Frequency Domain Analysis of linear Circuits Using Synchronous Detection

Physics 401, Spring 2015.
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Frequency Domain Analysis of Linear Circuits Using Synchronous Detection

Outline

2. Time domain and Frequency domain representation of the data.
3. Frequency domain spectroscopy (FDS)
4. Lock-in amplifiers
5. Practical application of lock-in’s in FDS
in 1822, Jean Baptiste Fourier developed the theory that shows that any real waveform can be represented by the sum of sinusoidal waves.

Let we try to create the square wave as a sum of sine waves of different frequencies

Square wave. F=40Hz, A=1.5V

Jean Baptiste Joseph Fourier
(1768 – 1830)
\[ A_1 \sin(2\pi\omega t) \]

\[ A_1 \sin(2\pi\omega t) + A_3 \sin(2\pi3\omega t + \varphi_3) \]

\[ A_1 \sin(2\pi\omega t) + A_3 \sin(2\pi3\omega t + \varphi_3) + A_5 \sin(2\pi5\omega t + \varphi_5) \]

\[ A_1 \sin(2\pi\omega t) + A_3 \sin(2\pi3\omega t + \varphi_3) + A_5 \sin(2\pi5\omega t + \varphi_5) + A_7 \sin(2\pi7\omega t + \varphi_7) \]
The continues Fourier transformation of the signal $h(t)$ can be written as:

$$H(f) = \int_{-\infty}^{+\infty} h(t)e^{2\pi jft} dt; \quad j=\sqrt{-1}$$

$H(f)$ represents in frequency domain mode the time domain signal $h(t)$.

Equation for inverse Fourier transform gives the correspondence of the infinite continues frequency spectra to the corresponding time domain signal.

$$h(t) = \int_{-\infty}^{+\infty} H(f)e^{-2\pi jft} df$$

In real life we working with discrete representation of the time domain signal recorded during a finite time.
It comes out that in practice more useful is the representation the frequency domain pattern of the time domain signal $h_k$ as sum of the frequency harmonic calculated as:

$$H_n = H(f_n) = \frac{1}{N} \sum_{k=0}^{N-1} h_k e^{2\pi kn/N}$$

$\Delta$ is the sampling interval, $N$ – number of collected points
Discrete Fourier Transform

For periodic signals with period $T_0$:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right)$$

$$a_n = \frac{2}{T_0} \int_{0}^{T_0} F(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt; \quad b_n = \frac{2}{T_0} \int_{0}^{T_0} F(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt;$$

$$a_0 = \frac{2}{T_0} \int_{0}^{T_0} F(t) dt;$$
Discrete Fourier Transform

Now how I found the amplitudes of the harmonics to compose the square wave signal from sine waves of different frequencies.

Time domain signal

Decomposition the signal into the sine wave harmonics. The only modulus's of the harmonics amplitudes are presented in this picture.
Frequency Domain Spectroscopy
(linear system)

Applied test signal

\[ A \sin(\omega t) \]

\textbf{Studied object}

Response of the studied system

\[ B_1 \sin(\omega t) + B_2 \cos(\omega t) \]

We applying the sine wave signal to tested object and measuring the response. Varying the frequency we can study the frequency properties of the system.
Now about the most powerful tool which can be used in frequency domain technique.

John H. Scofield
The DC output signal is a magnitude of the product of the input and reference signals. AC components of output signal are filtered out by the low-pass filter with time constant $\tau$ (here $\tau = RC$).
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

- Measuring equipment (DMM, lock-in etc.)
- AC
- Result as DC voltage corresponding $U_{\text{AMP}}$, $U_{\text{RMS}}$ ...

Diagram:
- Circuit diagram with AC input $U(t)$, filter circuit with $R$, $C$
- Signal and reference connections
- 1 and 2 numbered areas
We need to measure the amplitude/rms value of the sine wave
Lock-in amplifier. What is the advantage of using synchronous detecting?

"Noisy" sine wave

$U_{DC} = 0.64208\text{V}$

compare to

$U_{DC} = 0.63643\text{V}$
Lock-in amplifier. What is the advantage of using synchronous detecting?

Clear sine wave – no “noise”

“Noisy” sine wave

$U_{DC} = 0.63643V$
Lock-in amplifier. Phase shift.

\[ V_0 \sin(\omega t + \phi) \]

\( \phi = \pi/4, \ V_{out} = 0.72V_{in} \)

\[ V_{in} = \sin(\omega t + \pi/4) \]

\[ V_{in} \]

Reference

Output

Time (msec)
Lock-in amplifier. Two channels demodulation.

In many scientific applications it is a great advantage to measure both components \((E_x, E_y)\) of the input signal. We can use two lock-ins to do this or we can measure these value in two steps providing the phase shift of reference signal 0 and \(\pi/4\). Much better solution is to use the lock-in amplifier equipped by two demodulators.

\[
E_{\text{in}} = E_0 \sin(\omega t + \phi)
\]

\[
\begin{align*}
\sin(\omega t) & \quad \text{to } E_x \text{ channel} \\
\cos(\omega t) & \quad \text{to } E_y \text{ channel}
\end{align*}
\]
Digital Lock-in amplifier

- ADC
- DSP
- DAC

Input amplifier

Input: $e_{in}$

External reference signal: $A\sin(\omega t + \phi)$

Internal Function generator

Digital interface

Analog outputs
SR830. Digital Lock-in amplifier

In SR830 manual you can find the chapter dedicated to general description of the lock-in amplifier idea.

WHAT IS A LOCK-IN AMPLIFIER?

Lock-in amplifiers are used to detect and measure very small AC signals - all the way down to a few nanovolts! Accurate measurements may be made even when the small signal is obscured by noise experiment at the reference frequency. In the diagram below, the reference signal is a square wave at frequency $\omega_r$. This might be the sync output from a function generator. If the sine output from
SR830. Digital Lock-in amplifier
Digital Lock-in amplifier. SR830

- Time constant
- And output filter
- Sensitivity
- Channel#1
- Channel#2
- Auto functions
- Inputs
- Notch filter settings
- Analog outputs
- Function generator
- Interface settings
Experiments. Main idea.
Investigating the frequency response of circuit.

\[ \tilde{V}_{in}(\omega) \xrightarrow{H(\omega)} \tilde{V}_{out}(\omega) \]

Frequency domain representation of the system

Response function \[ \tilde{H}(\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} \]

Linear systems are those that can be modeled by linear differential equations.
Typical application of the lock-in amplifier

He4

Receiver

AC drive signal

Transmitter (heater)
Experiments. Main idea.
Calculation of the response function in frequency domain mode.

Example 1. High-pass filter.

\[ V_{\text{in}}(\omega) \quad \text{C} \quad V_{\text{out}}(\omega) \]

\[ V_{\text{in}}(\omega) \quad R \quad V_{\text{out}}(\omega) \]

Applying the Kirchhoff Law to this simple network

\[ V_{\text{out}}(\omega) = \tilde{H}(\omega) * \tilde{V}_{\text{in}}(\omega) = \tilde{V}_{\text{in}}(\omega) \frac{\tilde{Z}_2(\omega)}{\tilde{Z}_1(\omega) + \tilde{Z}_2(\omega)} \]
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ \tilde{V}_{\text{out}}(\omega) = \tilde{H}(\omega) * \tilde{V}_{\text{in}}(\omega) = \tilde{V}_{\text{in}}(\omega) \frac{\tilde{Z}(2)(\omega)}{\tilde{Z}(1)(\omega) + \tilde{Z}(2)(\omega)} \]

- **Ideal case**
  \[ \tilde{Z}_R = R \]
  \[ \tilde{Z}_L = j\omega L \]
  \[ \tilde{Z}_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} \]

- **More realistic**
  \[ \tilde{Z}_R = R + ... \]
  \[ \tilde{Z}_L = j\omega L + R_L \]
  \[ \tilde{Z}_C = \frac{1}{j\omega C} = \frac{1}{j\omega C + R_C^{-1}} \]
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ \tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega \tau}{1 + j\omega \tau} = \frac{\omega \tau}{1 + \omega^2 \tau^2}(\omega \tau + j); \]

where \( \tau = RC = \omega_c^{-1}; \)

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}}; \quad \theta(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = \arctan\left(\frac{1}{\omega \tau}\right) \]

\( \tau \) – time constant of the filter

\( \omega_c \) – cutoff frequency
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ V_{\text{out}}(\omega) = \frac{V_{0}}{\sqrt{2}} \]

\[ f_c \approx 398 \text{Hz} \]

\[ H(\omega) = \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}} \]

\[ f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi \tau} = \frac{1}{2\pi RC} \]
High-pass filter. Fitting

Fitting parameters: $V_0$, $\tau$, $V_{\text{off}}$

Fitting function

$$\tilde{V}_{\text{out}} = |\tilde{V}_{\text{in}}| \cdot |\tilde{H}(\omega)| = V_0 \cdot \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}}; \quad \tau = RC$$

Function Body (Dependent Variables: $y$)

$$y = V_0 \times 2 \pi \times \tau / \sqrt{1 + (2 \pi \tau)^2} + V_{\text{off}}$$
Experiments. Calculation of the response function in frequency domain mode. High-pass filter

\[ \theta(\omega_c) = \arctan(1) = 45^\circ \]

\[ f_c \approx 398 \text{ Hz} \]
Experiments. Calculation of the response function in frequency domain mode. Low-pass filter

\[ \tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega \tau} = \frac{1}{1 + \omega^2 \tau^2} ; \]

where \( \tau = RC = \omega_c^{-1} ; \)

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{1}{\sqrt{1 + (\omega \tau)^2}} ; \quad \theta(\omega) = \arctan \left( \frac{H_I(\omega)}{H_R(\omega)} \right) = -\arctan (\omega \tau) \]
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

• Use internal reference mode
• Do measurements on harmonic no1
• Take care about time constant – should be at least ~10 times larger than period of measuring frequency
• Avoid overloading of the lock-in

Setup for measuring of the response function of the Hi-pass filter
Setup for measurement of the transfer function of the RLC circuit.
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

\[ \tilde{H} = \left( \frac{1}{R_0} \right) \frac{R + j \omega L}{1 - \omega^2 LC + j \omega RC} = \frac{1}{R_0} \left( \frac{R + j \omega L}{1 - \frac{\omega^2}{\omega_0^2}} \right) + j \omega RC \]

\( R_0 \gg |Z_{RLC}| \)

\( \omega_0 = 2\pi f = \frac{1}{\sqrt{LC}} \); \( Q = \frac{1}{\omega_0 RC} = \frac{1}{R \sqrt{C}} \)
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

\[ \tilde{H} = \left( \frac{1}{R_0} \right) \frac{R + j\omega L}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right) + j\omega RC} = \left( \frac{R}{R_0} \right) \frac{1 + j\frac{\omega}{\omega_0} Q}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right) + j\frac{\omega}{\omega_0} \frac{1}{Q}} \]

\[ = \left( \frac{R}{R_0} \right) \frac{1 - j\frac{\omega}{\omega_0} \left( \frac{1}{Q} - Q \left( 1 - \frac{\omega^2}{\omega_0^2} \right) \right)}{\left( 1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2 \frac{1}{Q^2}} \]

\[ \omega_0 = 2\pi f = \frac{1}{\sqrt{LC}}; \quad Q = \frac{1}{\omega_0 RC} = \frac{1}{R} \sqrt{\frac{L}{C}} \]
Another style of the transfer function expressions

\[ V_{\text{out}} = \frac{\omega_0^2 L}{R_0} \times \frac{\delta \omega_0^2 + j \omega (\omega_0^2 - \omega^2 - \delta^2)}{\left( \omega_0^2 - \omega^2 \right)^2 + \omega^2 \delta^2} ; \]

\[ X_{\text{out}} = \frac{\omega_0^2 L}{R_0} \times \frac{\delta \omega_0^2}{\left( \omega_0^2 - \omega^2 \right)^2 + \omega^2 \delta^2} ; \]

\[ Y_{\text{out}} = \frac{\omega_0^2 L}{R_0} \times \frac{j \omega (\omega_0^2 - \omega^2 - \delta^2)}{\left( \omega_0^2 - \omega^2 \right)^2 + \omega^2 \delta^2} ; \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} ; \quad \delta = \frac{R}{L} ; \quad Q = \frac{\omega_0}{\delta} \]

Fitting parameters: \( \frac{\omega_0^2 L}{R_0} \) - scaling coefficient \( \omega_0 \) and \( \delta \)
Application of the lock-in amplifier for study of the transfer function of the RLC circuit

The resonance curves obtained on RLC circuits with two different damping resistors
Application of the lock-in amplifier for study of the transfer function of the RLC circuit
The example of fitting of the RLC circuit data to the analytical expression could be found in:
\textbackslash engr-file-03\phyinst\APL Courses\PHYCS401\Common\Simple Examples\Resonance fitting.opg

**Fitting function**

**Fitting parameters**

- $f_0 = \frac{w_0}{2\pi} = 1827.5\,\text{Hz}$
- $|U_c|$
- $\Delta f \approx 1500\,\text{Hz}$
- $Q_{\text{calc}} = \frac{\omega_0}{d} = 1.27$
- $Q_{\text{measured}} = \frac{f_0}{\Delta f} \approx 1.22$
From time domain to frequency domain.

Experiment.

Wavetek
Out
Sync

Lock-in SR830
input
Reference in

F(t) – periodic function $F(t) = F(t + T_0)$:

$$V = V_0 \left(0 < t \leq \frac{T_0}{2}\right);$$

$$-V_0 \left(\frac{T_0}{2} < t \leq T_0\right)$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos\left(\frac{2\pi nt}{T_0}\right)dt;$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin\left(\frac{2\pi nt}{T_0}\right)dt;$$

$$a_0 = \frac{2}{T_0} \int_0^{T_0} F(t)dt$$

Time domain pattern

Frequency domain?
From time domain to frequency domain. Experiment with SR830. Results.

Time domain pattern

Spectrum measured by SR 830 lock-in amplifier

Frequency domain
From time domain to frequency domain. FFT using Origin. Results.

Data file can be used to convert time domain to frequency domain.
From time domain to frequency domain.
FFT using Origin. Results.

Data file can be used to convert time domain to frequency domain.

Time domain taken by Tektronix scope. Spectrum calculated by Origin.
Accuracy is limited because of the limited resolution of the scope.
From time domain to frequency domain. Using of the Math option of the scope.

Time domain taken by Tektronix scope

Spectrum calculated by Tektronix scope.

Accuracy is limited because of the limited resolution of the scope.
From time domain to frequency domain. Using of the Math option of the scope.

Spectrum of the square wave signal

Spectrum of the pulse signal
From time domain to frequency domain. Different waveforms. Lock-in.

- ramp
- pulse
Appendix #1

Reminder: please submit the reports by e-mail in MsWord or pdf format. Strongly recommend the file name structure as:

L1_lab2_student1

Lab section   Lab number   Your name
Appendix #2

Origin templates for the Lab are available in:

\`\texttt{\textbackslash engr\textbackslash file\textbackslash 03\textbackslash PHYINST\textbackslash APL Courses\textbackslash PHYCS401\textbackslash Common\textbackslash Origin templates\textbackslash frequency domain analysis}\`
References:


   • You can find a soft copy of this book in:

   • \engr\file-03\phyinst\APL Courses\PHYCS401\Experiments