

Problem 0 : Not for points on this homework, but please do not forget to work through Discussion 3 Problem 2, which covers **degenerate modes**, if you did not manage to during discussion period.

Problem 1 : Normal Coordinates

The **normal coordinates** ξ_i of a coupled oscillator

problem are the coordinates that **decouple the**

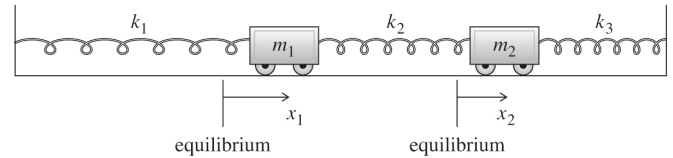
equations of motion. If you could figure out what they

are by just staring at your EOMs, these problems would be very simple! Unfortunately, in practice there are very

few cases where you can do this. One such case is problems with two DOFs, described by the normal

coordinates q_1 and q_2 , where the system is **symmetric under the exchange of q_1 and q_2** . Since the EOMs are *unchanged* when you swap q_1 and q_2 , here's what you do to decouple them:

- Add and subtract the EOMs to give two new EOMs.
- Identify the single linear combination of q 's that appears in each of these two EOMs
→ these are the normal coordinates ξ_+ and ξ_- . (For 1 □ 2 symmetric systems, they will be $\xi_+ \equiv q_1 + q_2$ and $\xi_- \equiv q_1 - q_2$.)
- Rewrite the EOMs in terms of ξ_+ and ξ_- ... et voilà! You have decoupled EOMs.



(a) The figure shows our standard 2-mass-3-spring system. For this problem, we will only consider the highly symmetric case of equal masses $m_1=m_2=m$ and equal springs $k_1=k_2=k_3=k$. Write down the EOMs of this simple system in terms of x_1 and x_2 then follow the bulleted procedure above to introduce coordinates ξ_+ and ξ_- that give you decoupled EOMs. Finally, write your decoupled EOMs in matrix form, $\mathbf{M}^\xi \ddot{\xi} = -\mathbf{K}^\xi \xi$. What elegant property do the mass (M) and spring (K) matrices have when they are written in “ ξ -space”, i.e. in terms of normal coordinates?

(b) Now let's subject each mass to a linear damping force $-b\vec{v} = -2\beta m\vec{v}$ (same β for both masses). Use the method of normal coordinates to solve this problem, i.e. identify two coordinates ξ_+ and ξ_- that are linear combinations of x_1 and x_2 that decouple the equations of motion.

(c) Using damped 1D-oscillator skills, solve your decoupled equations of motion to obtain the general solutions for the normal coordinates, $\xi_+(t)$ and $\xi_-(t)$. Assume that $\beta \ll k/m$ so that the oscillations are underdamped.

(d) Find $x_1(t)$ and $x_2(t)$ for the following initial conditions (ICs) : $x_1(0) = A$ and $x_2(0) = v_1(0) = v_2(0) = 0$.

TACTICS: Since the general solutions are so simple in ξ -space (only one mode for each normal coordinate!)

ξ -space is great for applying ICs to obtain a particular solution from a general one ... *if* you work efficiently.

You can do one of these two things to get the particular solutions $x_1(t)$ and $x_2(t)$ that you seek :

1. Transform the *general solutions* from ξ -space to x -space, then apply the ICs to get the particular solutions.
2. Transform the *initial conditions* from x -space to ξ -space, apply them in ξ -space, then transform the *particular solutions* from ξ -space to x -space.

As always, try both if you can, but you will find tactic #2 to be much more efficient!

(e-**NOT FOR POINTS**) For your edification, re-solve the damped-oscillator system (parts b,c,d) without using normal coordinates at all, i.e. using the same technique you used for last week's damped oscillator system.

Just write the equations of motion in matrix form including a damping term, $\mathbf{M}\ddot{\vec{x}} + \mathbf{D}\dot{\vec{x}} + \mathbf{K}\vec{x} = 0$, hypothesize normal-mode solution form, $\vec{x}(t) = \tilde{\vec{A}}e^{\tilde{\omega}t}$ with a complex exponent and complex amplitude, and solve for the (complex) eigenvalues $\tilde{\omega}$ and eigenvectors $\tilde{\vec{A}}$ using standard techniques. When you're done, ask yourself:

How much easier was the normal-coordinate solution? I think you will find the answer is “not much”. Normal coordinates are *conceptually* important – as we will see! – but not all that much help as a solving technique.

If you are working on this before Thursday’s lecture : you don’t have enough information yet to solve the last part of 2(b) or parts 3(d),(e),(f). (We are a bit behind the posted schedule; I have flagged the affected parts in red and the formulae you need are in the **Appendix**.) These parts are not long, but if your schedule makes things difficult, you can turn in the homework on Wednesday without penalty.

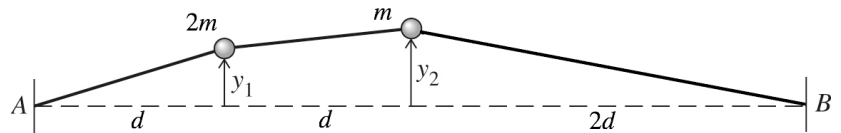
Problem 2 : 3 Beads and Springs on a Ring

Qual Problem

Consider a frictionless rigid horizontal hoop of radius R . Onto this hoop we thread three beads with masses $2m$, m , and m ; between the beads we thread three identical springs on the hoop, each with force constant k .

- (a) Solve for the three normal frequencies.
- (b) Find the three normal modes, describe them with sketches, and **express them in normalized form, i.e. so that their amplitude vectors obey the orthonormality relation**

$\langle \hat{a}_m | \hat{a}_n \rangle = \hat{a}_m^T \mathbf{M} \hat{a}_n = \delta_{mn}$. Take $R = 1$ for simplicity.



Problem 3 : Transverse Modes

Two particles, of masses $2m$ and m , are secured to a light string of total length $4d$ that is stretched to tension T_0 between two fixed supports. As shown, the masses are not evenly spaced along the string. The masses undergo small transverse oscillations, where their transverse displacements from equilibrium, y_1 and y_2 , are kept to very small values compared with the length-scale d of the string.

- (a) Find the normal frequencies of transverse oscillation for this system. You will find it useful throughout this problem to define the constant $\alpha \equiv T_0 / (dm) \rightarrow$ using it will greatly simplify your expressions!
- (b) Write down the general solution for $y_1(t)$ and $y_2(t)$.
- (c) Is the general motion you calculated in (b) periodic? Explain why or why not, and if it is, give the period of the general motion.
- (d) **Normalize the eigenvectors for the fast and slow modes to obtain an orthonormal basis $\{\hat{a}_F, \hat{a}_S\}$.**
- (e) Find the **normal coordinates** ξ_F and ξ_S in terms of the generalized coordinates y_1 and y_2 , and determine the matrices \mathbf{R} and \mathbf{R}^{-1} that relate them via $\xi = \mathbf{R} \vec{y}$ and $\vec{y} = \mathbf{R}^{-1} \xi$.
- (f) Explicitly transform the mass matrix \mathbf{M} and spring matrix \mathbf{K} to ξ -space (i.e., calculate \mathbf{M}^ξ and \mathbf{K}^ξ) using the matrix transformation formula $\mathbf{M}^\xi = (\mathbf{R}^{-1})^T \mathbf{M} (\mathbf{R}^{-1})$ derived in class, and verify that they are diagonal.

Problem 4 : 4-Atom Ring Molecule

Qual Problem

To study the vibrational spectrum of a ring molecule like benzene, one can reasonably approximate the molecule’s atoms / sub-molecules as beads placed on a ring with springs between them. Let’s try a 4-element ring molecule: consider four identical beads of mass m placed on a ring with springs of equal strength k running along the ring between the beads. Using as generalized coordinates the positions x_1, x_2, x_3, x_4 of the 4 beads measured along the ring relative to equilibrium, determine the four normal modes of the system. Provide a small sketch of each mode so you can visualize it, and make sure your four modes are orthogonal to each other.

Hint: the 4×4 matrix $\mathbf{M}\omega^2 - \mathbf{K}$ can be hugely simplified by introducing a variable $\alpha \equiv (m\omega^2 / k - 2)$.

Appendix : Complete Formula Set for Inner Product Space Description of Normal Mode Solutions

• **Space** : $|\vec{q}(t)\rangle \equiv$ all solutions of a particular linear oscillator system

• **Inner Product** : $\langle \vec{q}_1 | \vec{q}_2 \rangle \equiv \vec{q}_1^T \mathbf{M} \vec{q}_2$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q} | \vec{q} \rangle$

• **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K} \vec{a}_m = \omega_m^2 \mathbf{M} \vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$

• **Basis is Orthonormal** : $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$

• **Completeness** for $\vec{q}(t)$ and **Normal Coordinates** ξ_m :

$$\xi_m \text{ is the component of } \vec{q} \text{ along mode } m : |\vec{q}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle \langle \hat{a}_m | \vec{q}(t) \rangle \equiv \sum_m \hat{a}_m \xi_m(t) = \sum_m \hat{a}_m \tilde{A}_m e^{i\omega_m t}$$

$$\xi_m \text{ is projected out of } \vec{q} \text{ by inner product: } \xi_m(t) = \langle \hat{a}_m | \vec{q}(t) \rangle = \tilde{A}_m e^{i\omega_m t} = A_m \cos(\omega_m t - \delta_m)$$

• **Transformation** between q -space and ξ -space :

$$\text{vectors : } \vec{\xi} = \mathbf{R} \vec{q} \quad \vec{q} = \mathbf{R}^{-1} \vec{\xi} \quad \mathbf{R}^{-1} = \begin{pmatrix} | & | \\ \hat{a}_1 & \hat{a}_2 & \dots \\ | & | \end{pmatrix} \quad \mathbf{R} = (\mathbf{R}^{-1})^T \mathbf{M}$$

$$\text{tensors : } \mathbf{M}^\xi = (\mathbf{R}^{-1})^T \mathbf{M} \mathbf{R}^{-1} \rightarrow \mathbf{M}_{mn}^\xi = \delta_{mn} \quad \& \quad \mathbf{K}_{mn}^\xi = \omega_m^2 \delta_{mn}$$

$$\text{inhomogeneous EOM : } \mathbf{M} \ddot{\vec{x}} + \mathbf{K} \vec{x} = \vec{F} \text{ in } q\text{-space} \rightarrow \mathbf{M}^\xi \ddot{\vec{\xi}} + \mathbf{K}^\xi \vec{\xi} = (\mathbf{R}^{-1})^T \vec{F} \text{ in } \xi\text{-space}$$