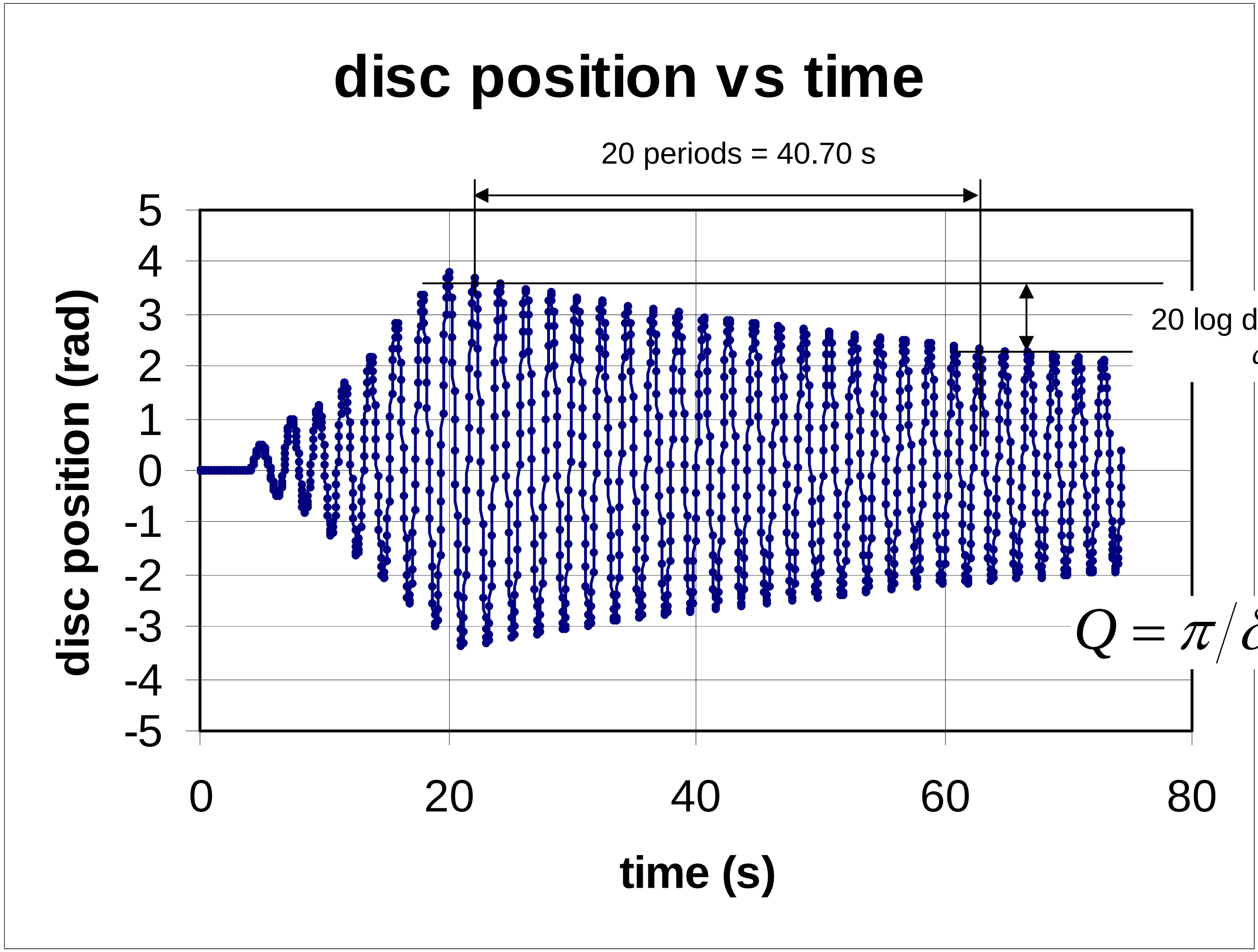
Torsional oscillator transient solution

$I\ddot{\theta} + R\dot{\theta} + K\theta = 0$	Equation of motion
$\theta(t) = Ae^{-at}\cos(\omega_1 t - \phi)$	Transient solution*
$a = R/2I$	Attenuation constant [1/s]
$\omega_o = \sqrt{K/I}$	Natural (angular) frequency [rad/s]
$\omega_1 = \sqrt{\omega_o^2 - a^2}$	Damped (angular) frequency [rad/s]

*Amplitude, A, and phase, ϕ, fixed by initial conditions transient solution data



$$Q = \pi/\delta = \pi/aT$$

$= 140$

Note exponential decay and oscillation

data file 101003_1.xls

Torsional oscillator
steady state solution

$$I\ddot{\theta} + R\dot{\theta} + K\theta = \lambda K\theta_o \cos\omega t$$

Equation of motion

$$\theta_s(t) = B(\omega)\cos\left(\omega t - \beta(\omega)\right)$$

Steady state solution

$$B(\omega) = \frac{\lambda\theta_o\,\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$$

Amplitude function

$$\tan\beta(\omega) = \frac{\omega\gamma}{\omega_o^2 - \omega^2}$$

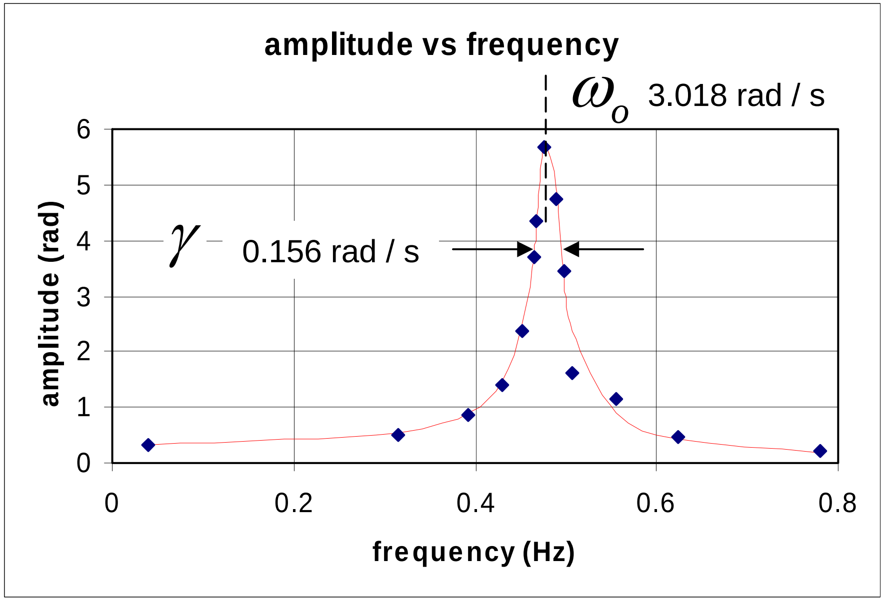
Phase function

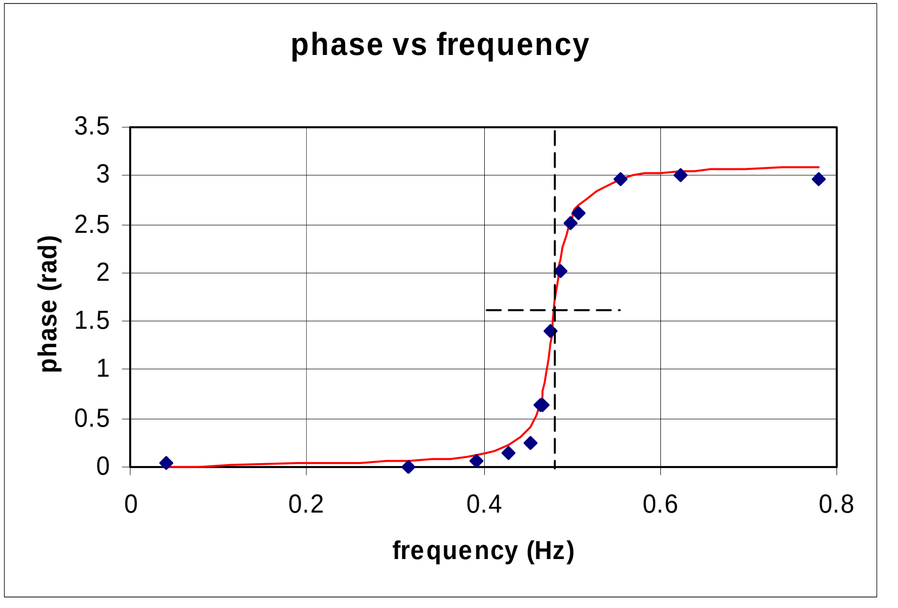
$$\gamma = \frac{R}{I} = 2\frac{R}{2I} = 2a$$

Damping constant

Note link between damping in transient and width of resonance curve

steady-state solution data





$$\omega_1 = \sqrt{\omega_o^2 - (\gamma/2)^2}$$ 3.017 rad / s

$$Q \approx \omega_o / \gamma$$ = 20

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Note $\omega_1 \approx \omega_o$

(Two) Resonance Definitions

$$\frac{1}{\omega_o^2 - \omega^2}$$

Resonance when denominator of phase function goes to zero

$$\tan\beta(\omega) = \frac{\pi}{2} \rightarrow \omega_{resonance} = \omega_o$$

$$\frac{1}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$$

Resonance when denominator of amplitude function goes to minimum

$$B(\omega) = \text{maximum} \rightarrow \omega_{resonance} = \sqrt{\omega_o^2 - \gamma^2/2}$$

if $\gamma \ll \omega_o$ resonance definitions give similar values

Q definition
from energy consideration in transient response, 1

$$Q \equiv 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}}$$

Total energy $$E(t) = KE(t) + PE(t) = \frac{1}{2} I \left[\dot{\theta}(t)\right]^2 + \frac{1}{2} K \left[\theta(t)\right]^2$$

Potential energy $$PE(t) = \frac{1}{2} K \left[\theta(t)\right]^2 = \frac{1}{2} K A^2 e^{-2at} \left[\cos(\omega_1 t - \phi)\right]^2$$

Kinetic energy $$KE(t) = \frac{1}{2} I A^2 e^{-2at} \begin{Bmatrix} \left[a\cos(\omega_1 t - \phi)\right]^2 + \\ \left[\omega_1 \sin(\omega_1 t - \phi)\right]^2 + \\ 2a\omega_1 \sin(\omega_1 t - \phi)\cos(\omega_1 t - \phi) \end{Bmatrix}$$

Q definition
from energy consideration in transient response, 2

$$\langle PE(t)\rangle \approx \frac{1}{2}\times\frac{1}{2}KA^2e^{-2at}$$

$$\langle KE(t)\rangle \approx \langle PE(t)\rangle$$

$$\langle E(t)\rangle \approx \frac{1}{2}KA^2e^{-2at}$$

$$\frac{d}{dt}\langle E(t)\rangle \approx -2a\langle E(t)\rangle$$

$$Q = 2\pi\frac{\langle E(t)\rangle}{\frac{d}{dt}\langle E(t)\rangle\times T_1} = 2\pi\frac{\langle E(t)\rangle}{2a\langle E(t)\rangle\times T_1} = \frac{2\pi f_1}{2a} = \frac{\omega_1}{2a}$$

Equivalent expressions $Q = \frac{\omega_1}{2a}$ and $Q = \frac{\omega_1}{\gamma}$ since $\gamma = 2a$

Alternate Q definition
from energy consideration in transient response

$\frac{Q}{2\pi} =$ Number of cycles for energy to decay by $\frac{1}{e}$

$$\langle E(t) \rangle \approx \frac{1}{2} K A^2 e^{-2at}$$

$$Q = \frac{\omega_1}{2a} \rightarrow 2a = \frac{\omega_1}{Q}$$

$$\langle E(t) \rangle \approx \frac{1}{2} K A^2 e^{-\omega_1 t/Q}$$

$$t_{1/e} = Q/\omega_1 = 2\pi T_1 Q$$

$$\frac{Q}{2\pi} = \frac{t_{1/e}}{T_1}$$

Alternate Q definition
from amplitude consideration in transient response

$$\frac{Q}{\pi} = \text{ Number of cycles for amplitude to decay by } \frac{1}{e}$$

$$\theta(t) = Ae^{-at}\cos(\omega_1 t - \phi)$$

$$Q = \frac{\omega_1}{2a} \rightarrow a = \frac{\omega_1}{2Q}$$

$$\theta(t) = Ae^{-\omega_1 t/2Q}\cos(\omega_1 t - \phi)$$

$$t_{1/e} = 2Q/\omega_1 = \pi T_1 Q$$

$$\frac{Q}{\pi} = \frac{t_{1/e}}{T_1}$$

Alternate Q definition from consideration of resonance curve,1

neglect difference between natural (angular) frequency and damped (angular) frequency

$$Q = \frac{\omega_1}{\gamma} \qquad \omega_1 = \sqrt{\omega_o^2 - \gamma^2/4} \approx \omega_o\left(1 - \gamma^2/8\omega_o^2\right)$$

$$\text{if } \gamma \ll \omega_o \qquad Q \approx \frac{\omega_o}{\gamma}$$

find amplitude function at resonance

$$B(\omega) = \frac{\theta_o\,\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}} \qquad B(\omega_o) = \frac{\theta_o\,\omega_o}{\gamma}$$

Alternate Q definition from consideration of resonance curve, 2

find (angular) frequencies at which amplitude function smaller by a factor of 0.717

$$B(\omega) \to \frac{1}{\sqrt{2}} B(\omega_o)$$

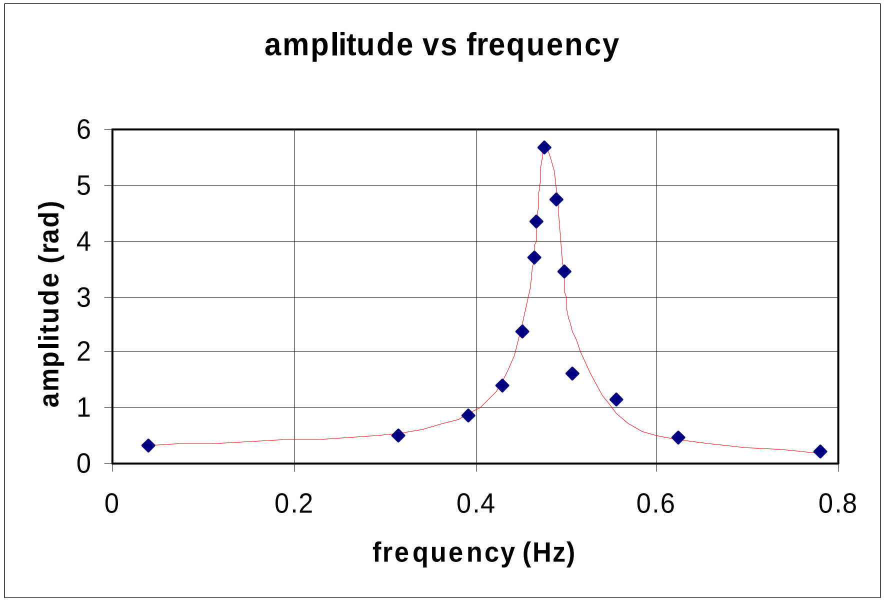
at $\omega_{LHP} = \omega_o - \gamma/2$

and $\omega_{UHP} = \omega_o + \gamma/2$

Define $\Delta\omega = \omega_{UHP} - \omega_{LHP}$

$$\Delta\omega = \gamma$$

$$Q \approx \frac{\omega_o}{\gamma} = \frac{\omega_o}{\Delta\omega} = \frac{f_o}{\Delta f}$$



Relate Q to energy input at resonance

Energy into system $\int \tau_{drive}(t)\, d\theta$ where $\tau_{drive}(t) = \lambda K \theta_o \cos \omega t$

change of variables $\int \tau_{drive}(t)\, d\theta = \int \tau_{drive}(t)\; \dot{\theta}_s(t)\, dt$

$$\theta_s(t) = B(\omega)\cos\left(\omega t - \beta(\omega)\right)$$

Integrate energy input over one cycle

$$\int_0^T \tau_{drive}(t)\; \dot{\theta}_s(t)\, dt = \lambda K \theta_o \omega B(\omega) \sin \beta(\omega) \frac{T}{2}$$

Relate Q to energy input at resonance

At resonance

energy input over one cycle

$$\int_0^T \tau_{drive}(t)\ \dot{\theta}_s(t)\, dt = \lambda K \theta_o \omega_o \frac{\lambda \theta_o \omega_o}{\gamma} \frac{2\pi}{2\omega_o} = \lambda^2 K \theta_o{}^2 \pi Q$$

kinetic plus potential

Stored energy at resonance

$$E = 2 \times \frac{1}{2} K \left[B(\omega_o) \right]^2 = \frac{1}{2} K \lambda^2 \theta_o{}^2 Q^2$$

$$\frac{\text{stored energy}}{\text{energy input per cycle}} = \frac{Q}{2\pi}$$