Driven Torsional Oscillator

Physics 401, FALL 2017
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1. Driven torsional oscillator. Equations
2. Setup. Kinematics
3. Resonance
4. Beats
5. Nonlinear effects
6. Comments
Before starting the torsional oscillator discussion let we take a look on some historical examples showing how dangerous the resonance in mechanical systems can be.
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge Disaster
Tacoma (WA) Narrows Bridge, 1940
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge, 1940
Tacoma (WA) Narrows Bridge, 1940
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
“Dancing Bridge” in Volgograd (Russia) (record from 2nd May 2010. 4.4 miles long).
In autumn 2011, 12 semi-active tuned mass dampers were installed in the bridge. Each one consists of a mass 5,200 kg (11,500 lb), a set of compression springs and a magnethoreological damper.
Milestones in Flight History
Dryden Flight Research Center

PA-30 Twin Commanche
Tail Flutter Test
April 5, 1966

AIRBOYD.TV
The goals: (i) analyze the response of the damped driven harmonic oscillator to a sinusoidal drive. (ii) transient response and (iii) steady-state solution.

\[ I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda \theta_0 \cos(\omega t) \]

- **I** is momentum of inertia, [kg·m²]
- **R** is a damping constant [N·m·s]
- **K** is the total spring constant [N·m]

Angular displacement:
\[ \theta_0 \cos(\omega t) \]

Torque by motor:
\[ K\lambda \theta_0 \cos(\omega t) \]

Viscous damping
Driven torsional oscillator

Motor

Pendulum
Transient solution

\[ I \ddot{\theta} + K \theta + R \dot{\theta} = \tau_m = K \lambda \theta_0 \cos(\omega t) \]

Solutions: sum of (1) Transient solution + (2) steady solution due to torque \( \tau_m \)

(1) Transient solution (1\(^{st}\) week experiment)

\[ \ddot{\theta} + \frac{R}{I} \dot{\theta} + \frac{K}{I} \theta = 0 \]

\[ \theta(t) = A e^{-at} \cos(\omega_1 t - \phi) \]

\( a = \frac{R}{2I} \)

\( \omega_0 = \sqrt{\frac{K}{I}} \)

\( \omega_1 = \sqrt{\omega_0^2 - a^2} \)

The homogeneous equation of motion

Transient solution

Attenuation constant

Natural (angular) frequency

Damped (angular) frequency
Steady-state solution

\[ \theta_t(t) = |A| e^{-at} \cos(\omega_1 t + \phi) \quad \rightarrow \quad \omega_1 = \sqrt{\omega_0^2 - a^2} \]

Initially the system responds on the characteristic frequency \( \omega_1 \).

Once this response dies away in time the system response only on the frequency of drive \( \omega \).

So the steady-state solution must have the similar time dependence as the drive.

\[ \theta_{ss}(t) = \text{Re}\left(\theta(\omega)e^{i\omega t}\right) \]

Substituting \( \theta_{ss}(t) \) in equation of motion we will find the equations for \( \theta(\omega) \).

\[ \theta(\omega) = \frac{\lambda \omega^2 \theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4 \omega^2 a^2}} e^{-i\beta(\omega)} \]

\[ \beta(\omega) = \tan^{-1}\left(\frac{2\omega a}{\omega_0^2 - \omega^2}\right) \]
\[ I \ddot{\theta} + K \theta + R \dot{\theta} = \tau_m = K \lambda \theta_0 \cos(\omega t) \]

(2) steady solution

\[ \theta_s(t) = B(\omega) \cos(\omega t - \beta(\omega)) \]

Steady state solution

\[ B(\omega) = \frac{\lambda \theta_0 \omega^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2 \gamma^2}} \]

Amplitude function

\[ \tan \beta(\omega) = \frac{\omega \gamma}{\omega_o^2 - \omega^2} \]

Phase function

\[ \gamma = \frac{R}{I} = 2 \frac{R}{2I} = 2a \]

Damping constant
General solution for equation of motion consist of the sum of sum of two components:

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega))$$

Coefficients $A$ and $\phi$ could be determined from initial conditions.
Resonance. Experiment. Amplitude

Fitting function:

\[ \theta(f) = \frac{A \cdot f_0^2}{\sqrt{(f_0^2 - f^2)^2 + \gamma^2 f^2}} \]

\( \omega = 2\pi f; \ \gamma = 2a \)

To create a new fitting function go “Tools” → “Fitting Function Builder” or press F8

<table>
<thead>
<tr>
<th>Model</th>
<th>Resonance1 (User)</th>
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<tbody>
<tr>
<td>Equation</td>
<td>( y = A \cdot f_0^2 / \sqrt{(f_0^2 - x^2)^2 + x^2 \gamma^2} )</td>
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<tr>
<td>Reduced Chi-Sqr</td>
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<tr>
<td>Adj. R-Square</td>
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<table>
<thead>
<tr>
<th>Value</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>pend A</td>
<td>0.286662</td>
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<tr>
<td>pend f0</td>
<td>0.500271</td>
</tr>
<tr>
<td>pend gamma</td>
<td>0.062856</td>
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</tbody>
</table>
Resonance. Experiment. Phase

Scanning the driving frequency we can measure the amplitude of the pendulum oscillating and the phase shift.

Both parameters Amplitude and phase can be defined by DAQ program or using Origin.
Resonance.
Amplitude of the Angular Displacement.

Amplitude

\[ |\theta_{ss}(t)| = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 a^2}} \]

At resonance \( \omega = \omega_0 \)

\[ |\theta_{ss}(t)| = \frac{\lambda \omega_0 \theta_0}{2a} = \lambda \theta_0 \cdot Q \]

Combination of high initial amplitude \( \theta_0 \), and high quality \( Q \) or low damping factor \( a \) could be result of the destruction of the mechanical system.
For correct representation of the resonance curve take care about choosing of the step size in frequency.
There are two parameters used to measure the rate at which the oscillations of a system are damped out. One parameter is the logarithmic decrement $\delta$, and the other is the quality factor, $Q$.

$\delta$, is defined by

$$\delta = \ln \left( \frac{\theta(t_{\text{max}})}{\theta(t_{\text{max}} + T_1)} \right) = \ln \left( \frac{e^{-a t_{\text{max}}}}{e^{-a (t_{\text{max}} + T_1)}} \right) = a T_1.$$ 

$$\delta = \ln \left( \frac{8.49}{7.35} \right) \approx 0.144$$

$Q = 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}}$

$$Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi}{a} \frac{\omega_1}{2\pi} = \frac{\pi}{a} \frac{1}{T_1} = \frac{\pi}{\delta}$$

$Q \sim 21.8$
It can be shown that $Q$ can be calculated as $\frac{\omega_1}{\Delta \omega}$ or $\frac{f_1}{\Delta f}$. $\Delta \omega$ is bandwidth of the resonance curve on the half power level or $\frac{\theta_{max}}{\sqrt{2}}$ for amplitude graph.

Here $Q \approx 7.9$
Consider sum of two harmonic signals of frequencies $\omega_1$ and $\omega_2$

$$y_1 = A\sin(\omega_1 t + \varphi_1); \quad y_2 = B\sin(\omega_2 t + \varphi_2)$$

In case $A=B$ $y = y_1 + y_2 = 2A\sin\left(\frac{\omega_1 + \omega_2}{2} t + \beta_1\right)\cos\left(\frac{\omega_1 - \omega_2}{2} t + \beta_2\right)$;

$$\beta_1 = \frac{\varphi_1 + \varphi_2}{2}; \quad \beta_2 = \frac{\varphi_1 - \varphi_2}{2}$$

If $\omega_1 \approx \omega_2 \approx \frac{\omega_1 + \omega_2}{2} = \omega$ and $\frac{\omega_1 - \omega_2}{2} = \Omega$

$$y = 2A\cos(\Omega t + \beta_2)\sin(\omega t + \beta_1)$$

![Graph showing the sum of two harmonic signals](image)
More general case $A \neq B$ \( \omega_1 \) and \( \omega_2 \)

\[ y_1 = A \sin(\omega_1 t); \quad y_2 = B \sin((\omega_1 + \alpha) t) \]

\[ y = y_1 + y_2 = C \sin((\omega + \beta) t) \quad \text{where} \quad C = \sqrt{A^2 + B^2 + 2AB \cos(\alpha t)} \]

\[ \beta = \tan^{-1} \left( \frac{B \sin(\alpha t)}{A + B \cos(\alpha t)} \right) + \begin{cases} 0 & \text{if } A + B \cos(\alpha t) \geq 0 \\ \pi & \text{if } A + B \cos(\alpha t) < 0 \end{cases} \]
Beats. Experiment

Time domain trace

Beating spectrum

Two peaks corresponding \( \omega \) and \( \omega_1 \)

Use Origin to analyze the frequency spectrum!
Beats dying in time. How fast – it depends on damping. When you will work on resonance data – wait until you will see the steady state oscillations.

\[ \theta(t) = \theta_\text{c}(t) + \theta_\text{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) \]
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) \]

**Beats. Experiment.**

\[ \theta_t(t) \rightarrow 0 \]

This can be seen well from “**envelope**” plot

**Origin 8.6: Analysis → Signal Processing → Envelope**
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = A e^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) + C \]

First let we apply FFT to find \( \omega_1 \) and \( \omega \)

Result: \( \omega_1 = 3.1402 \text{rad}^{-1} \) and \( \omega = 2.8298 \text{ rad}^{-1} \)
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-\frac{t}{t_0}} \cos(\omega t - \phi) + B \cos(\omega t - \beta(\omega)) + C \]

Result from FFT: \( \omega_1 = 3.1402 \text{ rad}^{-1} \) and \( \omega = 2.8298 \text{ rad}^{-1} \)

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<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Standard Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.65012</td>
<td>0.00161</td>
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<tr>
<td>(t_0)</td>
<td>199.64912</td>
<td>0.78484</td>
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<tr>
<td>(\omega_1)</td>
<td>3.13666</td>
<td>3.38011E-5</td>
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<tr>
<td>p1</td>
<td>0.33135</td>
<td>0.0019</td>
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<tr>
<td>B</td>
<td>-0.74076</td>
<td>5.86177E-4</td>
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<tr>
<td>(w_2)</td>
<td>2.82464</td>
<td>8.4184E-6</td>
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<td>p2</td>
<td>-0.88782</td>
<td>5.92212E-4</td>
</tr>
<tr>
<td>C</td>
<td>-0.11176</td>
<td>3.09237E-4</td>
</tr>
</tbody>
</table>

From fitting

Compare with original pendulum spectrum

Possible origin of “extra” peaks:
(i) Nonlinear behavior of pendulum
(ii) Not a single frequency driving force provided by motor
(iii) Not ideal fitting function
\[ \theta(t) = \theta_i(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) \]

\[ \theta_i(t) \to 0 \]

We also can analyze the decrease of the amplitude of the \( \omega_i \) component by analyzing the spectrum as a function of time.

Origin 9.0: Analysis → Signal Processing → FFT
From fitting

\[ \omega_1 = 3.13666 \]
\[ f_1 = 0.4992 \text{ Hz} \]
\[ \omega = 2.82464 \]
\[ f_2 = 0.4496 \text{ Hz} \]

From FFT

\[ f_1 = 0.499 \text{ Hz} \]
\[ f_2 = 0.451 \text{ Hz} \]
Beats. RLC Experiment.
Beats. RLC Experiment.

- **RLC Circuit**: Diagram showing R, L, and C in a series circuit.
- **Graph 1**: Time-domain waveform of $V_c(t)$, showing oscillations.
- **Graph 2**: FFT analysis with peaks at 310 kHz and 340 kHz.
- **Graph 3**: Magnitude plot indicating frequency components.

Envelopes and Find peaks annotations.
Beats. Experiment. More complicated case.

In the case of driving frequency $f_d = f_1/N$ where $N$ is integer we can observe more complicated motion of the pendulum.

In the case of driving frequency $f_d = f_1/N$ where $N$ is integer we can observe more complicated motion of the pendulum.

$$\omega_d \sim \omega_0 / 3$$

$$f_0 = 0.4891 \text{Hz}$$

$$f_d = 0.163 \text{ Hz}$$

$$f_0 - f_d$$

$$f_0 + f_d$$

$$f_d = 0.163$$
In the case of driving frequency $f_d = f_1/N$ where $N$ is integer we can observe more complicated motion of the pendulum.

\[ \omega_d \sim 0.5 \omega_0 \]
Detailed analyses show that even if $\phi = \phi_0 \sin(\omega t)$, the driving torque contains several harmonics of $\omega$.

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