variable instead of Schwarzschild time, t, since the whole point of the path equation is to *get rid of* the time variable and express r in terms of ϕ.) To use the path equation, we need a **force**. Calculate the force $F(r)$ that is described by the GR potential in the box above.

(b) If you plug your force into the path equation, you won't be able to get an analytic solution. Whenever a physicist is in that situation, they look for an approximation to make … and for Mercury we can find a good one: its orbit is <u>nearly circular</u>. Here is the trick:

- substitute $u(\phi) = u_0 + \xi(\phi)$ into the path equation where
- $r_0 \equiv 1/u_0$ is the radius of a <u>circular</u> orbit of angular momentum L

You will obtain a differential equation you can solve for $\xi(\phi)$, and from that, you can find the angle $\Delta\phi$ by which perihelion advances every year. One more suggestion: introduce the symbol

$$\varepsilon \equiv \frac{L^2}{m^2 c^2 r_0^2}$$

to replace L. That will simplify your expressions a lot. It is a very small number for Mercury, which you can probably tell my inspection.

→ The final answer you seek is $\Delta\phi = 6\pi\varepsilon$. Can you derive what Einstein did in 50 minutes? ☺