Pulses in transmission lines

Physics 401, Spring 2014
Eugene V. Colla
Transmission lines. Agenda.

- Definition
- Distributed parameters network
- Pulses in transmission line
- Wave equation and wave propagation
- Reflections. Resistive load
- Thévenin's theorem
- Reflection. Non resistive load
- Appendix. Error propagation
Transmission line is a specialized cable designed to carry alternating current of radio frequency, that is, currents with a frequency high enough that its wave nature must be taken into account.

Courtesy Wikipedia
Transmission lines. Distributed parameters network

Simplified equivalent circuit

Ideal case

Real situation

\[ L_{i-1} \quad L_i \quad L_{i+1} \]

\[ C_{i-1} \quad C_i \quad C_{i+1} \]

\[ R_i \quad L_i \]

\[ C_i \quad G_{i+1} \]
Transmission lines. Different types.

Coaxial cable

Courtesy Analog Devices

Twisted line

Courtesy Wikipedia

Twin lead

Courtesy Wikipedia
**Specification:**

**Impedance:** 53 Ω  
**Capacitance:** 83 pF/m  
**Conductor:** Bare Copper Wire (1/1.02mm)
Pulses in transmission line

\[ V(t) \]

\[ V(x,t) \]

\[ r_{\text{out}} \]

\[ i(x,t) \]

\[ v_0 \]

\[ z_L \]

\[ x \]

Forward

Reflected
Setup

**Wavetek 81**

- Sync output
- Signal output

**Tektronix 3012B**

- Triggering input

**RG8U**

**Load**
The Wave Equation

\[ V(t) \]

\[ V(x, t) \]

\[ i(x, t) \]

\[ C = \text{capacitance per unit length} \]

\[ L = \text{inductance per unit length} \]

\[ C \frac{dV}{dx} = -dq; \]

\[ C \frac{\partial V}{\partial t} = -\frac{\partial q}{\partial t} = i; \]

\[ \frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t} \]

\[ dV = -(Ldx) \frac{di}{dt}; \]

\[ \frac{\partial V}{\partial x} = -L \frac{\partial i}{dt} \]
The Wave Equation

\[
\frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t} \tag{1}
\]

\[
\frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t} \tag{2}
\]

Combining (1) and (2)

\[
\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} \\
\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}
\]

\[
\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} \quad \frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2}
\]

Looking for solution

\[
V(x,t) = V_0 \sin \omega \left( t - \frac{x}{v} \right)
\]

\[
i(x,t) = i_0 \sin \omega \left( t - \frac{x}{v} \right)
\]

Now substituting \(V(x,t)\) and \(i(x,t)\) in

\[
\frac{\partial V}{\partial x} = -L \frac{\partial i}{\partial t} \quad \frac{\partial i}{\partial x} = -C \frac{\partial V}{\partial t}
\]

We can find \(V_0 = i_0 \sqrt{\frac{L}{C}}\) or

\[
V(x,t) = \sqrt{\frac{L}{C}} i(x,t) = Z_k i(x,t)
\]

\[v = \frac{1}{\sqrt{LC}}\]

Speed of wave propagation

\(Z_k\) - characteristic Impedance

Equivalent to Ohm's law equation
Characteristic impedance

\[ Z_k = \sqrt{\frac{L}{C}} \]

\(C = \text{capacitance per unit length}\)

\(L = \text{inductance per unit length}\)

Cross-section of the coaxial cable

\[ C = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln\left(\frac{D}{d}\right)} \quad (\text{F/m}) \]

\[ L = \frac{\mu_0 \mu_r}{2\pi} \ln\left(\frac{D}{d}\right) \quad (\text{H/m}) \]

\(\varepsilon_r = \text{dielectric permittivity}\)

\(\mu_r = \text{magnetic permeability} \approx 1\)

\[ Z_k = \frac{138}{\sqrt{\varepsilon_r}} \log\left(\frac{D}{d}\right) \quad (\text{Ohms}) \]

\(\varepsilon_0 = 8.854 \times 10^{-12} \quad (\text{F/m})\)

\(\mu_0 = 4\pi \times 10^{-7} \quad (\text{H/m})\)

Finally for coaxial cable:
Speed of wave propagation, delay.

**For polyethylene $\varepsilon_r \approx 2.25$ (up to 1GHz)**

\[ v = \frac{1}{\sqrt{LC}} \]

**Speed of wave propagation**

\[ v = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} \approx \frac{c}{\sqrt{\varepsilon_r}} \]

\[ \tau = \frac{1}{v} \approx \frac{3.336 \times 10^{-9}}{\sqrt{\varepsilon_r}} \text{ (s/m)} = 3.336 \sqrt{\varepsilon_r} \text{ (ns/m)} \]

**Delay time**

**RG-8/U:**
- Inner Insulation Materials: Polyethylene
- Nominal Impedance: 52 ohm

**RG58U:**
- Delay time $\approx 5$ ns/m
Reflection in transmission line

- Solution for the traveling in opposite direction

\[ V(x,t) = V_0 \sin \omega \left( t - \frac{x}{v} \right) \]
\[ i(x,t) = i_0 \sin \omega \left( t - \frac{x}{v} \right) \]

For reflected wave \( V_r = -Z_k i_r \)

At any point of the transmission line:

1. Resistive load \( Z_L = R_L \)

\[ \frac{V}{i} = R_L \]

\[ \frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k} \quad \text{or} \quad V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i \]

\[ V = V_r + V_i \]
\[ i = i_r + i_i = \frac{V_i}{Z_k} - \frac{V_r}{Z_k} \]
Resistive load $Z_L = R_L$

Open line $R_L = \infty \implies V_r = V_i$ and $V = V_i + V_r$

![Graph showing incident and reflected pulses]

**Graph Equations:**

$$\frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k}$$

or

$$V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i$$
Reflection in transmission line. Loses.

**Theory:** $R_L = \infty \rightarrow V_r = V_i$

### Experiment RG 58U

<table>
<thead>
<tr>
<th>MHz</th>
<th>30</th>
<th>50</th>
<th>100</th>
<th>146</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>RG-58U</td>
<td>2.5</td>
<td>4.1</td>
<td>5.3</td>
<td>6.1</td>
<td>6.1</td>
</tr>
</tbody>
</table>

\[
ATTN\,(db) = 20\log\left(\frac{V_i}{V_r}\right)
\]

Important parameter for cable is attenuation per length.
In our case: \[ \text{Attn}(200 \text{ ft}) = 20 \log \left( \frac{4.18}{3.54} \right) \approx 1.46 dB \]
Different cables loses.

\[ Attn(200 \text{ft}) = 20 \log \left( \frac{4.18}{3.54} \right) \approx 1.46 \text{dB} \]

\[ > \]

\[ Attn(200 \text{ft}) = 20 \log \left( \frac{3.932}{3.78} \right) \approx 0.335 \text{dB} \]

Core \( \phi = 0.81 \text{ mm} \)

Dielectric \( \phi = 2.9 \text{ mm} \)

Core \( \phi = 2.17 \text{ mm} \)

Dielectric \( \phi = 7.2 \text{ mm} \)
Loses. Frequency dispersion.

Reflected pulse does not follow the shape of the incident pulse.

Frequency dependence of the attenuation RG-58U cable.
Loses. Frequency dispersion.

FFT

Spectrum correction

IFFT

Incident pulse spectrum
reflected pulse spectrum

2/10/2014
Spring 2014
Reflection in transmission line

Resistive load $Z_L = R_L$

Shorted line $R_L = 0$ $\rightarrow V_r = -V_i$

\[
\frac{V_i + V_r}{V_i - V_r} = \frac{R_L}{Z_k} \quad \text{or} \quad V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i
\]
Resistive load $Z_L = R_L$

Matching the load impedance $R_L \rightarrow Z_k$; $V_r \rightarrow 0$

$V_i + V_r = \frac{R_L}{Z_k}$ or $V_r = \frac{R_L - Z_k}{R_L + Z_k} V_i$

Incident pulse

Reflected pulse

End of the line
Thévenin's theorem

Any combination of batteries and resistances with two terminals can be replaced by a single voltage source $e$ and a single series resistor $r$. 

Hermann Ludwig Ferdinand von Helmholtz (1821-1894)

Léon Charles Thévenin (1857–1926)
From this equivalent equation we can find the maximum possible power delivered to $R_L$

$$P = i^2 R_L = \frac{(2V_i)^2}{(R_L + Z_k)^2} R_L$$

$P = P_{\text{max}}$ if $R_L = Z_k$ (no reflection)
This experiment better to perform on RG 8U cable because of lower attenuation. 

\[ RL = \infty, \text{ amplitude of the pulse at the end of line is expected to be } 2Vi, \text{ where } Vi \text{ is the amplitude of the incident pulse} \]
Reflection. Inductive load.

\[ i = \frac{2V_i}{Z_L + Z_k} \]

\[ 2V_i = iZ_k - L \frac{di}{dt}; \]

\[ i = i_0 \left( 1 - \exp \left( -\frac{t}{\tau} \right) \right); \]

\[ \tau = \frac{L}{Z_k} \]

\[ V_r = V_L - V_i \]
\[ i = \frac{2V_i}{Z_L + Z_k} \]

\[ 2V_i = iZ_k - L \frac{di}{dt} \]

\[ i = i_0 \left( 1 - \exp \left( -\frac{t}{\tau} \right) \right) \]

\[ \tau = \frac{L}{Z_k} \]

\[ \tau \approx 50\text{ns}, \quad L = \tau Z_k \approx 2.5\mu H \]
\[ i = \frac{2V_i}{Z_L + Z_k} \]

\[ \tau = Z_k C \]

\[ C = \frac{\tau}{Z_k} \approx 3.2 nF \]
Appendix #1. Export graphs from Origin

![Diagram of Origin software export graphs function]

- Click on "Export Graphs" to open the export dialog.
- Select the desired file format from the dropdown menu, such as Encapsulated Postscript (".eps").
- Choose additional settings as needed, such as "No Preview" or "Auto Preview".
- Click "Preview" to display updated preview and "Finish" to export.
Appendix #2. Reminders

1. The reports should be uploaded to the *proper* folder and *only* to the proper folder.

   For example folder *RLC Lab Report_L1* should be used by students from L1 section only.

   I would recommend the file name style as:

   **L1_lab2_student1**

   *Lab section*    *Lab number*    *Your name*

   You do not need to submit two copies in pdf and in MsWord formats.

2. Origin template for this week Lab:

   \engr-file-03\phyinst\APL Courses\PHYCS401\Common\Origin templates\Transmission line\Time trace.otp
Appendix #3.  LCR resonance circuit. Fitting results.

Data courtesy Tsung-Lin Hsieh. Physics 401, Fall 2011.
Appendix #4.

**Error propagation.**

\[ y = f(x_1, x_2 \ldots x_n) \]

\[ \Delta f(x_i, \Delta x_i) = \sqrt{\sum_{i=1}^{n} \left[ \frac{\partial f}{\partial x_i} \right]^2} \cdot \Delta x_i^2 \]
Derive resonance frequency $f$ from measured inductance $L \pm \Delta L$ and capacitance $C \pm \Delta C$

$$f_0(L, C) = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$L_1 = 10 \pm 1\text{mH}, \quad C_1 = 10 \pm 2\mu\text{F}$

$$\Delta f(L, C, \Delta L, \Delta C) = \sqrt{\left[\frac{\partial f}{\partial L}\right]^2 \cdot \Delta L^2 + \left[\frac{\partial f}{\partial C}\right]^2 \Delta C^2}$$

$$\frac{\partial f}{\partial L} = -\frac{1}{4\pi} C\frac{1}{2} L^{-\frac{3}{2}};$$
$$\frac{\partial f}{\partial C} = -\frac{1}{4\pi} L^2 C^{-\frac{3}{2}}$$

Results:
$$f(L_1, C_1) = 503.29212104487\text{Hz}$$
$$\Delta f = 56.26977\text{Hz}$$

$$f(L_1, C_1) = 503 \pm 56\text{Hz}$$
$L_1 = 10 \pm 1\text{mH}, \quad C_1 = 10 \pm 1\text{μF}$  

Where these numbers are coming from?

1. Using commercial resistors, capacitors, inductances...

   \[ C = 500\text{pF} \pm 5\% \]
   \[ L = 35\text{mH} \pm 10\% \]

2. Measuring the parameters using standard equipment

   **SENCORE “Z” meter model LC53**
   Capacitance measuring accuracy $\pm 5\%$
   Inductance measuring accuracy $\pm 2\%$

   **Agilent E4980A Precision LCR Meter**
   Basic accuracy $\pm 0.05\%$
Origin uses the **Levenberg–Marquardt** algorithm for nonlinear fitting.

From experiment you have the array \((x_i, y_i)\) of independent and dependent variables: \(x_i\) (e.g. f- frequency) and \(y_i\) (e.g. magnitude of the signal) and you have optimize the vector of fitting parameters \(\beta\) of your model function \(f(x, \beta)\) in order to minimize the sum of squares of deviations:

\[
S(\beta) = \sum_{i=1}^{m} [y_i - f(x_i, \beta)]^2
\]

Important point is the choice of fitting parameters. In some cases the algorithm will work with \(\beta=(1,1...1)\), but in many situations the choice of more realistic parameters will lead to solution.

For details go to: [http://en.wikipedia.org/wiki/Levenberg%E2%80%93Marquardt_algorithm](http://en.wikipedia.org/wiki/Levenberg%E2%80%93Marquardt_algorithm)