Frequency domain analysis of linear circuits using synchronous detection

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Frequency domain analysis of linear circuits using synchronous detection

Outline

1. Fourier transform, Discrete Fourier transform. Some properties.
2. Time domain and Frequency domain representation of the data.
3. Frequency domain spectroscopy (FDS)
4. Lock-in amplifiers
5. Practical application of lock-in’s in FDS
in 1822, Jean Baptiste Fourier developed the theory that shows that any real waveform can be represented by the sum of sinusoidal waves.

Let we try to create the square wave as a sum of sine waves of different frequencies

Jean Baptiste Joseph Fourier (1768 – 1830)
$$A_1 \sin(2\pi \omega t)$$

$$A_1 \sin(2\pi \omega t) + A_3 \sin(2\pi 3\omega t + \varphi_3)$$

$$A_1 \sin(2\pi \omega t) + A_3 \sin(2\pi 3\omega t + \varphi_3) + A_5 \sin(2\pi 5\omega t + \varphi_5)$$

$$A_1 \sin(2\pi \omega t) + A_3 \sin(2\pi 3\omega t + \varphi_3) + A_5 \sin(2\pi 5\omega t + \varphi_5) + A_7 \sin(2\pi 7\omega t + \varphi_7)$$
Fourier Transform

The continuous Fourier transformation of the signal \( h(t) \) can be written as:

\[
H(f) = \int_{-\infty}^{+\infty} h(t) e^{2\pi j ft} \, dt; \quad j = \sqrt{-1}
\]

\( H(f) \) represents in frequency domain mode the time domain signal \( h(t) \).

Equation for inverse Fourier transform gives the correspondence of the infinite continues frequency spectra to the corresponding time domain signal.

\[
h(t) = \int_{-\infty}^{+\infty} H(f) e^{-2\pi j ft} \, df
\]

In real life we working with discrete representation of the time domain signal recorded during a finite time.
Discrete Fourier Transform

It comes out that in practice more useful is the representation the frequency domain pattern of the time domain signal $h_k$ as sum of the frequency harmonic calculated as:

$$H_n = H(f_n) = \frac{1}{N} \sum_{k=0}^{N-1} h_k e^{2\pi kn/N}$$

$\Delta$ is the sampling interval, $N$ – number of collected points
Discrete Fourier Transform

For periodic signals with period $T_0$:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T_0} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T_0} \right)$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos \left( \frac{2\pi nt}{T_0} \right) dt; \quad b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin \left( \frac{2\pi nt}{T_0} \right) dt;$$

$$a_0 = \frac{2}{T_0} \int_0^{T_0} F(t) dt;$$
Discrete Fourier Transform

Now how I found the amplitudes of the harmonics to compose the square wave signal from sine waves of different frequencies.

Time domain signal

Decomposition the signal into the sine wave harmonics. The only modulus's of the harmonics amplitudes are presented in this picture.
**Frequency Domain Spectroscopy**  
(linear system)

Applied test signal: $A \sin(\omega t)$

**Studied object**

Response of the studied system: $B_1 \sin(\omega t) + B_2 \cos(\omega t)$

We are applying the sine wave signal to the tested object and measuring the response. Varying the frequency, we can study the frequency properties of the system.
Lock-in amplifier

Now about the most powerful tool which can be used in frequency domain technique.

Signal in \rightarrow Signal amplifier \rightarrow PSD* \rightarrow Low-pass filter \rightarrow DC amplifier \rightarrow output

Reference in \rightarrow VCO** \rightarrow Reference out

*PSD - phase sensitive detector; **VCO - voltage controlled oscillator

John H. Scofield
The DC output signal is a magnitude of the product of the input and reference signals. AC components of output signal are filtered out by the low-pass filter with time constant $\tau$ (her $\tau=RC$)
Lock-in amplifier. Phase shift.

\[ V_{in} = \sin(\omega t + \pi/4) \]

\[ \phi = \pi/4, \ V_{out} = 0.72V_{in} \]
**Lock-in amplifier. Two channels demodulation.**

In many scientific applications it is a great advantage to measure both components \((E_x, E_y)\) of the input signal. We can use two lock-ins to do this or we can measure these value in two steps providing the phase shift of reference signal 0 and \(\pi/4\). Much better solution is to use the lock-in amplifier equipped by two demodulators.

\[
E_{\text{in}} = E_0 \sin(\omega t + \phi) = E_x \sin(\omega t) \cos(\phi) + E_y \cos(\omega t) \sin(\phi)
\]

To \(E_x\) channel

To \(E_y\) channel
Digital Lock-in amplifier

\[ e_{in} \]

Input amplifier

ADC

DSP

DAC

Digital interface

External reference signal

Asin(\(\omega t + \phi\))

Internal Function generator

Analog outputs
Digital Lock-in amplifier

Digital lock-in SR830

Lock-in demo
Digital Lock-in amplifier. SR830

- Channel#1
- Channel#2
- Time constant and output filter
- Sensitivity
- Auto functions
- Inputs
- Notch filter settings
- Analog outputs
- Function generator
- Interface settings
Experiments. Main idea.
Investigating the frequency response of circuit.

$\tilde{V}_{in}(\omega) \xrightarrow{H(\omega)} \tilde{V}_{out}(\omega)$

Frequency domain representation of the system

Response function $\tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$

Linear systems are those that can be modeled by linear differential equations.
Typical application of the lock-in amplifier

- Transmitter (heater)
- Receiver
- He4
- AC drive signal
Example 1. High-pass filter.

\[ \tilde{V}_{in}(\omega) \rightarrow \tilde{V}_{out}(\omega) \]

Applying the Kirchhoff Law to this simple network

\[ \tilde{V}_{out}(\omega) = \tilde{H}(\omega) \ast \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega) \frac{\tilde{Z}2(\omega)}{\tilde{Z}1(\omega) + \tilde{Z}2(\omega)} \]
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ \tilde{V}_{\text{out}}(\omega) = \tilde{H}(\omega) * \tilde{V}_{\text{in}}(\omega) = \tilde{V}_{\text{in}}(\omega) \frac{\tilde{Z}_2(\omega)}{\tilde{Z}_1(\omega) + \tilde{Z}_2(\omega)} \]

**Ideal case**

\[ \tilde{Z}_R = R \]
\[ \tilde{Z}_L = j \omega L \]
\[ \tilde{Z}_C = \frac{1}{j \omega C} = -\frac{j}{\omega C} \]

**More realistic**

\[ \tilde{Z}_R = R + \ldots \]
\[ \tilde{Z}_L = j \omega L + R_L \]
\[ \tilde{Z}_C = \frac{1}{j \omega C} = \frac{1}{j \omega C + R_C^{-1}} \]
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ \tau - \text{time constant of the filter} \]

\[ \omega_c - \text{cutoff frequency} \]

\[ \tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega \tau}{1 + j\omega \tau} = \frac{\omega \tau}{(1 + \omega^2 \tau^2)} (\omega \tau + j); \]

where \( \tau = RC = \omega_c^{-1} \);

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}}; \quad \theta(\omega) = \arctan \left( \frac{H_I(\omega)}{H_R(\omega)} \right) = \arctan \left( \frac{1}{\omega \tau} \right) \]
Experiments. Calculation of the response function in frequency domain mode. High-pass filter
High-pass filter. Fitting

Fitting function:

\[ \tilde{V}_{out} = \tilde{V}_{in} \ast \tilde{H}(\omega) = V_0 \ast \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}}; \quad \tau = RC \]

Fitting parameters: \( V_0, \tau, \text{Voff} \)

Function Body (Dependent Variables: \( y \))

\[ y = V_0 \times 2 \times \pi \times \tau / \sqrt{1 + (2 \times \pi \times \tau)^2} + \text{Voff} \]
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ V_{\text{in}}(\omega) \rightarrow C \rightarrow V_{\text{out}}(\omega) \]

\[ \theta(\omega_c) = \arctan(1) = 45^\circ \]

\[ f_c \approx 398 \text{Hz} \]
Experiments. Calculation of the response function in frequency domain mode. Low-pass filter

\[ \tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega \tau} = \left(\frac{1 - j\omega \tau}{1 + \omega^2 \tau^2}\right); \]

where \( \tau = RC = \omega_c^{-1} \);

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{1}{\sqrt{1 + (\omega \tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = -\arctan(\omega \tau) \]
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

Circuit under the study

\[ \vec{V}_{in}(\omega) \]

FG output

FG

R

C

Lock-in SR830

input

FG output

Setup for measuring of the response function of the Hi-pass filter

- Use internal reference mode
- Do measurements on harmonic no1
- Take care about time constant – should be at least \(~10\) times larger than period of measuring frequency
- Avoid overloading of the lock-in
Setup for measurement of the transfer function of the RLC circuit.

SR830

input

FG output

FG

R

L

C
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

\[ \tilde{H} = \left( \frac{1}{R_0} \right) \frac{R + j\omega L}{1 - \omega^2 LC + j\omega RC} = \]

\[ \left( \frac{1}{R_0} \right) \frac{R + j\omega L}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + j\omega RC} \]

\[ R_0 \gg |z_{RLC}| \]

\[ \omega_o = 2\pi f = \frac{1}{\sqrt{LC}}; \quad Q = \frac{1}{\omega_0 RC} = \frac{1}{R} \sqrt{\frac{L}{C}} \]
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

\[
\tilde{H} = \left( \frac{1}{R_0} \right) \frac{R + j\omega L}{1 - \frac{\omega^2}{\omega_0^2}} + j\omega RC = \left( \frac{R}{R_0} \right) \frac{1 + j\frac{\omega}{\omega_0} Q}{1 - \frac{\omega^2}{\omega_0^2}} + j\frac{\omega}{\omega_0} \frac{1}{Q}
\]

\[
= \left( \frac{R}{R_0} \right) \frac{1 - j\frac{\omega}{\omega_0} \left( \frac{1}{Q} - Q \left( 1 - \frac{\omega^2}{\omega_0^2} \right) \right)}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 \frac{1}{Q^2}}
\]

\[
\omega_0 = 2\pi f = \frac{1}{\sqrt{LC}}; \quad Q = \frac{1}{\omega_0 RC} = \frac{1}{R} \sqrt{\frac{L}{C}}
\]
Another style of the transfer function expressions

\[ V_{out} = \frac{\omega_0^2 L}{R_0} \times \frac{\delta \omega_0^2 + j \omega \left( \omega_0^2 - \omega^2 - \omega \delta \right)}{\left( \omega_0^2 - \omega^2 \right)^2 + \omega^2 \delta^2}; \]

\[ X_{out} = \frac{\omega_0^2 L}{R_0} \times \frac{\delta \omega_0^2}{\left( \omega_0^2 - \omega^2 \right)^2 + \omega^2 \delta^2}; \]

\[ Y_{out} = \frac{\omega_0^2 L}{R_0} \times \frac{j \omega \left( \omega_0^2 - \omega^2 - \omega \delta \right)}{\left( \omega_0^2 - \omega^2 \right)^2 + \omega^2 \delta^2}; \]

\[ \omega_0 = \frac{1}{\sqrt{LC}}; \quad \delta = \frac{R}{L}; \quad Q = \frac{\omega_0}{\delta} \]

Fitting parameters: \( \frac{\omega_0^2 L}{R_0} \) - scaling coefficient, \( \omega_0 \) and \( \delta \)
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

The resonance curves obtained on RLC circuits with two different damping resistors.
Application of the lock-in amplifier for study of the transfer function of the RLC circuit
The example of fitting of the RLC circuit data to the analytical expression could be found in: 
\(\text{engr-file-03\textbackslash PHYINST\textbackslash APL Courses\textbackslash PHYCS401\textbackslash Common\textbackslash Simple Examples\textbackslash Lab 3 Frequency Domain Analysis_example.opjj}\)

**Fitting function**

**Fitting parameters**

![](attachment:Analysis_example.opjj)
From time domain to frequency domain.
Experiment.

Wavetek
Out
Sync

Lock-in SR830
input
Reference in

F(t) – periodic function \( F(t) = F(t + T_0) \):

\[
V = V_0 \begin{cases} 
0 < t \leq \frac{T_0}{2} \\
-V_0 \left( \frac{T_0}{2} < t \leq T_0 \right)
\end{cases}
\]

\[
a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos \left( \frac{2\pi nt}{T_0} \right) dt;
\]

\[
b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin \left( \frac{2\pi nt}{T_0} \right) dt;
\]

\[
a_0 = \frac{2}{T_0} \int_0^{T_0} F(t) dt
\]

Time domain pattern

Frequency domain?
From time domain to frequency domain. Experiment with SR830. Results.

Time domain pattern

Spectrum measured by SR 830 lock-in amplifier

Frequency domain
From time domain to frequency domain. FFT using Origin. Results.

Data file can be used to convert time domain to frequency domain.
From time domain to frequency domain. FFT using Origin. Results.

Time domain taken by Tektronix scope

Spectrum calculated by Origin.

Accuracy is limited because of the limited resolution of the scope
From time domain to frequency domain. Using of the Math option of the scope.

Time domain taken by Tektronix scope

Spectrum calculated by Tektronix scope.

Accuracy is limited because of the limited resolution of the scope.
From time domain to frequency domain. Using of the Math option of the scope.

Spectrum of the square wave signal

Spectrum of the pulse signal
From time domain to frequency domain. Different waveforms. Lock-in.

- Ramp
- Pulse
Appendix #1

Reminder: please submit the reports by e-mail in MsWord or pdf format. Strongly recommend the file name structure as:

L1_lab2_student1

Lab section Lab number Your name
Appendix #2

Origin templates for the Lab are available in:

\`\`\`\engr-file-03\PHYINST\APL Courses\PHYCS401\Common\Origin templates\frequency domain analysis

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**Graph 1:**

- **X-axis:** Frequency (Hz)
- **Y-axis:** Voltage (V)

**Graph 2:**

- **X-axis:** Time (ms)
- **Y-axis:** Voltage (V)

_Time trace: input signal_
Appendix #3

Lab Server access

\"engr-file-03\PHYINST\APL Courses\PHYCS401

http://it.engineering.illinois.edu/user-guides/remote-access/how-do-i-connect-network-drive-or-file-share

If still there are questions – send the question to the help desk: engrit-help@illinois.edu
References:


• You can find a soft copy of this book in:
• \engr-file-03\phyinst\APL Courses\PHYCS401\Experiments