

Experiment 2: Transients and Oscillations in RLC Circuits

Will Chemelewski

Partner: Brian Enders

TA: Nielsen

See laboratory book #1 pages 5-7, data taken September 1, 2009

September 7, 2009

Abstract

Transient responses of RLC circuits are examined when subjected to both long time scale (relative to the decay time) square wave voltages and sinusoidally varying voltages over a range of frequencies about the resonant frequency. In general, a good correspondence is found between theory - describing the charge in the system in terms of a 2nd order differential equation with a harmonic oscillator form - and experiment. It is demonstrated that the inductance can be accurately measured using period of oscillation versus capacitance measurements. Furthermore, the exponential decay of the response is described well by the model and the resonant frequency of a sinusoidal external voltage is accurately predicted. However, some discrepancies were found though not necessarily a result of theoretical failures. One problem is the failure to predict the inductance of a circuit based on the critical resistance variation with capacitance, although the problem could lie in how the measurement is conducted. Additionally, the quality and bandwidth of a RLC element is poorly predicted but this could also be a result of experimental problems.

1 Purpose

The purpose of this experiment was to observe and measure the transient response of RLC circuits to external voltages. We measured the time varying voltage across the capacitor in a RLC loop when an external voltage was applied. The capacitance was varied and the periods of the oscillations were used to determine the inductance in the circuit. Next we measured the log decrement as a function of resistance to verify the response is approximately linear and to estimate the total resistance of the circuit including the inductor and the function generator. Following that we determined the resistance required for critical damping as a function of capacitance. Using this we verified the theoretical result that the critical resistance is proportional to $1/\sqrt{C}$. Finally, we measured the voltage across the capacitor in a different RLC circuit driven by a sinusoidally varying voltage. The peak-to-peak voltage was measured as a function

of frequency to determine the resonant frequency, the bandwidth, and the quality factor Q . We also compared the resonant frequency with the theoretical value.

2 Theory

The governing equation for a resistor, inductor and capacitor in series with a voltage source is

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V(t) \quad (1)$$

This is the equation for an oscillator with damping and a driving function. Solving the characteristic equation gives two roots $s_1, s_2 = a \pm b$ with

$$a = \frac{R}{2L} \quad (2)$$

$$b = \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} \quad (3)$$

There are three distinct types of solutions depending on whether b^2 is positive, negative or zero.

When $b^2 = 0$ the circuit is said to be critically damped. In this case the two roots of the characteristic equation are real and the same value. Therefore, the charge in the capacitor falls to zero exponentially and quicker than for any other value of b .

When $b^2 > 0$ the two roots of the characteristic equation are real and again the charge drops to zero in an exponential fashion. However, the falloff of charge is slower than for $b^2 = 0$. This can be seen by observing that at later times the decay constant is $(a - b) < a$. With this type of response the circuit is said to be over-damped.

Finally, when $b^2 < 0$ the two roots are imaginary and thus the charge oscillates about 0 before finally decaying to 0 (assuming $a \neq 0$). Under these conditions the circuit is said to be under-damped. The frequency of the oscillation is

$$f_1 = \frac{1}{T_1} = \left(\frac{1}{2\pi}\right) \sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2} \quad (4)$$

where T_1 is the period of the oscillation. Because the charge is still decaying the logarithmic decrement δ can be defined as the natural log of the ratio between two successive peaks of charge

$$\delta = \ln \frac{q(t_{max})}{q(t_{max} + T_1)} = aT_1 \quad (5)$$

The last equality in equation 5 can be made based on the form of $q(t)$ where e^{-at} is the attenuating factor. Lastly, the quality factor Q of the circuit is defined as

$$Q = 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}} = \frac{\pi}{\delta} = \frac{\omega_1 L}{R} \quad (6)$$

with $\omega_1 = 2\pi f_1$ being the damped angular frequency. As can be seen, a lower resistance leads to a higher quality while a higher inductance increases Q .

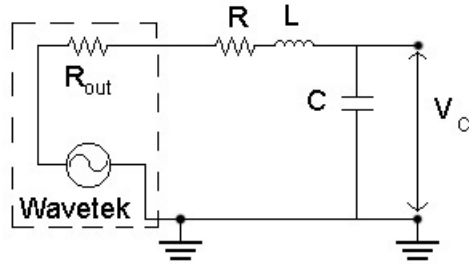


Figure 1: Circuit layout for parts A, B, and C.

3 Experiment

3.1 Equipment

A decade capacitor and decade resistor were used in parts A-C. The serial number for the capacitor was A-1678 and for the resistor was 48288. A HP 34401A Digital Multimeter (DMM) with serial number Phys-943034 was used to measure the resistance of the inductor coil #7. A Wavetek function generator was used in all parts and had the serial number Phys-943026. Additionally a Tektronix TDS3012B oscilloscope with serial number F28819 was used to measure the transient response in all parts.

3.2 Part A: Frequency dependence on capacitance

In this section, along with parts B and C, the Wavetek and circuit were connected as shown in Figure 1. V_C was measured using the Tektronix scope and R_{out} was assumed to be 50Ω . R in the figure was a decade resistance box and C was a decade capacitance box. L was a large coil inductor (#7) which we measured the resistance of using the DMM before turning on the Wavetek. The resistance was found to be 19.36Ω . The Wavetek was set to output a 8 V unipolar square waveform with a frequency of 10 Hz and a duty cycle of 50%. The oscilloscope was set to trigger on the leading edge of the unipolar output of the Wavetek.

After all the elements had been setup, the resistance box was set to 0Ω and it was verified that the period was constant for each oscillation as shown in Table 1. Additionally a representative scope output is shown in Figure 2.

Table 1: Verification of constant period.

Cycle #	Period (ms)
1	$1.86 \pm .02$
2	$1.86 \pm .02$
3	$1.86 \pm .02$

Then we measured the period of oscillation for different capacitances by determining the time to complete multiple cycles and dividing the result by the number of cycles

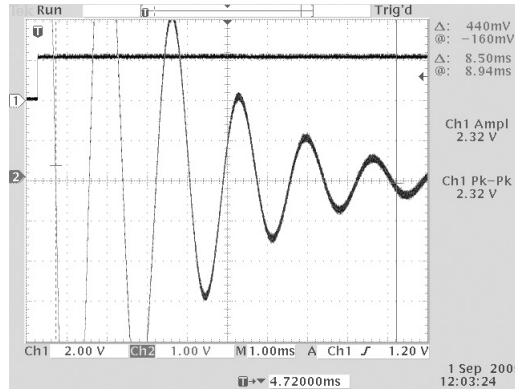


Figure 2: Representative transient signal of voltage across capacitor in this part of the experiment.

used. This was done for 11 different capacitances over the range of $0.05\text{-}1\ \mu\text{F}$. The results are shown in Table 2. The error for the total time was taken to be the resolution

Table 2: Measurement of period of oscillation as a function of capacitance.

$C\ (\mu\text{F})$	# of periods	Total time (ms)	Single period (ms)
1	5	$9.40\pm.08$	$1.88\pm.02$
0.9	5	$8.86\pm.04$	$1.77\pm.01$
0.8	4	$6.72\pm.04$	$1.68\pm.01$
0.7	5	$7.82\pm.04$	$1.56\pm.01$
0.6	6	$8.64\pm.04$	$1.44\pm.01$
0.5	5	$6.62\pm.04$	$1.32\pm.01$
0.4	6	$7.08\pm.04$	$1.18\pm.01$
0.3	7	$7.14\pm.04$	$1.02\pm.01$
0.2	9	$7.70\pm.04$	$0.856\pm.004$
0.1	13	$7.64\pm.04$	$0.588\pm.004$
0.05	9	$3.75\pm.02$	$0.417\pm.002$

of horizontal axis of the Tektronix multiplied by 2. We assumed that this large range gives a 95% confidence in the value. Then the error of the single period was just the error of the total time divided by the number of periods for that measurement.

According to equation 4 a plot of $1/T^2$ versus $1/C$ should give a straight line with a slope

$$m = \frac{1}{4\pi^2 L} \quad (7)$$

Figure 3 shows this relationship along with the best fit line. The error bars were calculated using the propagation of errors formula:

$$Error = \sqrt{\frac{4}{T^6} T_{error}^2} \quad (8)$$

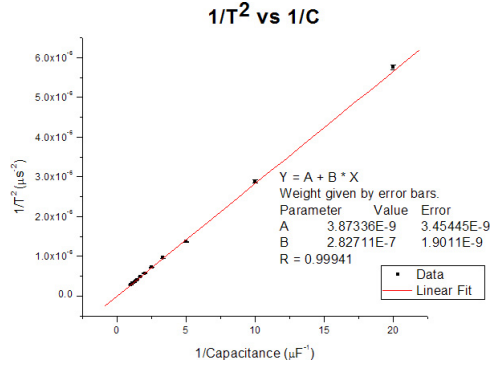


Figure 3: Plot of $1/T^2$ versus $1/C$ showing a straight line as predicted by theory.

In this plot the 'Error' column refers to the standard error and it can be seen that the R value indicates a strong linear relationship. Using the relationship in equation 7 it was determined that the inductance of the circuit was 90 ± 1 mH to 95% confidence¹.

Based on this value for the inductance it is determined that for $C = 1 \mu F$ and $R = 70 \Omega$

$$\left(\frac{1}{LC}\right) = 1.11 \times 10^7 \gg 1.51 \times 10^5 = \left(\frac{R}{2L}\right)^2 \quad (9)$$

and therefore, according to equation 4

$$T^2 \approx (2\pi)^2 LC \quad (10)$$

This relationship is plotted in Figure 4². Again, the R value shows a strong linear relationship between the values as predicted and the error in the slope is only 0.9% with a 95% confidence interval.

Finally, the inductance of the coil was measured using a Z-meter and found to be 92 ± 1 mH. This is in good agreement with the value determined from the $1/T^2$ versus $1/C$ relationship.

3.3 Part B: Log decrement dependence on resistance

For this section of the experiment the same circuit and waveform was used as in part A. The capacitance was set to $1 \mu F$. First we observed the voltage transient at $R = 0, 50, 100, 150, 200, 250,$ and 300Ω . Above 100Ω it became difficult to see more than 2 oscillations. Next we verified that the log decrement (δ) did not change going from one set of peaks to the next. We did this by setting the resistance to 0Ω and measuring the height of the first 4 peaks allowing 3 successive δ s to be calculated. The results are summarized in Table 3 with the errors in voltage determined by multiplying the minimum change by 2.³ The error becomes worse as the peak number increases

¹Error determined using $\frac{1}{4\pi^2 m^2} 1.812 * m_{error}$

²Error bars determined using $2TT_{error}$

³Error for δ calculated using $\sqrt{1/V_1^2 * V_{1error} + 1/V_2^2 * V_{2error}}$

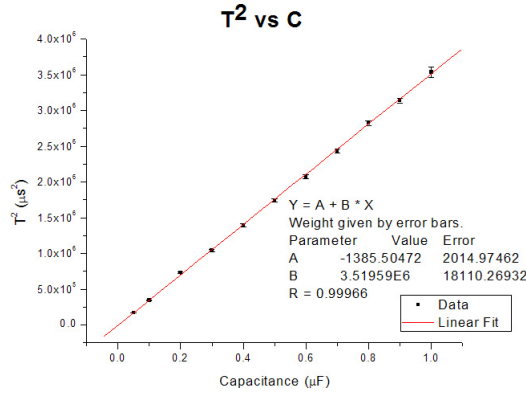


Figure 4: Plot of T^2 versus C showing a straight line as predicted by theory.

Table 3: Verification of constant δ

Peak #	Voltage (V)	δ
1	$7.48 \pm .04$	N/A
2	$3.44 \pm .04$	0.78 ± 0.01
3	$1.56 \pm .04$	0.79 ± 0.03
4	$0.76 \pm .04$	0.72 ± 0.06

because the data was only acquired once with a set vertical resolution on the scope, and thus the uncertainty as a percentage of the value increased as well.

With the consistency of δ established we proceeded to measure δ as a function of R . The first and second peaks were measured and used to determine δ . This data appears in Table 4. Again the error was determined by multiplying the smallest change in V on the scope by 2. Plugging equations 2 and 4 to equation 5 gives

$$\delta = \left(\frac{R}{2L} \right) \frac{2\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L} \right)^2}} = \pi \frac{R}{L} \sqrt{LC} \frac{1}{\sqrt{1 - \frac{R^2C}{4L}}} \approx \pi \sqrt{\frac{C}{L}} R \quad (11)$$

when $R^2C/4L \ll 1$. In this experiment, the maximum value of $R^2C/4L$ is 0.08 and can fairly safely be ignored to test for linearity. Figure 5 shows δ versus R and a strong linear relationship is observed. Based on the fitting parameters the x intercept of the line is not $R = 0 \Omega$. This is because R is referring only to the resistance of the decade box and ignores the resistance of the coil and the function generator output. This residual resistance is calculated to be $70 \pm 4 \Omega$ with 95% confidence. The combined resistance of the inductor coil with the stated output resistance of the Wavetek is 69.4Ω giving a good agreement with the above measurement.

Table 4: δ versus resistance.

R (Ω)	V1 (V)	V2 (V)	δ
100	-6.28 \pm .04	-1.00 \pm .04	1.84 \pm .04
90	-6.72 \pm .04	-1.24 \pm .04	1.69 \pm .03
80	-7.16 \pm .04	-1.46 \pm .04	1.59 \pm .03
70	-7.56 \pm .04	-1.70 \pm .04	1.49 \pm .02
60	-8.00 \pm .04	-2.00 \pm .04	1.39 \pm .02
50	-8.44 \pm .04	-2.36 \pm .04	1.27 \pm .02
40	-8.96 \pm .08	-2.80 \pm .08	1.16 \pm .03
30	-9.44 \pm .08	-3.28 \pm .08	1.06 \pm .03
20	-10.1 \pm .2	-3.88 \pm .08	0.96 \pm .03
10	-10.6 \pm .2	-4.52 \pm .08	0.85 \pm .03
0	-11.1 \pm .2	-5.24 \pm .08	0.75 \pm .02

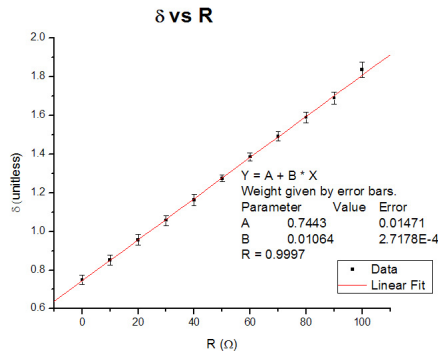


Figure 5: Graph of δ versus R showing a good fit and match to theory.

3.4 Part C: Dependence of resistance for critical damping on capacitance

For this part of the experiment the same circuit and waveform is used as in part B. The critical resistance is the resistance at which the circuit is critically damped. Therefore $b^2 = 0$ and by solving equation 3 we find

$$R_{critical} = 2\sqrt{\frac{L}{C}} \quad (12)$$

So a plot of R vs $1/\sqrt{C}$ should be linear. We determined the critical resistance by adjusting the decade resistance box and observing the resulting waveform. Three example waveforms are shown in Figure 6. To find where the circuit was critically damped we allowed it to be slightly under-damped and then increased the resistance until no overshoot was observed. We did this for 5 different capacitances and our results are summarized in Table 5. Unfortunately we did not have one definitive way to determine the error in our measurement. However, when the resistances were adjusted by

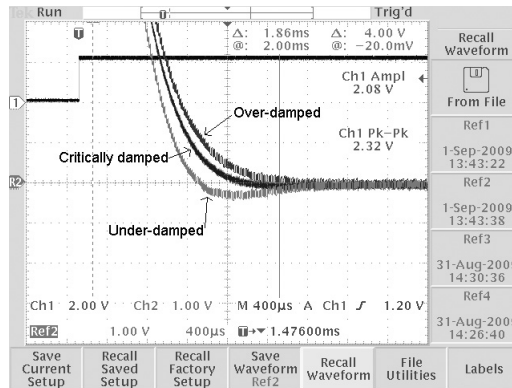


Figure 6: Sample circuit response for this part of the experiment

Table 5: $R_{critical}$ versus capacitance.

C (μF)	$R_{critical}$ (Ω)
1	460 ± 10
0.5	678 ± 10
0.1	1620 ± 30
0.05	2320 ± 30
0.01	5300 ± 100

the error values a change in the response was observed that we believe was significant enough to warrant a high degree of confidence (assumed to be 95%) in that range.

A plot of these critical resistances versus $1/\sqrt{C}$ is shown in Figure 7. Additionally, the ideal behavior is plotted using $L = 90$ mH. The first result to notice is that the linear fit is very strong and that the error in the slope is only 2.3% with 95% confidence. However, it is also clear that the plot implies a different value for the inductance of the circuit than previously found. Based on the slope of the fit, the inductance of the circuit is 72 ± 3 mH with 95% confidence⁴. This gives a 20% difference and based on this it would seem all of our measurements of $R_{critical}$ were too low by at least 15%. However, when the resistance is adjusted upwards that much it is readily apparent that the circuit has moved into being over-damped based on the waveform. Therefore, it must be concluded that this is not a reliable way to measure an inductance at least partially due to the subjectivity of the measurement.

3.5 Part D: RLC response to sinusoidal signal

For this section a different circuit was used than in parts A, B, and C. The circuit is shown in Figure 8. R_M was 101 k Ω , L was 93 mH, R_B was 5.69 Ω , and C was 1.03 μF . Also, the Wavetek was set to output a sine wave with an amplitude of 16 V.

⁴ $L = B^2/4$ and the error is $\frac{B}{2} 1.812 B_{error}$

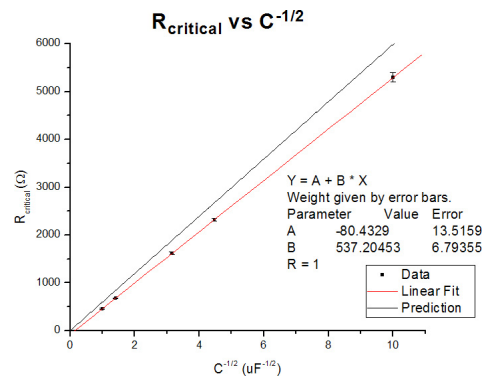


Figure 7: Plot of $R_{critical}$ versus $1/\sqrt{C}$ showing a strong linear relationship. Additionally, a plot of the theoretical behavior given the measured inductance is shown.

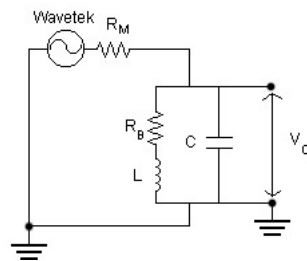


Figure 8: Circuit layout for part D

When analyzing AC circuits it is convenient to use impedances to determine various aspects of the response. For example, with this circuit, the impedance can be used to determine the magnitude of the voltage across the RLC element as

$$V_C = V_0 \frac{R}{R_M + Z} \quad (13)$$

Impedances are complex numbers in general and for a capacitor $Z = 1/j\omega C$, for an inductor $Z = j\omega L$, and for a resistor $Z = R$ where $j = \sqrt{-1}$. As can be seen in Figure 8 the RLC loop has the capacitor in parallel with a series addition of the resistor and inductor. Using the addition rules for circuits the result is that

$$|Z| = \frac{\sqrt{R^2 + (\omega L)^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + (\omega RC)^2}} \quad (14)$$

where ω_0 is the resonant frequency.

With this in mind we set out to observe this response by measuring the peak to peak voltage across the capacitor as a function of frequency. We observed frequencies between 100 and 1000 Hz and set the oscilloscope to average 16 samples and measure the amplitude automatically. The results are shown in Table 6. These values are then

Table 6: Voltage versus frequency. Errors for all values are ± 2 mV.

Frequency (Hz)	Vpp (mV)	Frequency (Hz)	Vpp (mV)	Frequency (Hz)	Vpp (mV)
100	59	450	142	600	138
200	72	480	149	630	129
270	86	490	150	650	123
300	94	500	151	680	114
330	103	510	152	690	112
340	106	515	150	700	110
350	109	520	152	710	107
360	113	530	150	720	105
370	117	540	150	770	92
380	121	550	149	800	85
400	126	560	147	900	71
410	130	580	143	1000	59
430	137				

plotted along with equation 14 in Figure 9. Equation 14 is important because R_M is so large that the peak to peak voltage (V_{pp}) should approximately equal $V_0 Z / R_M$ and thus is directly related V_{pp} . Using this plot we found the resonant frequency to be 515 ± 10 Hz. Based on the measurements of the inductor, resistor, and capacitor the theoretical resonance occurs at 514 Hz which is in good agreement with our data. We also measured the bandwidth by finding the two frequencies at which V_{pp} fell to 70.7% of its peak value. These two frequencies were 350 ± 10 Hz and 710 ± 10 Hz giving a

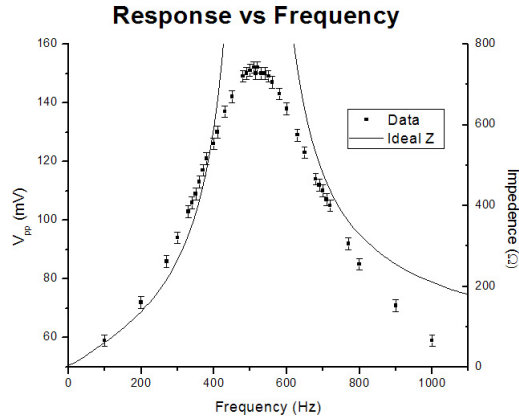


Figure 9: Peak to peak voltage across capacitor as a function of frequency. The data points go to the left axis and the Ideal Z goes with the right axis.

bandwidth of $360 \pm 15 \text{ Hz}^5$. Then, using $Q = f_0/\Delta f$ it was found that the quality of the circuit was 1.4 ± 0.1 .

The theoretical Q is 52.8 and it is therefore quite obvious why the theoretical impedance curve is much sharper than the measured curve. The plot shows a poor agreement between the theoretical response and the actual results. The peak is much sharper for the ideal case and the bandwidth is much lower as seen in a comparison of Q values. These discrepancies are due to components not behaving ideally and other losses in the circuit such as resistance in the wires or imperfect connections on the bread board. Additionally, the actual voltage falls off faster than expected for the same reasons. Building on this idea, if we assume that the inductance and capacitance of the circuit are the same as measured and fit the function to the data by changing the resistance in equation 14, we find the best fit around $R = 202 \Omega$. This fit is shown in Figure 10. It can be seen that this does shift the resonance from where measured (to 485 Hz) but the Q factor is the same as in the experiment. So while not entirely responsible for the difference between measurement and theory, as evidenced by the resonant frequency difference, uncounted resistances play a part in the disparity.

4 Conclusions

RLC circuit behavior is described well by oscillating system theory which describes the charge on the capacitor as a 2nd order differential equation. It is demonstrated in this experiment that the theory can be used to reliably find the inductance of a RLC loop by measuring the period of oscillation as a function of capacitance. The predicted exponential damping dependence on resistance is also verified and it is shown that that total resistance of the circuit can be found accurately using a resistance versus log decrement plot. However, when applying the theory to measure the inductance using

⁵Error calculated using $\sqrt{10^2 + 10^2}$

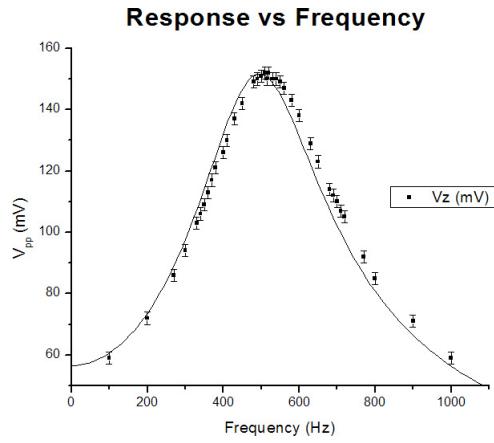


Figure 10: Peak to peak voltage across capacitor as a function of frequency with theoretical voltage fit to data with $R = 202 \Omega$.

the critical resistance as a function of capacitance there is a problem. But the issue is most likely due to the nature of the measurement as it is difficult to tell exactly when the circuit transitions from under-damped to over-damped. Finally, the model accurately predicts the resonant frequency of a RLC circuit, however, other details of the behavior such as the Q factor are not described correctly in this experiment.