Physics 326 – Homework #2   due Friday, 1 pm

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: "No Work = No Points". However you may always use any relation on the 1DMath, 3DMath or exam formula sheets or derived in lecture / discussion. Write your __NAME__ and __DISCUSSION SECTION__ on your solutions.

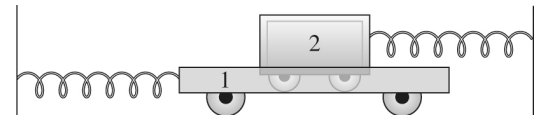## Problem 1 : Triple Pendulum (a classic qual-exam problem)

A triple pendulum consists of masses $\beta m, m$, and m suspended to each other in a line by three massless rods of length a. The whole thing is suspended from a stationary pivot, as shown in the figure. Throughout this problem, you may assume that the displacements of the masses from equilibrium are <u>small</u>.
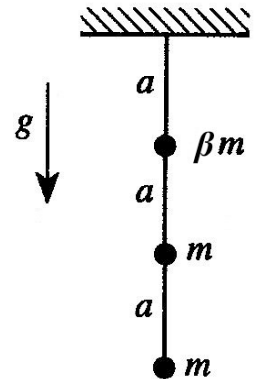
(a) Find the value of β such that one of the normal frequencies of this system will equal the frequency of a simple pendulum of length $a/2$ and mass m.

(b) Find the mode corresponding to this frequency and sketch it.

Problem 2 : Drag Coupling

From PHYS 325 and MATH 285 you know how to solve a **damped oscillator** with one degree of freedom. A quick recap: The EOM for such an oscillator is $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, with β being a damping constant and ω_0 being the oscillator's undamped frequency. To solve, you <u>guess</u> the solution form $\tilde{x}(t) = \tilde{A}e^{\tilde{\omega}t}$, i.e. an exponential where the frequency $\tilde{\omega}$ in the exponent and/or the amplitude $\tilde{A}$ may be <u>complex numbers</u>. When you <u>plug</u> that form into the EOM and solve for $\tilde{\omega}$, you will get quite different results if the oscillator is weakly damped ($\beta < \omega_0$, which produces damped oscillations) or strongly damped ($\beta > \omega_0$, which doesn't oscillate at all). You can review the blackboards from PHYS 325 lecture 14B for a reminder; they are available in a folder on the PHYS 326 website.

In this problem, we tackle a coupled oscillator (*two* degrees of freedom) with damping involved. As you will see, the <u>matrix notation</u> we have been using to write and solve the EOMs for coupled linear oscillators can be readily extended to a damped system, you just have to use complex exponentials instead of cosines for your normal-form solutions. No problem! ☺

The two carts in the figure above have equal masses m. They are joined by identical but separate springs of force constant k to separate walls. Cart 2 rides in cart 1 as shows, and cart 1 is filled with molasses, whose viscous drag supplies the coupling between the two carts. The drag force has magnitude $\beta m v$ where v is the <u>relative</u> velocity of the two carts.

(a) Write down the equations of motion of the two carts using as coordinates x_1 and x_2, the displacements of the carts to the right of their equilibrium positions. Show that the EOM can be written in matrix form as $\mathbf{1}\ddot{\vec{x}} + \beta\mathbf{D}\dot{\vec{x}} + \omega_0^2\mathbf{1}\vec{x} = 0$, where $\vec{x}$ is the column vector made up of x_1 and x_2, $\omega_0 \equiv \sqrt{k/m}$, $\mathbf{1}$ is the unit matrix, and $\mathbf{D}$ is a certain 2×2 square matrix for you to determine.

(b) The next step is to "guess the solution form". Let's try normal mode form, but with a slight variation. **Normal mode form** means a solution where all the coordinates are oscillating at the <u>same frequency</u> and the <u>same phase</u>. This system has damping, however, so its oscillations will decay with time. That suggests a solution form $\vec{x}(t) = \vec{A}e^{\tilde{\omega}t}$ where we hypothesize a **common frequency** $\tilde{\omega}$ that is **complex** instead of the usual purely-imaginary exponent $i\omega$ that you get for an undamped oscillator system. Assuming that the drag force is

weak ($\beta < \omega_0$), show that you do get two solutions of this form with $\tilde{\omega} = i\omega_0$ or $\tilde{\omega} = -\beta + i\sqrt{\omega_0^2 - \beta^2}$.

HINT: The determinant does *not* require much algebra *at all* … $a^2 - b^2 = (a+b)(a-b)$ …

NOTE: You may be wondering what happened to the minus option in the $\tilde{\omega} = \pm i\omega_0$ and $\tilde{\omega} = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$ solutions that you probably obtained. Answer: it doesn't matter which sign you choose in these two cases … but why? That is for you to figure out!

(c) Describe the motions corresponding to each normal mode, using words or sketches. Also explain physically why one of the modes is damped but the other is not.

Problem 3 : Bead on a Swinging Hoop

A bead of mass m is threaded on a frictionless circular wire hoop of radius R and the same mass m as the hoop. The hoop is suspended at the point A and is free to swing in its own vertical plane as shown in the figure.

(a) Using the angles ϕ_1 and ϕ_2 as generalized coordinates, solve for the normal frequencies of small oscillations, and find and describe (e.g. with sketches) the motion in the corresponding normal modes.

(b) Find two sets of <u>initial conditions</u> that allow the system to oscillate in each of its two normal modes.
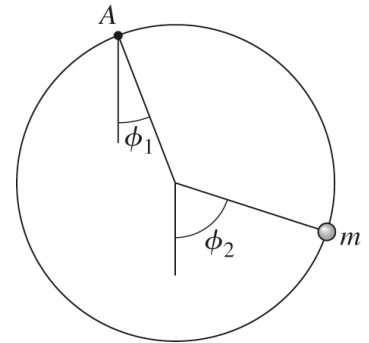
Problem 4 : Pendulum on a Cart

A simple pendulum (mass M and length L) is suspended from a cart of mass m that moves freely along a horizontal track. You will find it helpful to introduce the parameters $\eta \equiv m/M$ and $\omega_0 = \sqrt{g/L}$.

(a) What are the normal frequencies of small oscillations of the system?

(b) Find and describe (e.g. with sketches) the corresponding normal modes of the system. If you are doing this before the lecture on **DC modes**, please read the brief Appendix.

(c) The cart / pendulum system is held at rest in the configuration $x = 0$ and $\phi = \phi_0$, where ϕ_0 is small. At time $t = 0$, the system is released from rest. Write down the subsequent motion $x(t)$ of the cart and $\phi(t)$ of the pendulum.
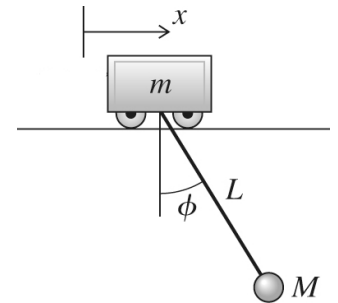
Appendix : DC Modes

If you find that one of the eigenfrequencies, ω, of your system is <u>zero</u> then your system has a **zero-frequency mode**, a.k.a. a **DC mode**. The normal-mode solution form is $q_i(t) = A_i \cos(\omega t - \delta)$; if you plug $\omega=0$ into that, you get a solution of all constants: $q_i(t) = A_i \cos(\delta) = B_i$. That's not enough free parameters: we need *two* free parameters *per coordinate*, and we only have one! What to do? → Go back to one step *after* we postulated normal mode form, when we wrote down $\ddot{\vec{q}} = -\omega^2 \vec{q}$. If you have a DC mode, then $\ddot{\vec{q}} = 0$ → none of the coordinates are accelerating at all in this mode. Well we know the solution of $\ddot{q}_i = 0$ → $q_i(t) = A_i + B_i t$. That's got the two free parameters we need, great. That's the solution form you use for a DC mode.