Driven Torsional Oscillator

Physics 401, Spring 2016
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Agenda

1. Driven torsional oscillator. Equations
2. Setup. Kinematics
3. Resonance
4. Beats
5. Nonlinear effects
6. Comments
Before starting the torsional oscillator discussion let we take a look on some historical examples showing how dangerous the resonance in mechanical systems can be.
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge Disaster
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge, 1940
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge, 1940
Tacoma (WA) Narrows Bridge, 1940
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
“Dancing Bridge” in Volgograd (Russia) (record from 2\textsuperscript{st} May 2010. 4.4 miles long).
In autumn 2011, 12 semi-active tuned mass dampers were installed in the bridge. Each one consists of a mass 5,200 kg (11,500 lb), a set of compression springs and a magnethoreological damper.
Torsional oscillations. Flutter. Aviation.

Milestones in Flight History
Dryden Flight Research Center

PA-30 Twin Commanche
Tail Flutter Test

April 5, 1966
The goals: (i) analyze the response of the damped driven harmonic oscillator to a sinusoidal drive. (ii) transient response and (iii) steady-state solution.

\[
I \ddot{\theta} + K \theta + R \dot{\theta} = \tau_m = K \lambda \theta_0 \cos(\omega t)
\]

Viscous damping

Torque by motor

Angular displacement:
\[\theta_0 \cos(\omega t)\]

torque:
\[K \lambda \theta_0 \cos(\omega t)\]

\[\lambda = \frac{L_1}{L_1 + L_2}\]

I is momentum of inertia, [kg\cdot m^2]
R is a damping constant [N\cdot m\cdot s].
K is the total spring constant [N\cdot m]
Driven torsional oscillator

Motor

Pendulum
Transient solution

\[ I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t) \]

Solutions: sum of (1) Transient solution + (2) steady solution due to torque \( \tau_m \)

(1) Transient solution (1st week experiment)

\[ I\ddot{\theta} + R\dot{\theta} + K\theta = 0 \]
\[ \theta(t) = A e^{-\alpha t}\cos(\omega_1 t - \phi) \]
\[ a = \frac{R}{2I} \]
\[ \omega_0 = \sqrt{\frac{K}{I}} \]
\[ \omega_1 = \sqrt{\omega_0^2 - \alpha^2} \]

The homogeneous equation of motion

Transient solution

Attenuation constant

Natural (angular) frequency

Damped (angular) frequency
\[
\theta_i(t) = |A| e^{-at} \cos(\omega_1 t + \phi) \quad \rightarrow \quad \omega_1 = \sqrt{\omega_0^2 - a^2}
\]

Transient solution

Initially the system responds on the characteristic frequency \( \omega_1 \)

Once this response dies away in time the system response only on the frequency of drive \( \omega \)

\[
\theta_{ss}(t) = \text{Re}\left(\theta(\omega)e^{i\omega t}\right)
\]

So the steady-state solution must have the similar time dependence as the drive

\[
\tau_m = K\lambda \theta_0 \cos(\omega t)
\]

Substituting \( \theta_{ss}(t) \) in equation of motion we will find the equations for \( \theta(\omega) \)

\[
\theta(\omega) = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 a^2}} e^{-i\beta(\omega)}
\]

and

\[
\beta(\omega) = \tan^{-1}\left(\frac{2\omega a}{\omega_0^2 - \omega^2}\right)
\]
Steady-state solution. Summary.

\[ I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_\text{m} = K\lambda\theta_0 \cos(\omega t) \]

(2) steady solution

\[ \theta_s(t) = B(\omega)\cos(\omega t - \beta(\omega)) \]

\[ B(\omega) = \frac{\lambda\theta_0\omega^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}} \]

\[ \tan \beta(\omega) = \frac{\omega\gamma}{\omega_0^2 - \omega^2} \]

\[ \gamma = \frac{R}{I} = 2 \frac{R}{2I} = 2a \]
General solution for equation of motion consist of the sum of sum of two components:

$$\theta(t) = \theta_t(t) + \theta_{ss}(t)$$

$$\theta_{ss}(t) = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 a^2}} \cos(\omega t - \beta(\omega))$$

Amplitude $B(\omega)$

Phase

Coefficients $A$ and $\phi$ could be determined from initial conditions.
**Resonance. Experiment. Amplitude**

**Fitting function:**

\[ \theta(f) = \frac{A \cdot f_0^2}{\sqrt{(f_0^2 - f^2)^2 + \gamma^2 f^2}} \]

\( f_0 = 0.50 \text{Hz} \) (fitting)

\( \omega = 2\pi f; \ \gamma = 2a \)

To create a new fitting function go "Tools"→"Fitting Function Builder" or press F8

<table>
<thead>
<tr>
<th>Model</th>
<th>Resonance1 (User)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation</td>
<td>( y = A \cdot f_0^2 / \sqrt{(f_0^2 - x^2)^2 + x^2 \gamma^2} )</td>
</tr>
<tr>
<td>Reduced Chi-Sqr</td>
<td>3.00E-04</td>
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<tr>
<td>Adj. R-Square</td>
<td>0.999411988</td>
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<table>
<thead>
<tr>
<th>pend</th>
<th>Value</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>pend</td>
<td>A</td>
<td>0.286662</td>
</tr>
<tr>
<td>pend</td>
<td>f0</td>
<td>0.500271</td>
</tr>
<tr>
<td>pend</td>
<td>gamma</td>
<td>0.062856</td>
</tr>
</tbody>
</table>
Scanning the driving frequency we can measure the amplitude of the pendulum oscillating and the phase shift.

Both parameters Amplitude and phase can be defined by DAQ program or using Origin.
Resonance.
Amplitude of the Angular Displacement.

\[ |\theta_{ss}(t)| = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 a^2}} \]

At resonance \( \omega = \omega_0 \)

\[ |\theta_{ss}(t)| = \frac{\lambda \omega_0 \theta_0}{2a} = \lambda \theta_0 \cdot Q \]

Combination of high initial amplitude \( \theta_0 \), and high quality \( Q \) or low damping factor \( a \) could be result of the destruction of the mechanical system.
For correct representation of the resonance curve take care about choosing of the step size in frequency.

\[ f_0 = 0.495 \text{Hz} \]
There are two parameters used to measure the rate at which the oscillations of a system are damped out. One parameter is the logarithmic decrement $\delta$, and the other is the quality factor, $Q$.

$\delta$, is defined by

$$\delta = \ln \left( \frac{\theta(t_{\text{max}})}{\theta(t_{\text{max}} + T_1)} \right) = \ln \left( \frac{e^{-at_{\text{max}}}}{e^{-a(t_{\text{max}} + T_1)}} \right) = aT_1.$$  

$$\delta = \ln \left( \frac{8.49}{7.35} \right) \approx 0.144$$

$$Q = 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}}$$

$$Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi \omega_1}{2\pi} = \frac{\pi}{a} T_1 = \frac{\pi}{\delta}$$

$Q \sim 21.8$
It can be shown that Q can be calculated as $\omega_1/\Delta \omega$ or $f_1/\Delta f$. $\Delta \omega$ is bandwidth of the resonance curve on the half power level or $\frac{\theta_{\text{max}}}{\sqrt{2}}$ for amplitude graph.

Here $Q \approx 7.9$
Beats. Theory.

Consider sum of two harmonic signals of frequencies $\omega_1$ and $\omega_2$

$$y_1 = A\sin(\omega_1 t + \varphi_1); \quad y_2 = B\sin(\omega_2 t + \varphi_2)$$

In case $A=B$ $y = y_1 + y_2 = 2A\sin\left(\frac{\omega_1 + \omega_2}{2} t + \beta_1\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t + \beta_2\right)$;

$\beta_1 = \frac{\varphi_1 + \varphi_2}{2}; \quad \beta_2 = \frac{\varphi_1 - \varphi_2}{2}$

If $\omega_1 \approx \omega_2 \approx \frac{\omega_1 + \omega_2}{2} = \omega$ and $\frac{\omega_1 - \omega_2}{2} = \Omega$

$$y = 2A\cos(\Omega t + \beta_2)\sin(\omega t + \beta_1)$$
More general case $A \neq B$ $\omega_1$ and $\omega_2$

$y_1 = A \sin(\omega_1 t)$; $y_2 = B \sin((\omega_1 + \alpha)t)$

$y = y_1 + y_2 = C \sin((\omega + \beta)t)$ where $C = \sqrt{A^2 + B^2 + 2AB\cos(\alpha t)}$

$\beta = \tan^{-1}\left(\frac{B \sin(\alpha t)}{A + B \cos(\alpha t)}\right) + \begin{cases} 0 & \text{if } A + B \cos(\alpha t) \geq 0 \\ \pi & \text{if } A + B \cos(\alpha t) < 0 \end{cases}$
Two peaks corresponding $\omega$ and $\omega_1$

Use Origin to analyze the frequency spectrum!
Beats dying in time.
How fast – it depends on damping. When you will work on resonance data – wait until you will see the steady state oscillations.
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) \]

\[ \theta_t(t) \to 0 \quad \text{This can be seen well from “envelope” plot} \]

**Origin 8.6: Analysis → Signal Processing → Envelope**
Beats. Experiment. Fitting.

\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = A e^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) + C \]

First let we apply FFT to find $\omega_1$ and $\omega$

Result: $\omega_1 = 3.1402 \text{rad}^{-1}$ and $\omega = 2.8298 \text{ rad}^{-1}$
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-\frac{t}{t_0}} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) + C \]

Result from FFT: \( \omega_1 = 3.1402 \text{ rad}^{-1} \) and \( \omega = 2.8298 \text{ rad}^{-1} \)
**Beats. Experiment. Fitting. Residuals.**

**Possible origin of “extra” peaks:**

(i) Nonlinear behavior of pendulum

(ii) Not a single frequency driving force provided by motor

(iii) Not ideal fitting function

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**Compare with original pendulum spectrum**

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**FFT**

**Residuals**
\[ \theta(t) = \theta_i(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) \]

\[ \theta_i(t) \rightarrow 0 \]

We also can analyze the decrease of the amplitude of the \( \omega_1 \) component by analyzing the spectrum as a function of time.
Beats. RLC Experiment.
In the case of driving frequency $f_d = f_1 / N$ where $N$ is integer we can observe more complicated motion of the pendulum.
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Detailed analyzes* shows that even if $\phi = \phi_0 \sin(\omega t)$ the driving torque contains several harmonics of $\omega$

*P. Debevec (UIUC, Department of Physics)