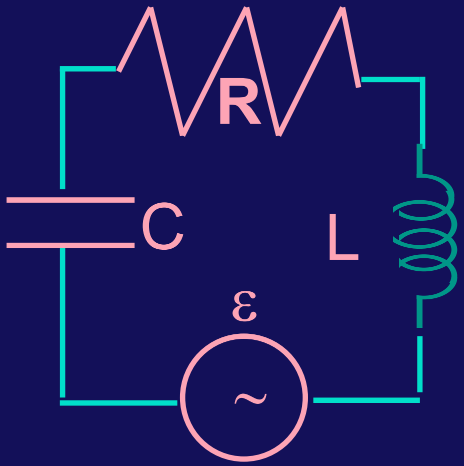


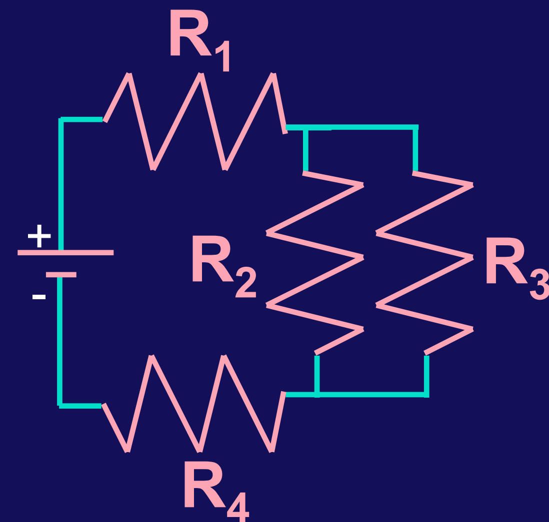
AC Circuits using complex impedances

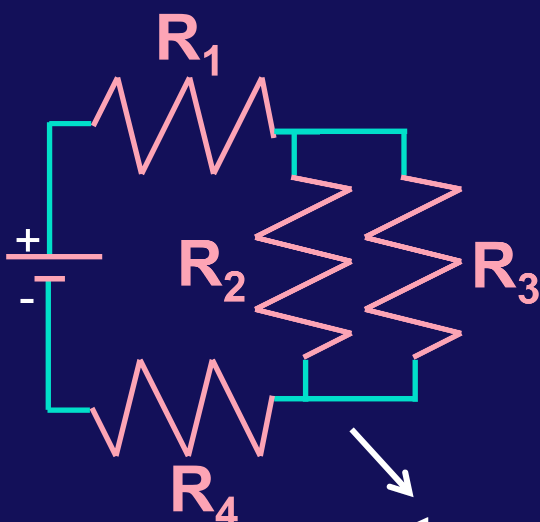
Lecture Outline

- **Special Lecture on Complex Impedance:**
  - $V_{ac}=\varepsilon_o e^{i\omega t}$ $Z_R=R$, $Z_L=i\omega L$, $Z_C=1/i\omega C$
- **Introduction:**
  - resistor networks, equivalent R_{eff}, Kirchoff's Laws
  - purpose of lecture: same at ac for R, C, and L elements
- **Complex Numbers and Functions:**
  - $e^{i\omega t}$ is a rotating 2D *unit* vector in complex plane
  - complex impedances relate current & voltage for Resistors, Capacitors, and (L)inductors.
- **Application to Any Driven LCR Circuit:**
  - series RLC circuit (like in last/next lecture)
  - parallel RLC circuit
  - any network with RLC at any freq.
    - low pass filter

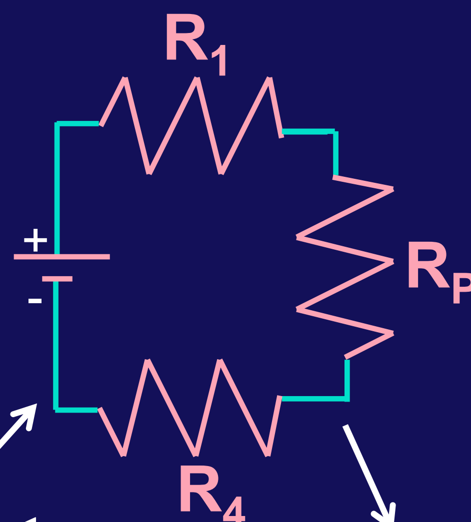
Introduction

- Remember good old dc circuits involving only batteries and resistors?
  - All branch currents proportional to voltage across branch
  - solve this with Kirchoff's voltage & current laws
  - resistor networks expressed as equivalent resistances

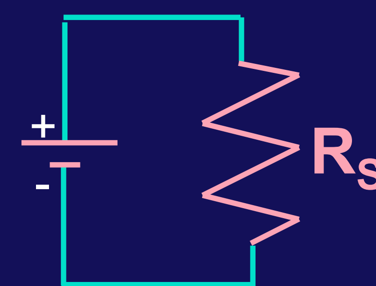




$$\frac{1}{R_P} = \frac{1}{R_2} + \frac{1}{R_3}$$



$$R_S = R_1 + R_P + R_4$$

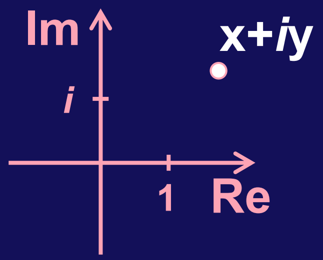


Introduction

- **Problem extending this idea to ac circuits with R, L and C:**
  - **current through R is proportional to voltage across R**
  - **current through L and C not proportional to voltage (using real numbers/functions)**
  - **phase between I and V**
  - **for series RLC circuit phasor algebra gives a clumsy answer**
- **New Idea:**
  - **represent currents and voltages in circuits as complex functions of time**
  - **measured value of voltage or current is real part of complex value you get at the end of the calculation**
  - **current through L or C is proportional to voltage across L or C**
  - **complex impedance plays role of resistance as in dc circuits but *automatically* contain phase shift of voltage *vis a vis* current**
  - **how does this happen -> via rules of complex number algebra!**

complex number algebra

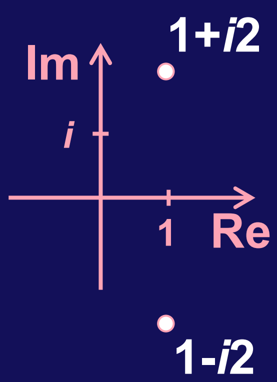
- definition: $i=\sqrt{-1}$
  - so that $i^2=-1$ $i^3=-i$ $i^4=1$



- definition: $z=x+iy$
  - looks like a 2-D vector space (x,y)
  - provides a natural description of phasors
  - length or modulus $|z|=\sqrt{x^2+y^2}=\sqrt{(\mathrm{Re}\,z)^2+(\mathrm{Im}\,z)^2}$
- complex numbers provide solution to algebraic equations that have none with only real numbers

$$z^2-2z+5=0$$

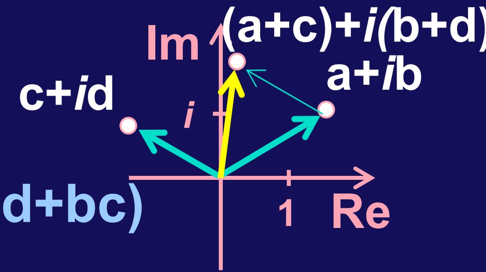
$$z=1\pm\frac{1}{2}\sqrt{4-20}=1\pm\sqrt{-4}=1\pm i2$$



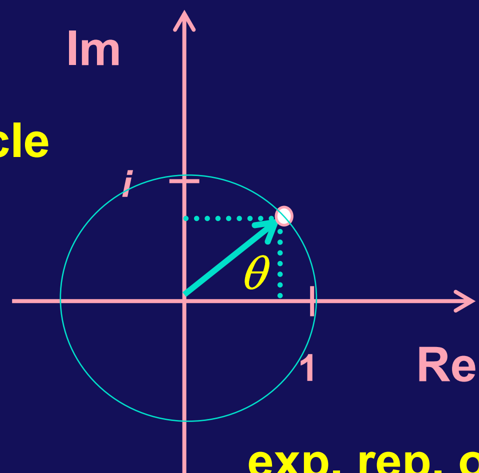
- Next:
  - including trig functions in complex functions -> cos(ωt)

complex number algebra

- **addition:** $(a+ib) + (c+id) = (a+c) + i(b+d)$
  - **so that addition is component-wise or vectorial**



- multiplication: $(a+ib) \times (c+id) = (ac- bd) + i(ad+bc)$
  - **remember i^2=-1**
- Trigonometric functions, the unit circle:
  - **if z = cosθ + $i$sinθ, then length is 1 - unit circle**
  - **exponential representation:**



prove using expansion formulae for exp, sin and cos functions.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

they multiply like exponents:

$$e^{i\theta} \cdot e^{i\phi} = (\cos\theta + i\sin\theta)\cdot(\cos\phi + i\sin\phi)$$
$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$
$$= e^{i(\theta+\phi)}$$

exp. rep. of any point

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$\theta = \tan^{-1}\left(\frac{\mathrm{Im}z}{\mathrm{Re}z}\right)$$

Expanding exponential and sin & cos functions

- Write $e^{i\theta}$ and $\cos\theta + i\sin\theta$ in power series expansion and show they are equal

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n + \cdots$$

$$e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \cdots + \frac{i^n}{n!}\theta^n + \cdots$$

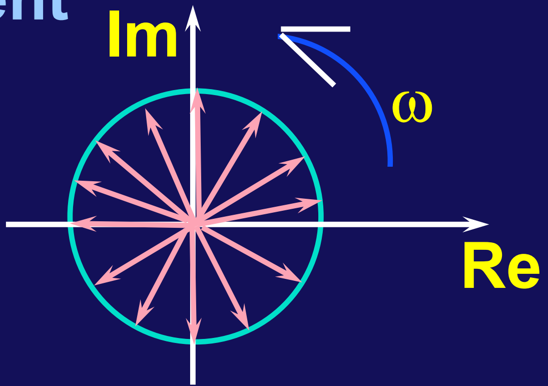
$$e^{i\theta} = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + i\left(\theta - \frac{1}{3!}\theta^3\right) + \cdots$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

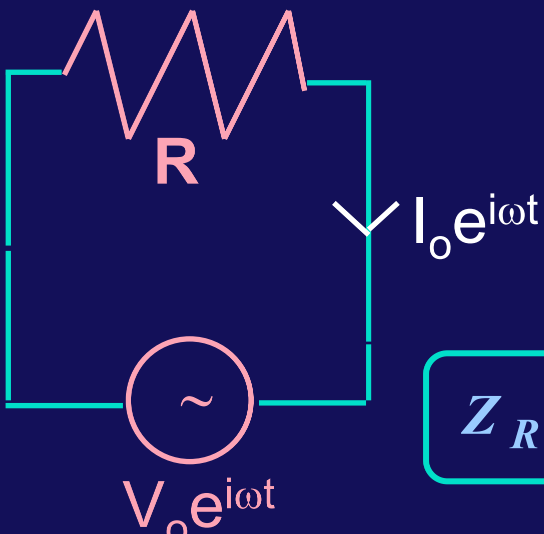
complex number algebra/circuit application

- Our sinusoidal phasor voltage and current functions have convenient exponential representations:

$$V(t) = V_o e^{i\omega t} = V_o(\cos \omega t + i \sin \omega t)$$



Use this in a circuit, first a resistor:



$$Z_R = R$$

In general case, I_o is complex with a magnitude and phase

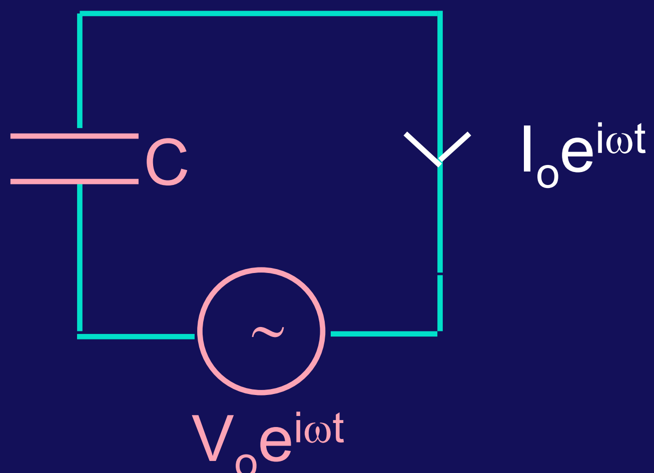
Here, I_o is real with phase = 0, and magnitude = V_o/R

Capacitor and inductor are much more interesting….

Impedances of capacitors and inductors

$$V(t) = V_o e^{i\omega t} = V_o(\cos\omega t + i\sin\omega t)$$

- Consider a capacitor driven by V(t):



In this case, I_o is complex with a magnitude and phase

Recall that Q(t) = CV(t). Since I(t)=dQ/dt, we must differentiate V(t) to get current.

$$I_o e^{i\omega t} \equiv I(t)$$

$$= \frac{d}{dt} C V_o e^{i\omega t} = i\omega C V_o e^{i\omega t}$$

$$I_o \cancel{e^{i\omega t}} = i\omega C V_o \cancel{e^{i\omega t}}$$

$$V_o = I_o \frac{1}{i\omega C}$$

voltage lags current

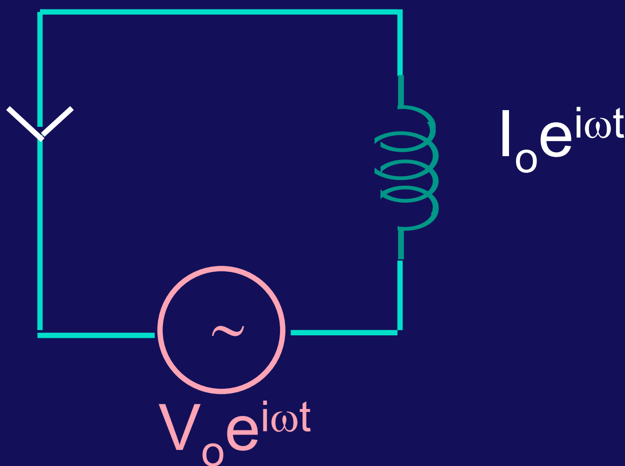
Looks like "resistor": current proportional to voltage, but….

$$Z_C = \frac{1}{i\omega C}$$

Impedances of capacitors and inductors

$$V(t) = V_o e^{i\omega t} = V_o(\cos\omega t + i\sin\omega t)$$

- Consider an inductor driven by V(t):



In this case, too, I_o is complex with a magnitude and phase

Recall that V(t) = LdI/dt. Now we must differentiate I(t) to get voltage.

$$V_o e^{i\omega t} = L\frac{d}{dt} I_o e^{i\omega t}$$

$$V_o \cancel{e^{i\omega t}} = i\omega L I_o \cancel{e^{i\omega t}}$$

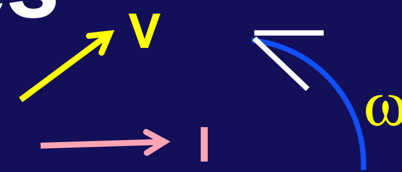
$$V_o = i\omega L I_o$$

voltage leads current

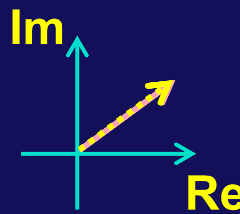
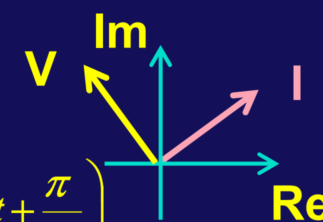
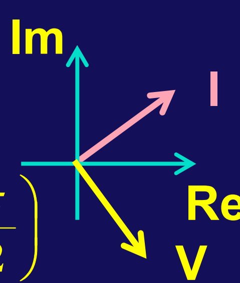
Looks like resistor: current proportional to voltage

$$Z_L = i\omega L$$

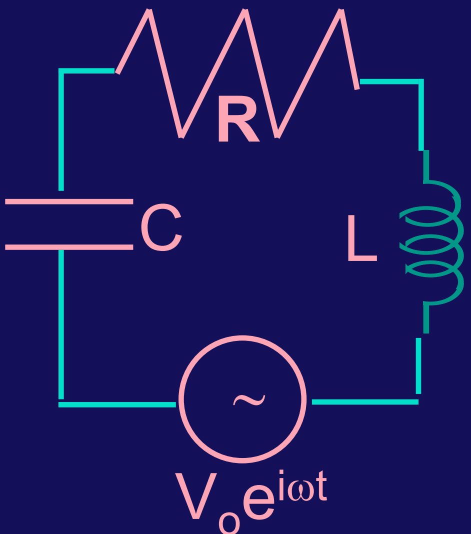
Complex impedances



- **Make a table to show impedance for different circuit elements:**

circuit element	complex impedance, Z	If $I=I_o e^{i\omega t}$ what is V_o	math	phasor picture
R	$Z_R = R$	$V_o = I_o R$		
L	$Z_L = i\omega L$	$V_o = I_o i\omega L$	$i = e^{i\frac{\pi}{2}}$ $V_L \propto e^{i\left(\omega t+\frac{\pi}{2}\right)}$	
C	$Z_C = \frac{1}{i\omega C}$	$V_o = I_o \frac{1}{i\omega C}$	$-i = e^{-i\frac{\pi}{2}}$ $V_C \propto e^{i\left(\omega t-\frac{\pi}{2}\right)}$	

Series LCR Circuit



- **Back to the original problem: the loop equation gave us before:**

$$L\frac{d^2Q}{dt^2}+\frac{Q}{C}+R\frac{dQ}{dt}=V_o\sin\omega t$$

- **Now use complex impedances and assume an ac current of the form $I(t)=I_o e^{i\omega t}$**

- **KVL gives us:**

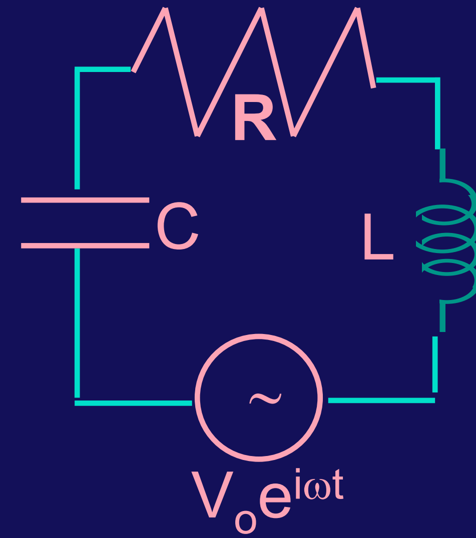
$$V_o=I_o\left(R+\frac{1}{i\omega C}+i\omega L\right)$$

$$V_o=I_o\left(R+iL\left(-\frac{1}{\omega LC}+\omega\right)\right)$$

$$V_o=I_o\left(R+i\frac{L}{\omega}\left(\omega^2-\omega_o{}^2\right)\right)$$

$$\omega_o{}^2=\frac{1}{LC}$$

Series LCR
AC Circuit



- using complex impedances and assuming an ac current of the form $I(t)=I_o e^{i\omega t}$
- KVL gave us: $V_o = I_o \left(R - i \frac{1-\omega^2 LC}{\omega C} \right)$

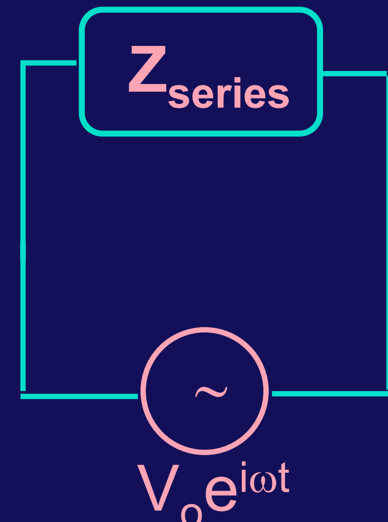
Impedance $Z(\omega)$ is a complex number

$$\phi = \tan^{-1}\left(\frac{\omega^2 LC - 1}{\omega RC} \right)$$

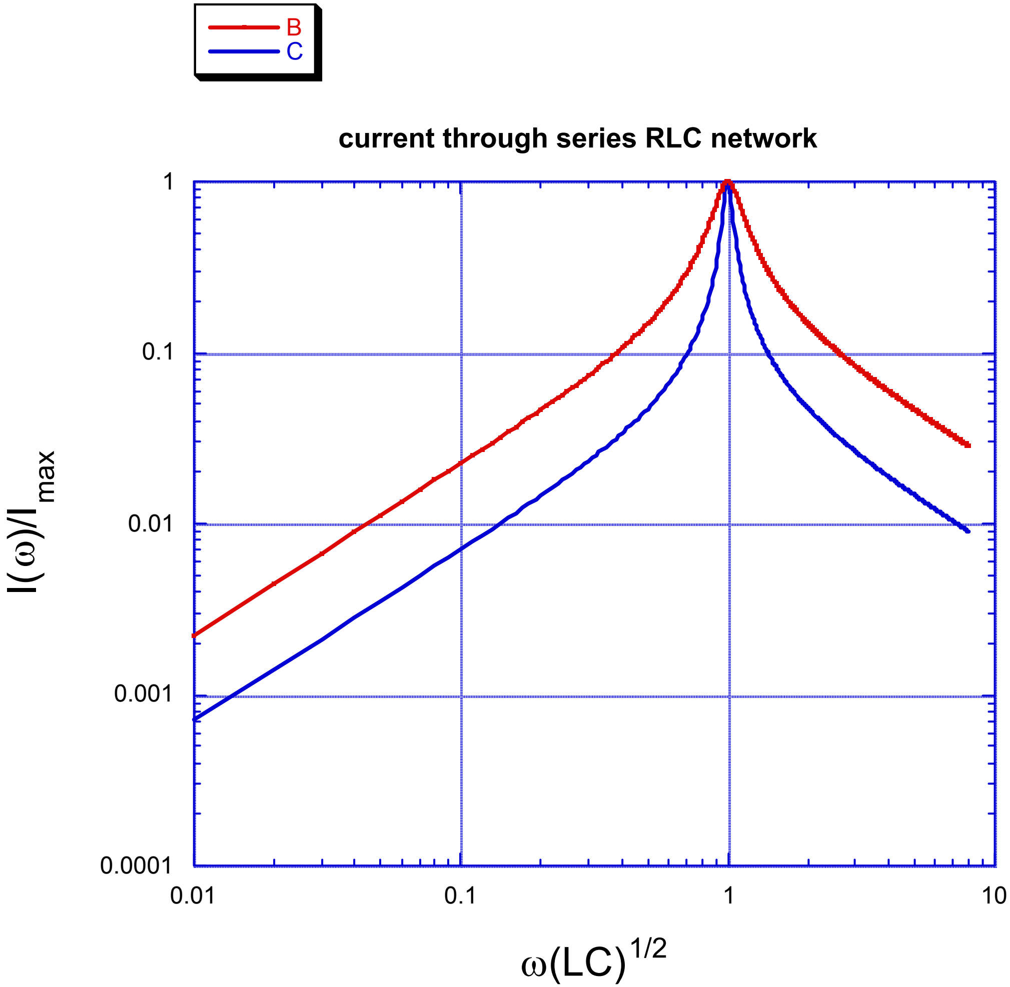
$$Z_{series} = Z \cdot e^{i\phi}$$

$$Z = \sqrt{R^2 + \left(\frac{1-\omega^2 LC}{\omega C} \right)^2}$$

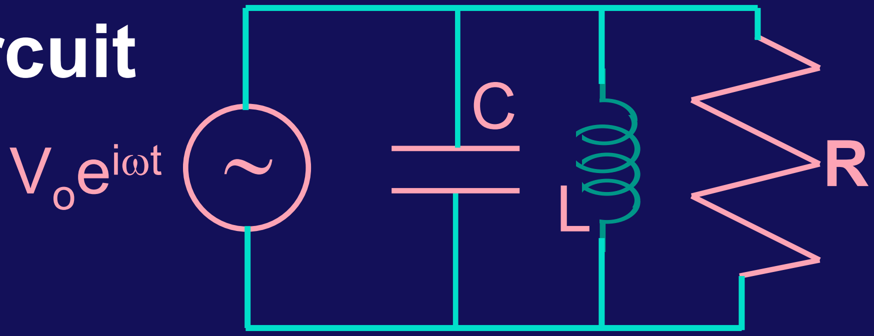
resonance when the numerator -> 0



- On resonance, the current is a maximum.
- If R is very small, the current will tend to go to infinity on resonance


B
C
current through series RLC network
1
0.1
0.01
0.001
0.0001
0.01
0.1
1
10
I(ω)/I_max
ω(LC)^1/2

Parallel LCR AC Circuit



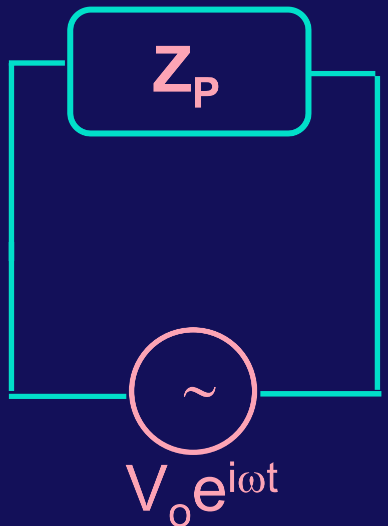
- Now, let's use complex impedances to solve a new problem:

- This time figure out an equivalent impedance from the parallel combination of the capacitor, inductor and resistor

$$\frac{1}{Z_P} = \frac{1}{R} + \frac{1}{i\omega L} + i\omega C$$

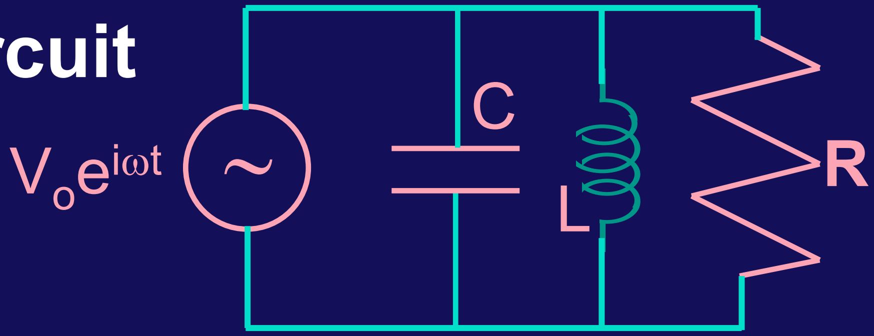
$$I_o = \frac{V_o}{Z_P} = V_o\left(\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)\right)$$

Inverse impedance $1/Z(\omega)$ is a complex number



Inverse impedance shows how current and voltage are related in this problem

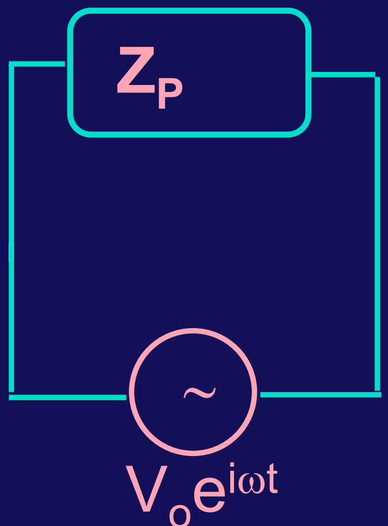
Parallel LCR AC Circuit



- Let's examine the solution from previous slide:

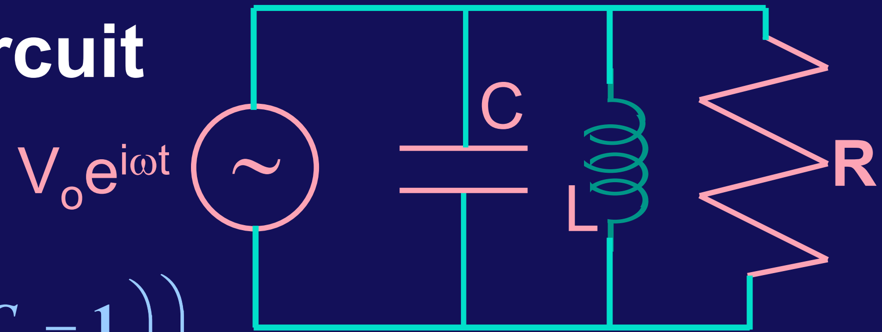
$$I_o = \frac{V_o}{Z_P} = V_o\left(\frac{1}{R} + i\left(\omega C - \frac{1}{\omega L}\right)\right)$$

$$I_o = \frac{V_o}{Z_P} = V_o\left(\frac{1}{R} + i\left(\frac{\omega^2 LC - 1}{\omega L}\right)\right)$$



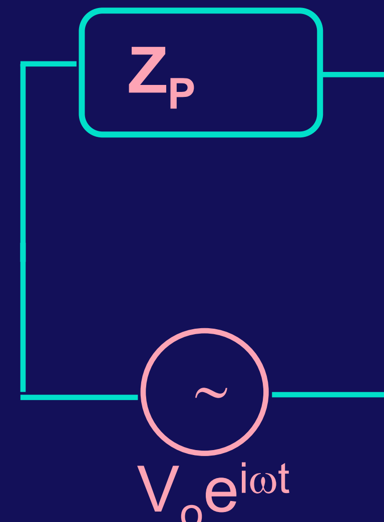
- **As in the series case, there is a resonance at** $\omega = 1/\sqrt{LC}$
- On resonance, the current is a minimum.
- If R is very large, the current will tend to go to zero on resonance

Parallel LCR AC Circuit



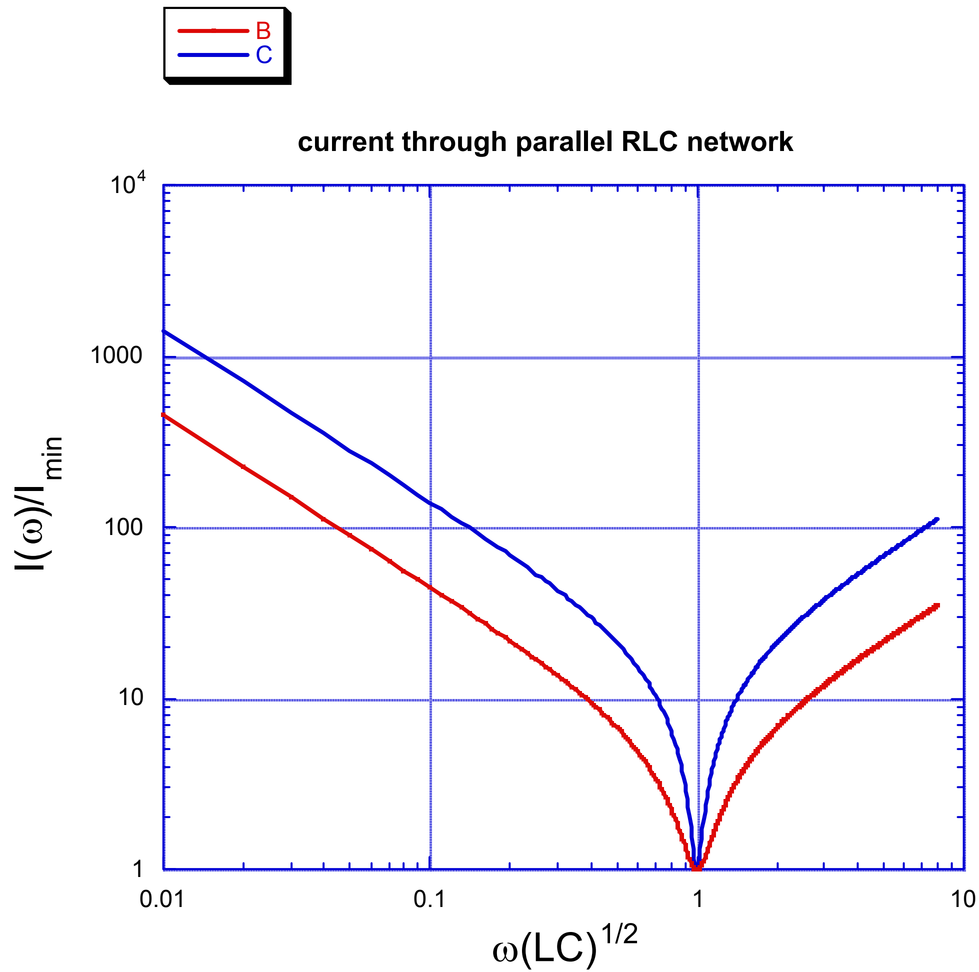
$$I_o = \frac{V_o}{Z_P} = V_o\left(\frac{1}{R} + i\left(\frac{\omega^2 LC - 1}{\omega L}\right)\right)$$

this is 1/Z, a complex function of ω



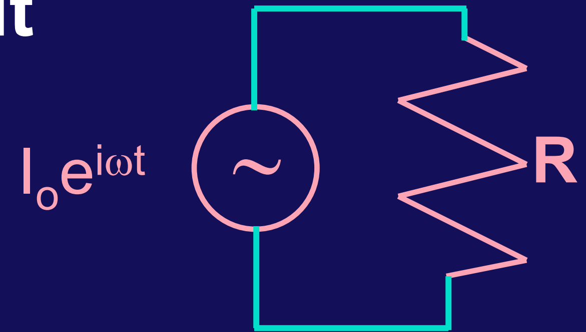
- On resonance, the current is a minimum.
- Magnitude of current is proportional to magnitude of 1/Z

$$\text{mod}\left(\frac{1}{Z}\right) = \frac{1}{R}\sqrt{1 + \frac{\left(\omega^2 LC - 1\right)^2}{\left(\omega \frac{L}{R}\right)^2}}$$

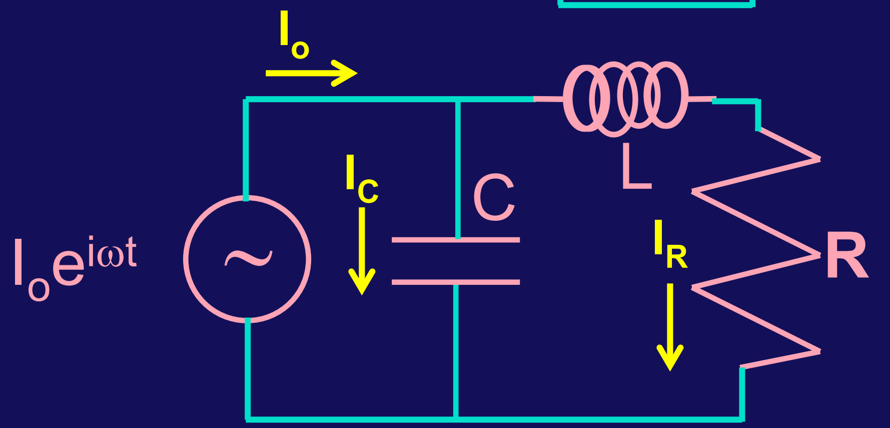

B
C
current through parallel RLC network
10^4
1000
100
10
1
0.01
0.1
1
10
$I(\omega)/I_{min}$
$\omega(LC)^{1/2}$

Low pass filter   AC Circuit

- Suppose we want to limit the current to be at "low frequency" into a resistor from an ac "current source"

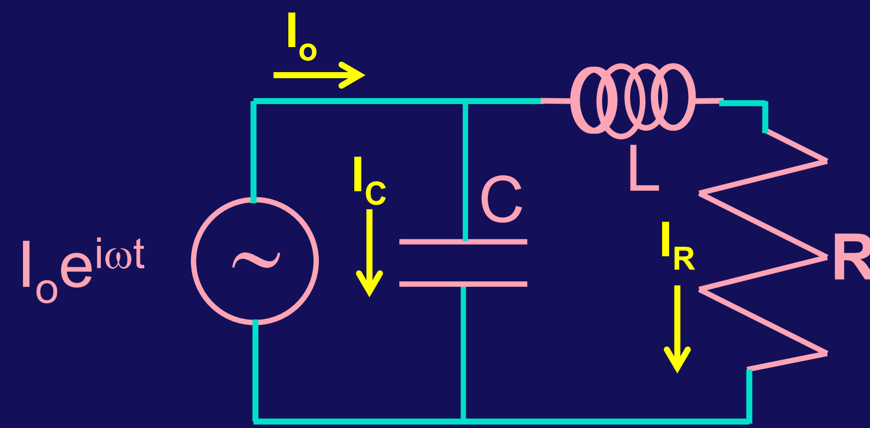


- Design a low pass filter
- Current at low ω should go through R
- Current at high ω should not go through R



- At high frequencies, the inductor will block current: $Z_L = i\omega L$
- At low frequencies, the capacitor will block current: $Z_C = 1/i\omega C$
- Maximize I_R/I_o at low freq. and minimize it at high freq.

Low pass filter   AC Circuit



- **Voltage across C must equal voltage across L+R**

$$I_C \frac{1}{i\omega C} = I_R\left(i\omega L + R\right)$$

- **And current through cap. is total current minus current through R**

$$I_C = I_o - I_R$$

$$I_o \frac{1}{i\omega C} = I_R\left(i\omega L + R + \frac{1}{i\omega C}\right)$$

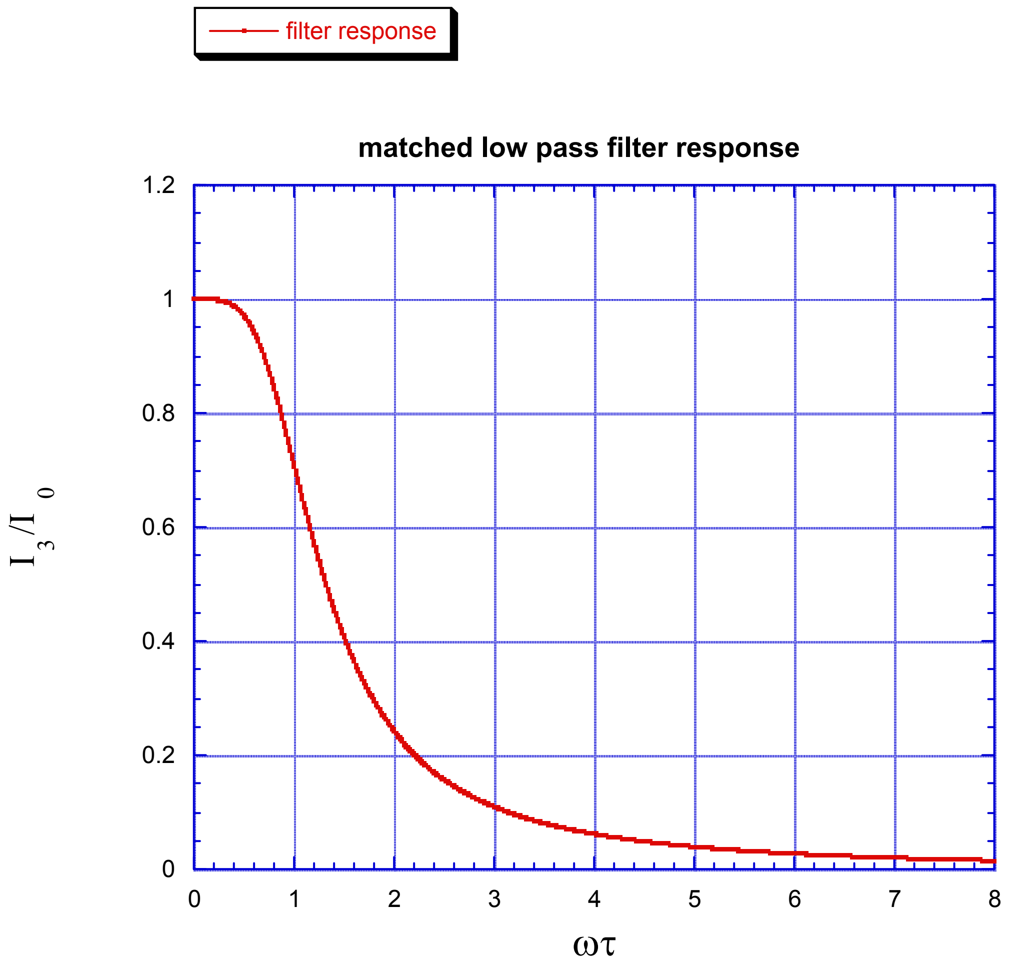
$$I_o = I_R\left(1 - \omega^2 LC + i\omega RC\right)$$

- **Solve for magnitude of I_R/I_o**

$$mag\,\frac{I_R}{I_o} = \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}}$$

- **Choose $C = 2L/R^2$**

$$mag\,\frac{I_R}{I_o} = \frac{1}{\sqrt{1+\left(\omega\tau\right)^4}} \qquad \tau = \sqrt{2}\,\frac{L}{R}$$



filter response
matched low pass filter response
1.2
1
0.8
0.6
0.4
0.2
0
I_3/I_0
0
1
2
3
4
5
6
7
8
ωτ