Transients and Oscillations in RLC Circuits

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Physics 401

Spring 2020

ILLINOIS

Goals of this Lab

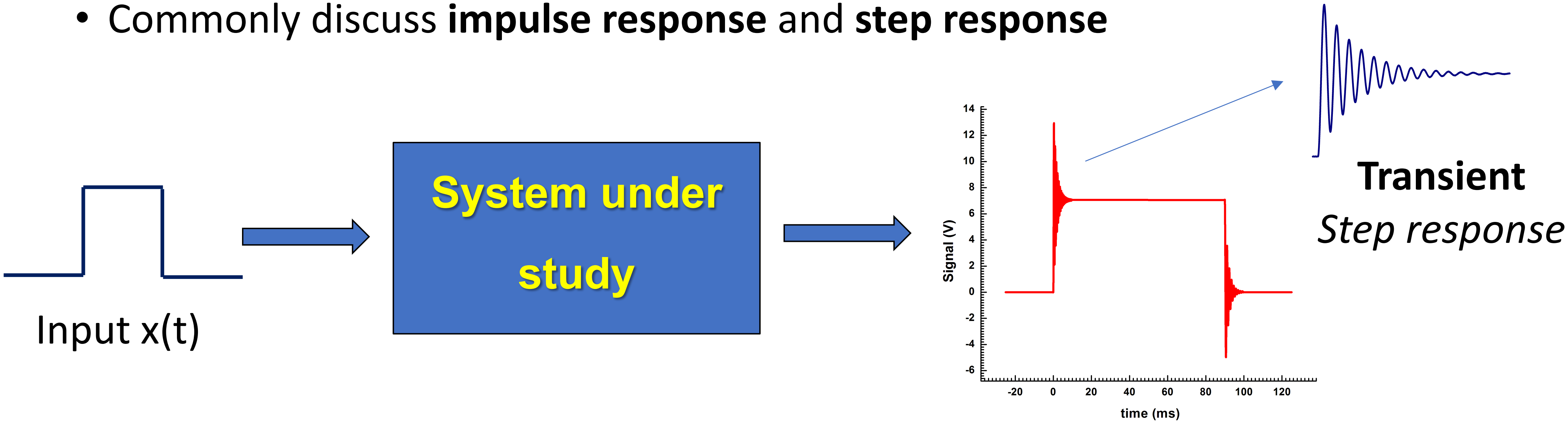
- Concepts: Oscillators in the time domain
  - Transients
  - Resonances
  - Damping regimes

- Implementation: RLC electrical circuits

- Data analysis using Origin

Driven Systems and Transients

- Consider a **system** that takes an input x(t) and yields output y(t)
  - We'll focus mostly on (approx.) **linear time-invariant (LTI)** systems, governed by constant-coefficient linear homogeneous differential equations
- The **transient response** of such a system is its ("short-lived") response to a change in input from an **equilibrium** state
  - Commonly discuss **impulse response** and **step response**



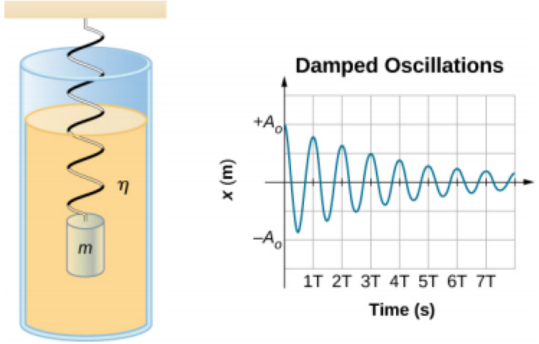
Output y(t)

From Harmonic Oscillator to RLC Circuit

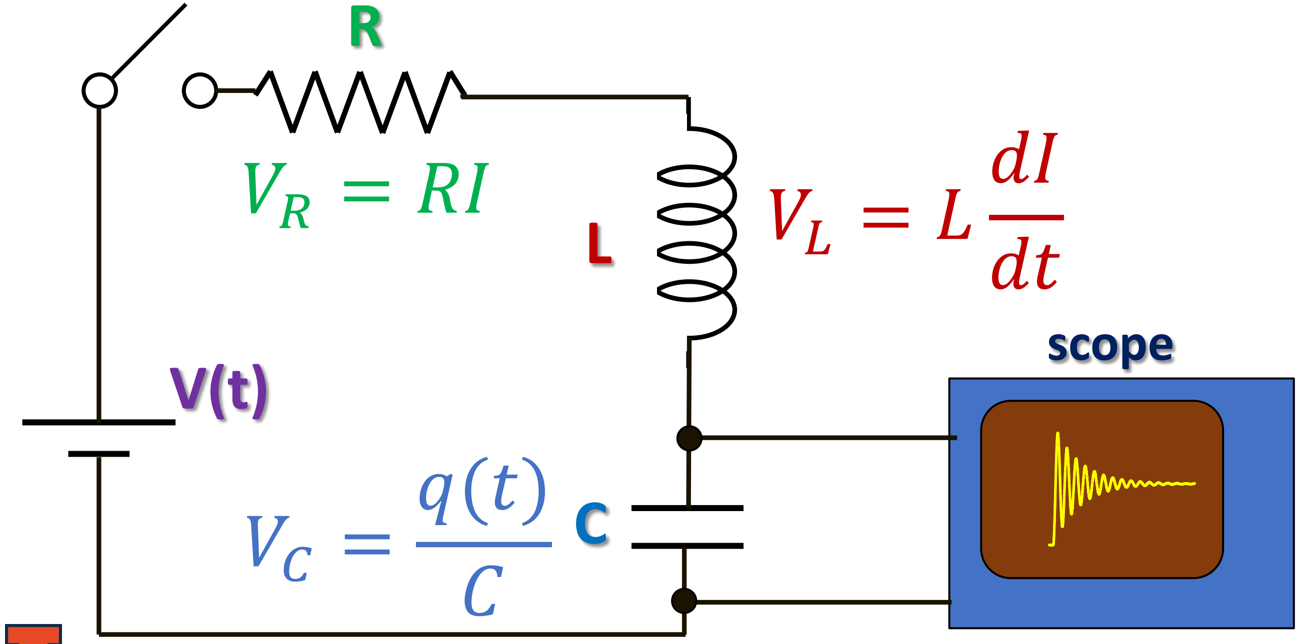
- A good reference LTI system is a **driven damped harmonic oscillator**

$$\underbrace{m\frac{d^2x}{dt^2}}_{\text{Inertia}} + \underbrace{c\frac{dx}{dt}}_{\text{Damping force}} + \underbrace{kx}_{\text{Restoring force}} = \underbrace{\mathrm{F(t)}}_{\text{Driving force}}$$

Inertia *Damping force* *Restoring force* *Driving force*



- A useful implementation of this is an **RLC circuit**



$V_R = RI$

$V_L = L\frac{dI}{dt}$

$V_C = \frac{q(t)}{C}$

$$V_R + V_L + V_C = \mathrm{V(t)}$$

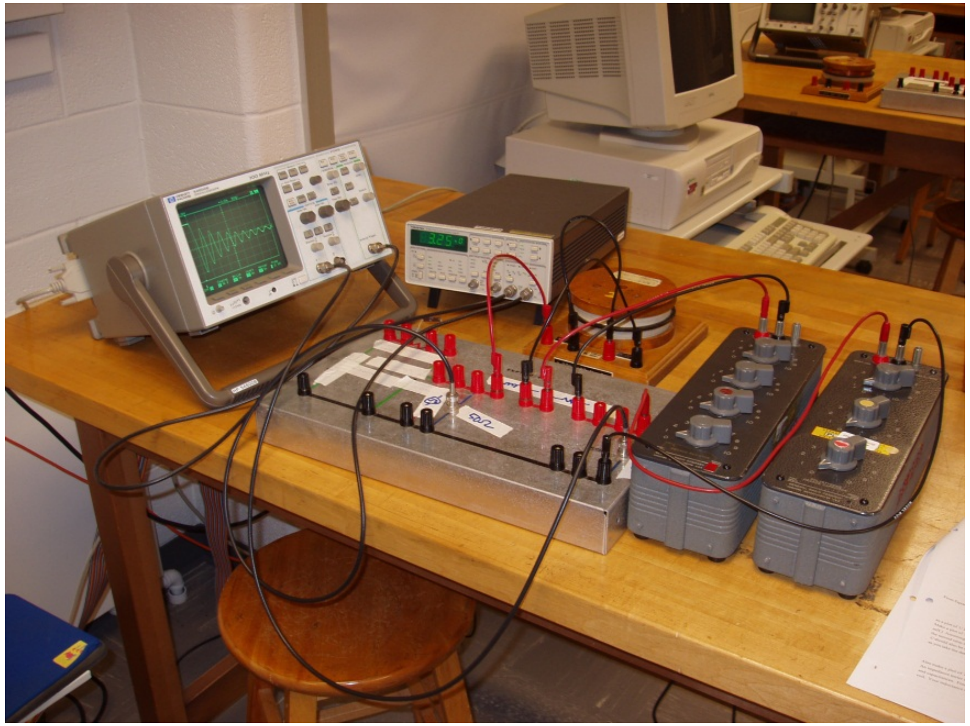
$$\mathrm{L}\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V\mathrm{(t)}$$

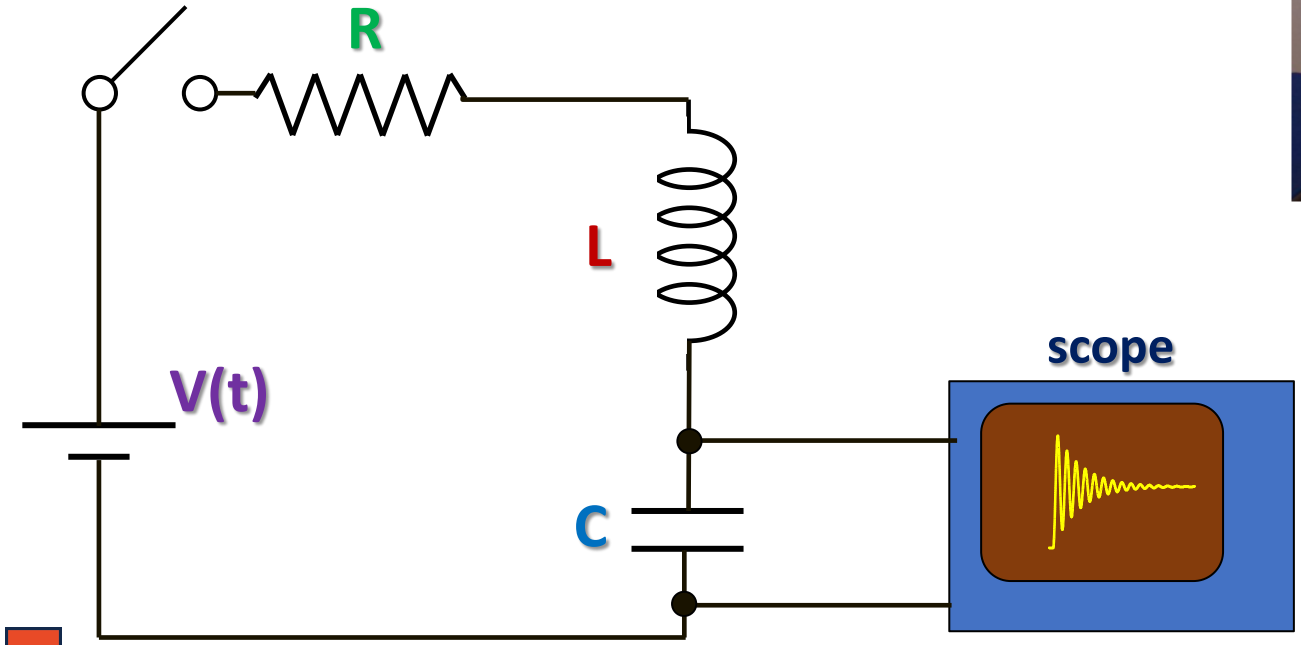
Where…

- q(t) is the charge on the capacitor
- The scope measures $V_C(t) = \frac{q(t)}{C}$

RLCs in the 401 Lab

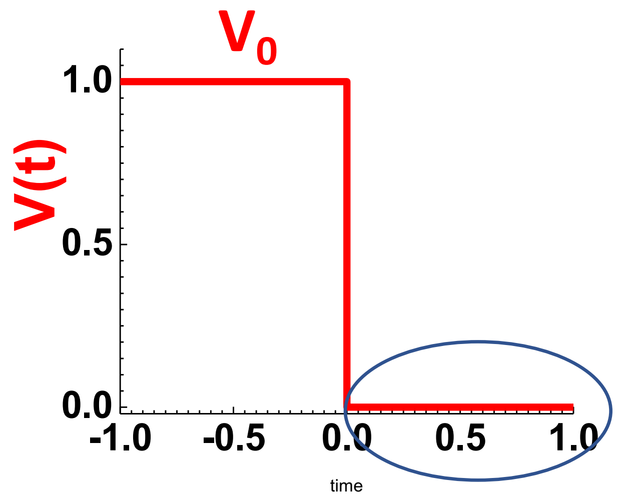
• Voltage	V	V (Volt)
• Resistance	R	Ω (Ohm)
• Inductance	L	mH (10^{-3} Henry)
• Capacitance	C	μF (10^{-6} Farad)





$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

RLC Transients: Three Solutions



- What happens after input voltage drops to zero?
- Solutions have the form: $q(t) = Ae^{st}$
- This turns our diff. eq. into a quadratic equation:

$$s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$$

- … with solutions:

$$s_\pm = -\underbrace{\frac{R}{2L}}_{\mathbf{a}} \pm \underbrace{\sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}}_{\mathbf{b}} \equiv -a \pm b$$

- … and boundary conditions $q(0) = CV_0, \;\; i(0) = \dot{q}(0) = 0$

$b^2>0$: Overdamped
$b^2=0$: Critically damped
Exponential decay

$b^2<0$: Underdamped
Oscillation / ringing

RLC Transients: Over-Damped Solutions

- **b²>0 ($R^2 > {}^{4L}/_{C}$)**: aperiodic **exponential decay**

- Solutions have the form:

$$q(t) = e^{-at}\left(A_1 e^{bt} + A_2 e^{-bt}\right)$$

$$\mathrm{i(t)} = \dot{q}(t) = -ae^{-at}\left(A_1 e^{bt} + A_2 e^{-bt}\right) + be^{-at}\left(A_1 e^{bt} - A_2 e^{-bt}\right)$$

- Applying boundary conditions $q(0) = CV_0, \;\; i(0) = \dot{q}(0) = 0$

$$q(t) = q(0)e^{-at}(\cosh bt + \sinh bt)$$

$$q(t) \xRightarrow[(a-b)t \gg 1]{} \frac{q(0)}{2}\left(1 + \frac{a}{b}\right)e^{-(a-b)t}$$

RLC Transients: Critical Damping

- **$b^2=0$ $(R^2 = {}^{4L}/_{C})$**: fastest possible **exponential decay**

- Solutions have the form:

$$q(t) = e^{-at}(A_1 + A_2 t)$$
$$\mathrm{i}(\mathrm{t}) = \dot{q}(t) = -ae^{-at}(A_1 + A_2 t) + A_2 e^{-at}$$

- Applying boundary conditions $q(0) = CV_0, \;\; i(0) = \dot{q}(0) = 0$

$$q(t) = q(0)e^{-at}(1 + at)$$
$$i(t) = -a^2 q(0) t e^{-at}$$

Ex: Real Data Analysis for Critical Damping

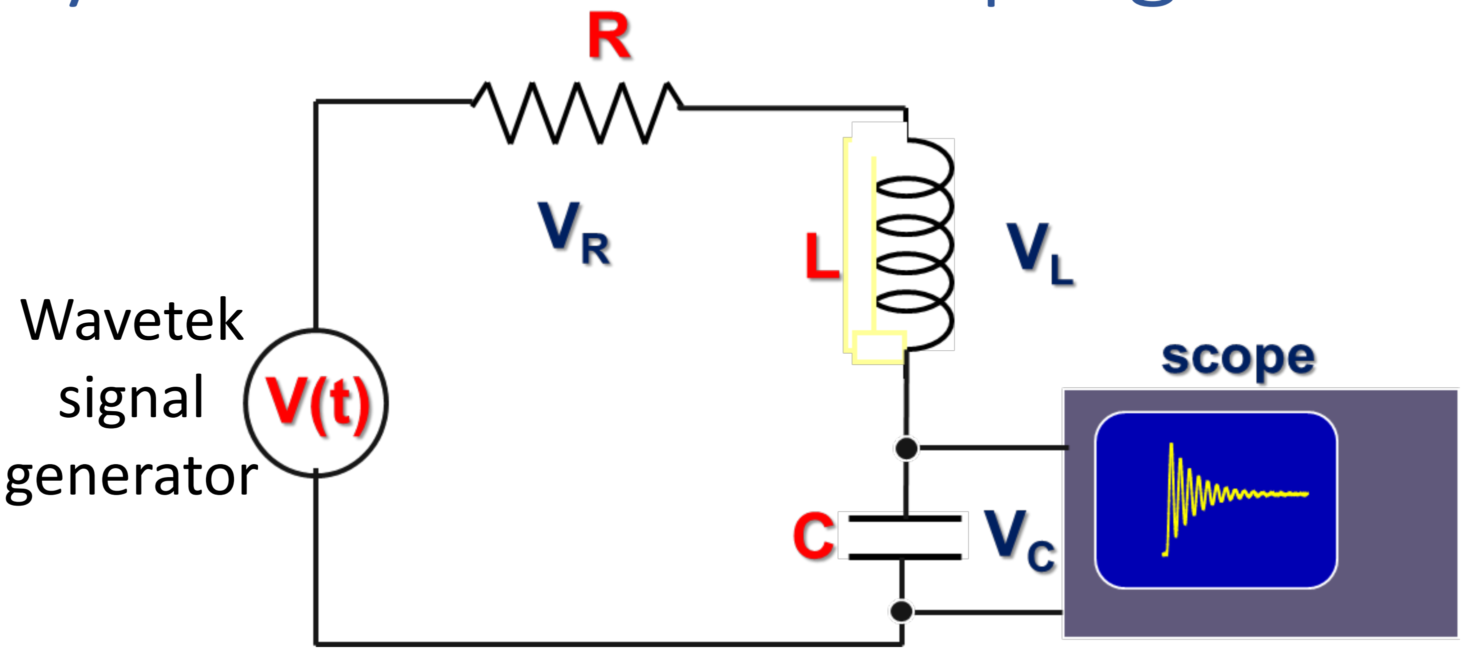
In this experiment:

- R = 300 ohm
- C = 1 μF
- L = 33.43 mH

… plus practical reality:

- Wavetek has 50 ohm output resistance
- Inductor coil has 8.7 ohm measured R

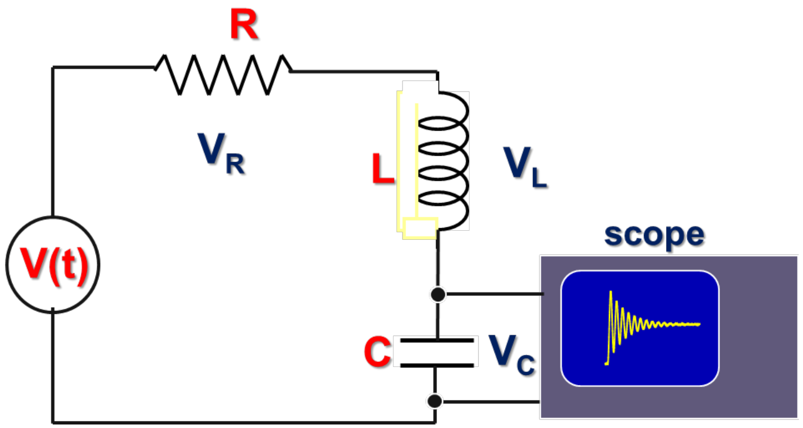
=> Loop R_{tot} = 358.7 ohm



Decay coefficient

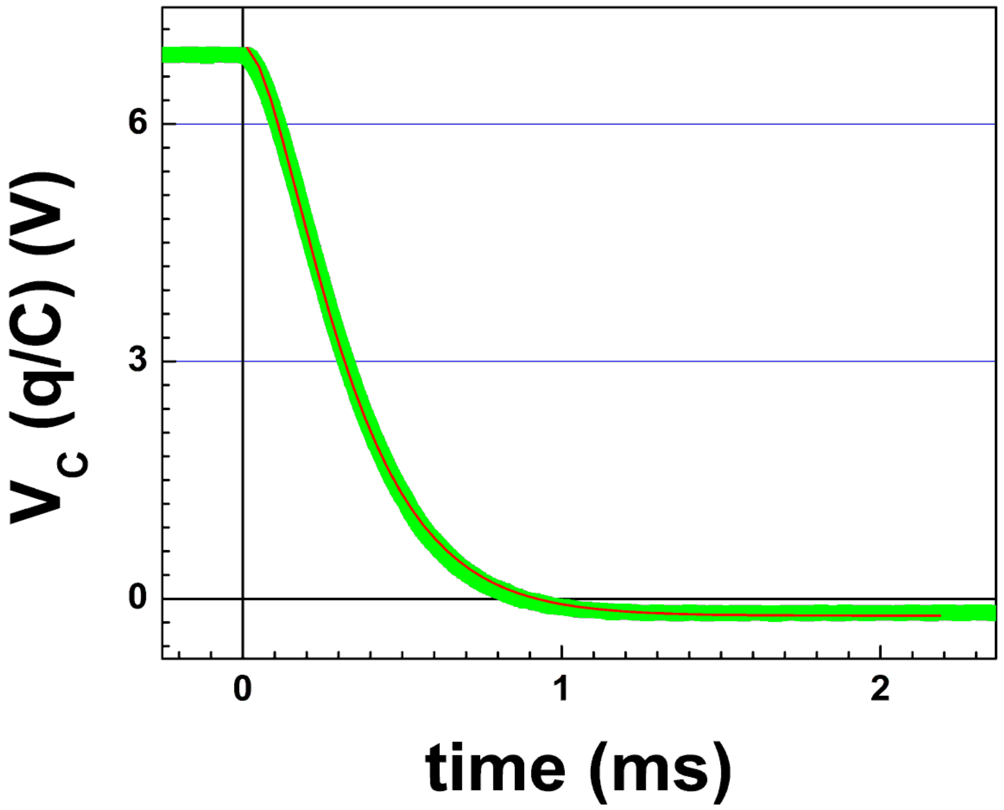
$$a = \frac{R_{tot}}{2L} = \frac{358.7}{2*33.43\times10^{-3}} \approx 5365\ s^{-1} \approx \frac{1}{0.2\ ms}$$

Ex: Real Data Analysis for Critical Damping



Fitting function for measured $V_C \propto q$:

$$V_C = \boldsymbol{V_{c0}}(1 + \boldsymbol{a}t)e^{-\boldsymbol{a}t}$$



Delay coefficient:

- Calculated: a=5385 s^{-1}
- Fitted: a= 5820 s^{-1} *(+8%)*

Possible cause for discrepancy?
Perhaps slightly overdamped?
Calculated b^2 = 2.99e7 - 2.90e7 > 0

RLC Transients: Underdamped Case

- **b^2<0 ($R^2 < {}^{4L}/_{C}$): decaying oscillation**

- Solutions (*see write-up!*):

$$q(t) = q(0)e^{-at}\sqrt{1-\frac{a^2}{\omega^2}}\ \sin(\omega t+\varphi)$$

$$\mathrm{i(t)} = q(0)e^{-at}\left(\frac{a^2-\omega^2}{\omega}\right)\sin\omega t$$

$$a = \frac{R}{2L};\ \ \omega = 2\pi f = \sqrt{\left(\frac{1}{LC}\right)-\left(\frac{R}{2L}\right)^2}$$

Quantifying Damping

General idea: How many *oscillation periods* ($T \equiv 1/f$) does it take for the *oscillation amplitude* to decay "substantially"?

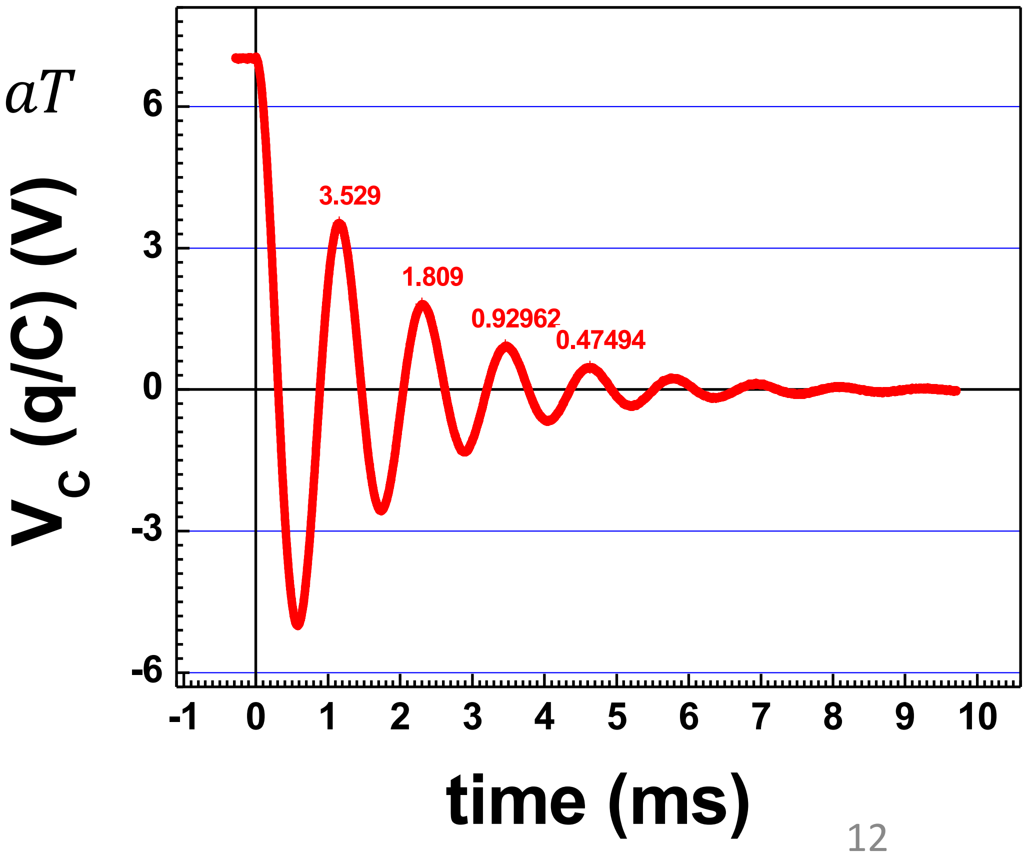
- **Log-decrement** can be defined as:

$$\delta \equiv \ln\left(\frac{q(t_{max})}{q(t_{max}+T)}\right) = \ln\left(\frac{e^{-at_{max}}}{e^{-a(t_{max}+T)}}\right) = aT$$

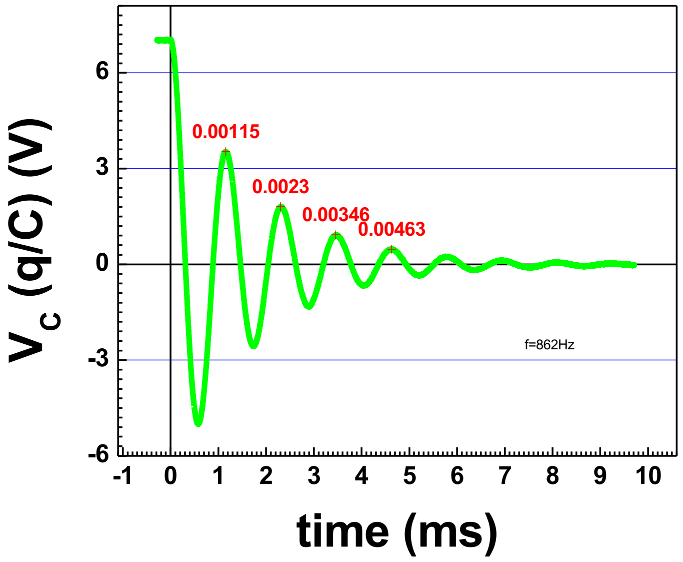
- More commonly, define **Quality Factor**

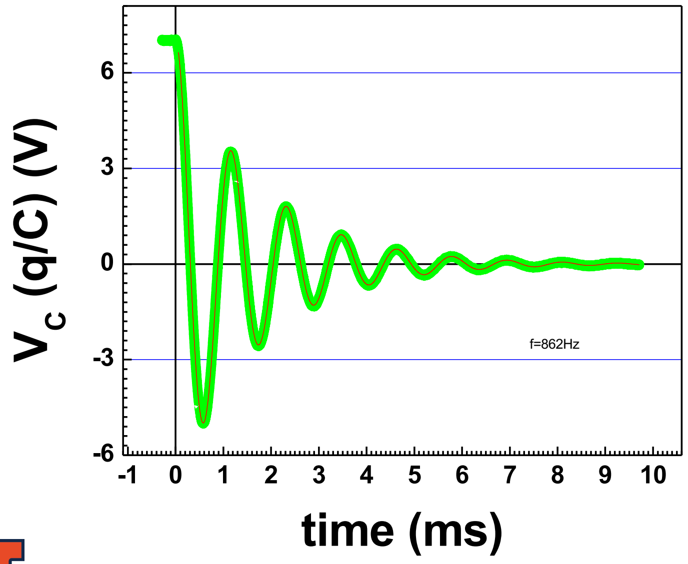
$$Q \equiv 2\pi \frac{E}{\Delta E} = \frac{\pi}{\delta}$$

- From this plot, $\delta \approx 0.67,\ Q \approx 4.7$



Analysis Using Origin





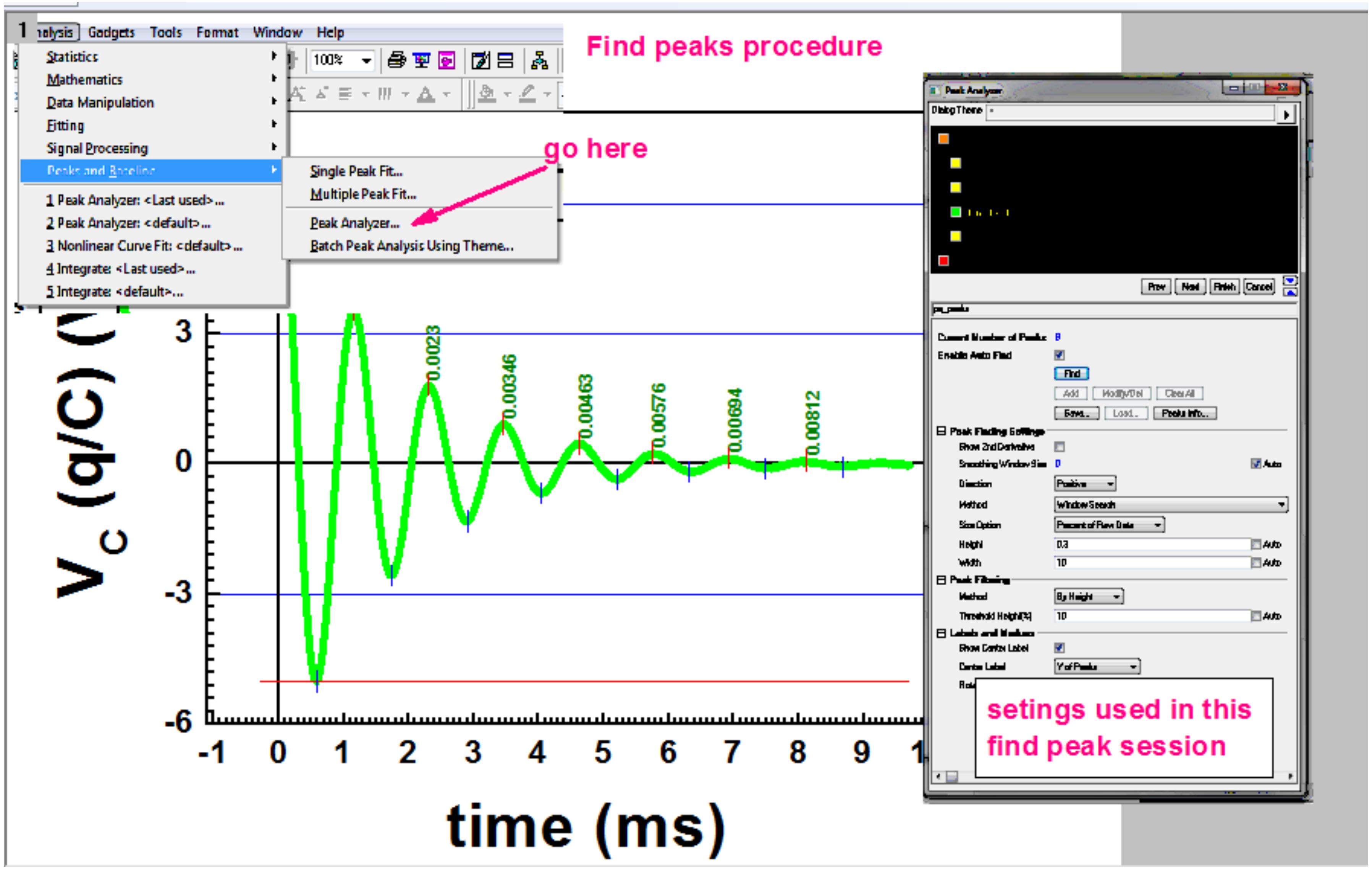
Keep in mind:

- Fitting multi-parameter **linear** models to data is generally pretty robust
- Fitting **non-linear** models to data is all about making **good initial guesses**

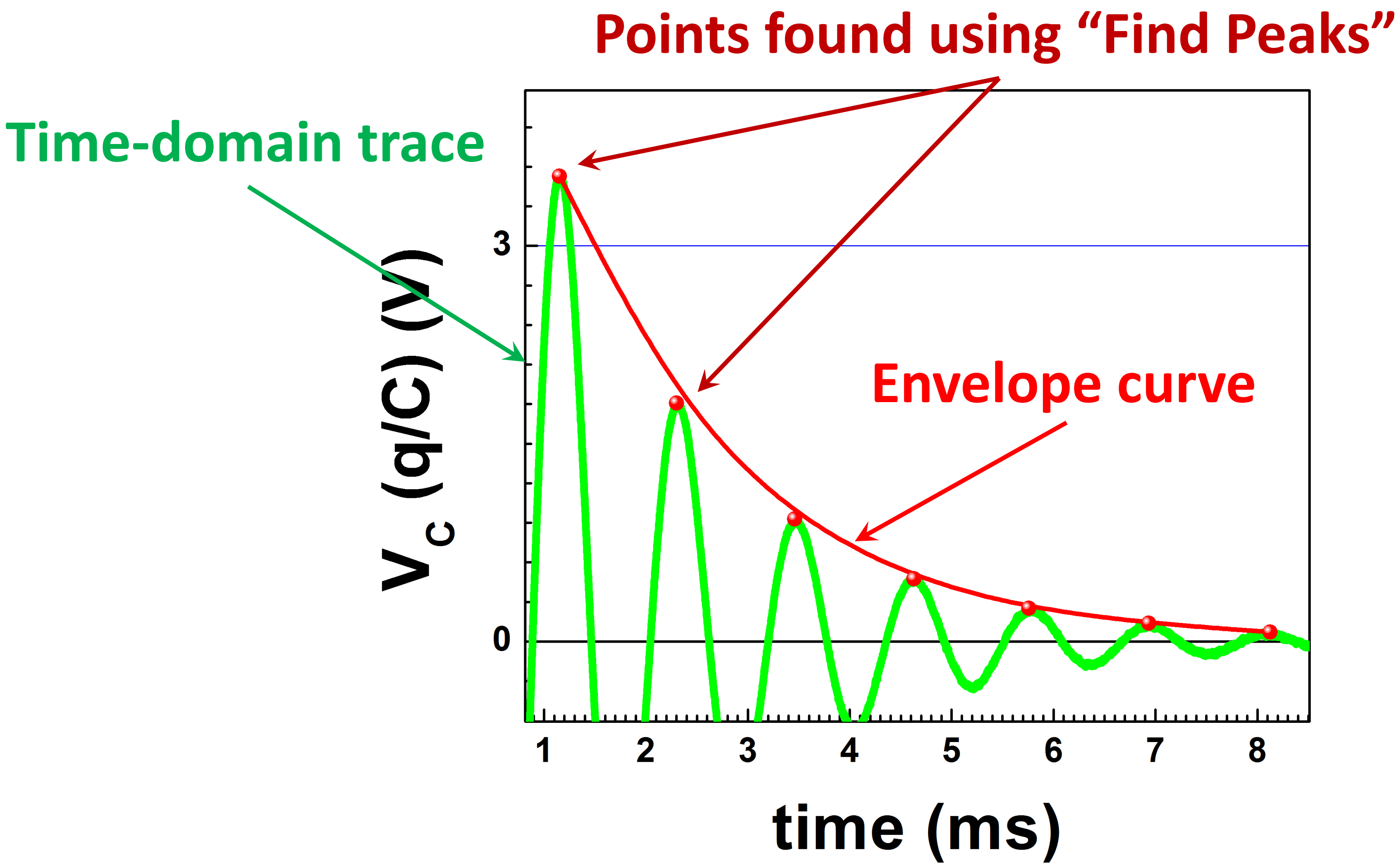
Practical procedure:

1. Identify peaks
2. Fit "envelope"
3. Perform nonlinear fit

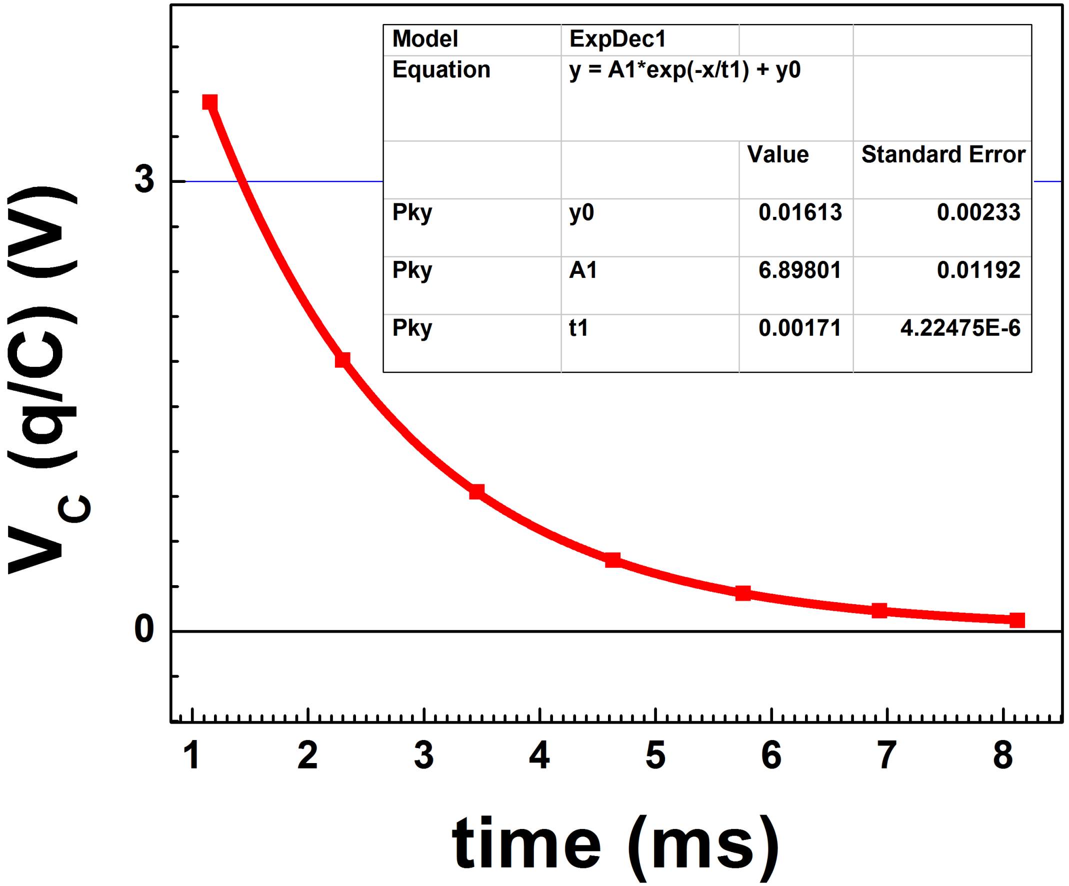
Analysis Using Origin: Identify Peaks

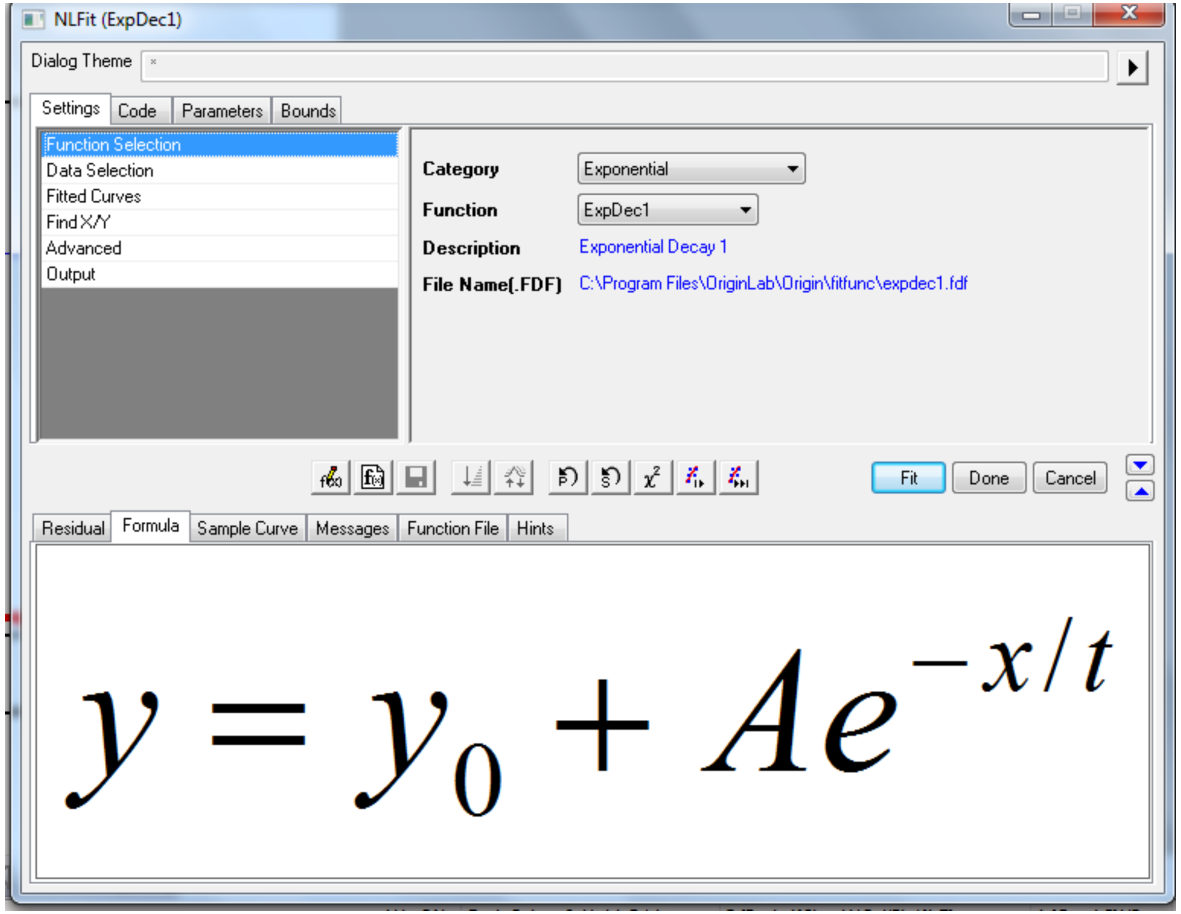


Analysis Using Origin: Fit Decay Envelope

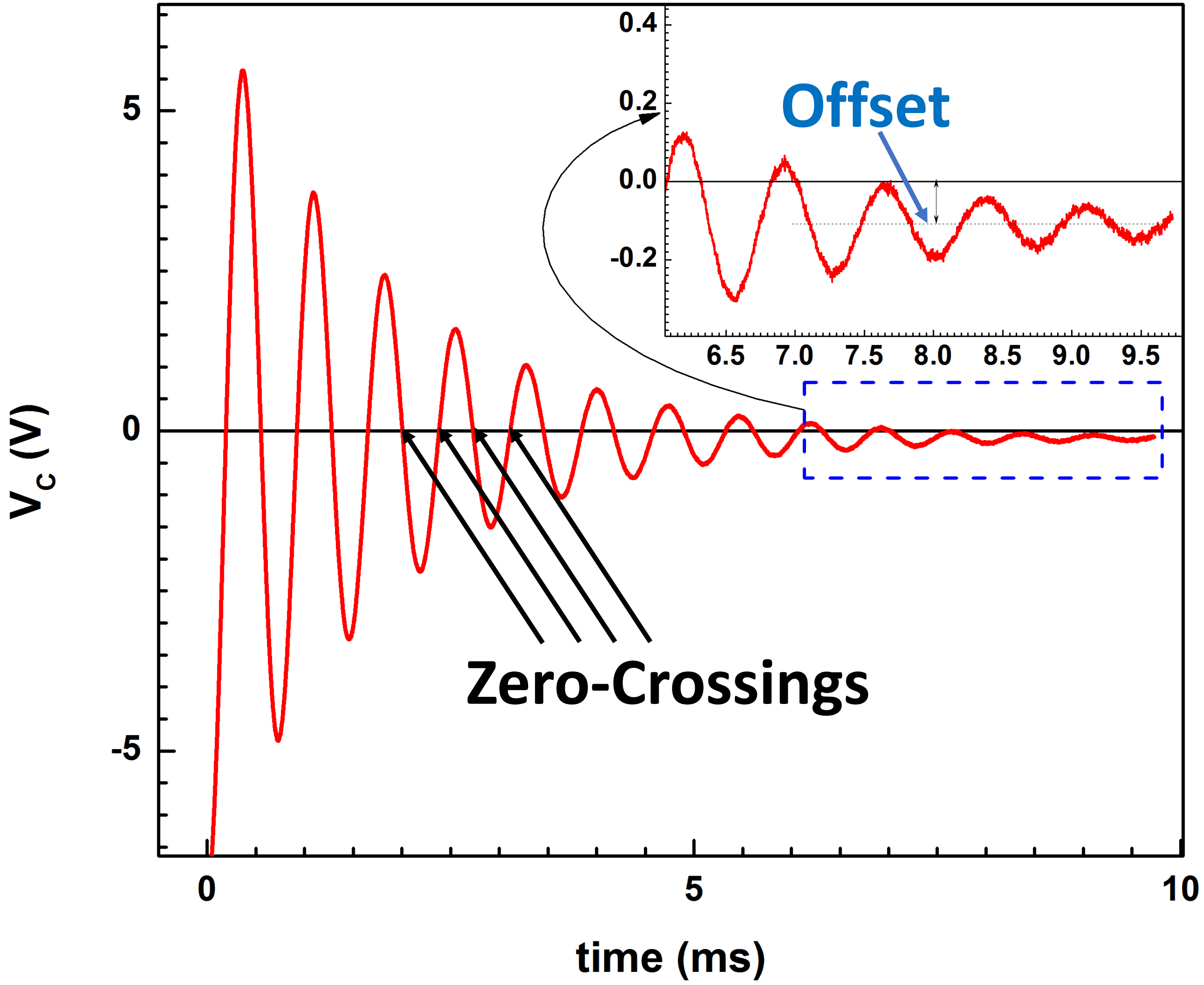


Analysis Using Origin: Fit Decay Envelope





Analysis Using Origin: Periods and Offsets

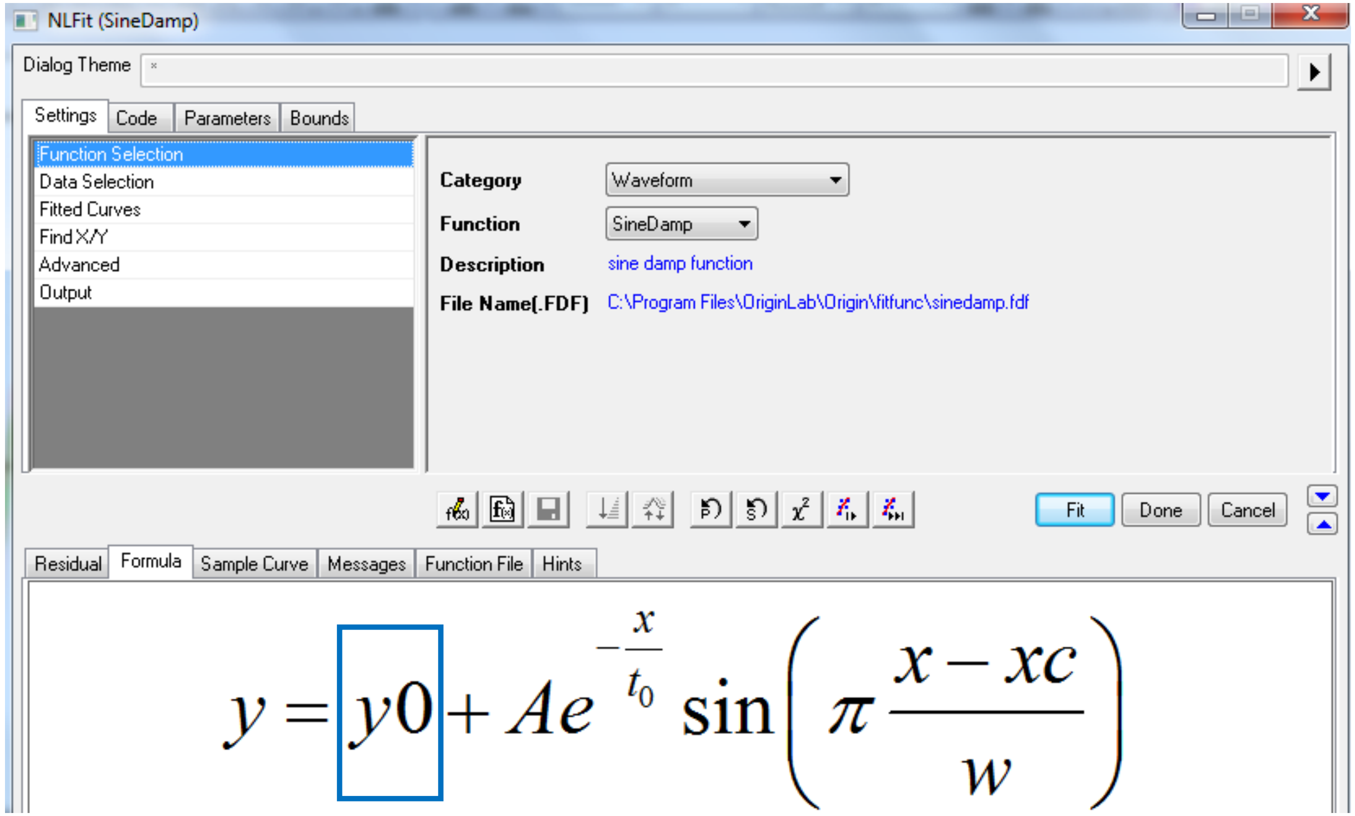


$$q(t) = Ae^{-at}\sin(\omega t + \varphi) + \boxed{K}$$

Offset

- Manually evaluate period of the oscillations
- Limited accuracy!
- Answer can be biased by **DC offset**

Analysis Using Origin: Non-Linear Fit



Offset

$$q(t) = Ae^{-at} \sin(\omega t + \varphi)$$

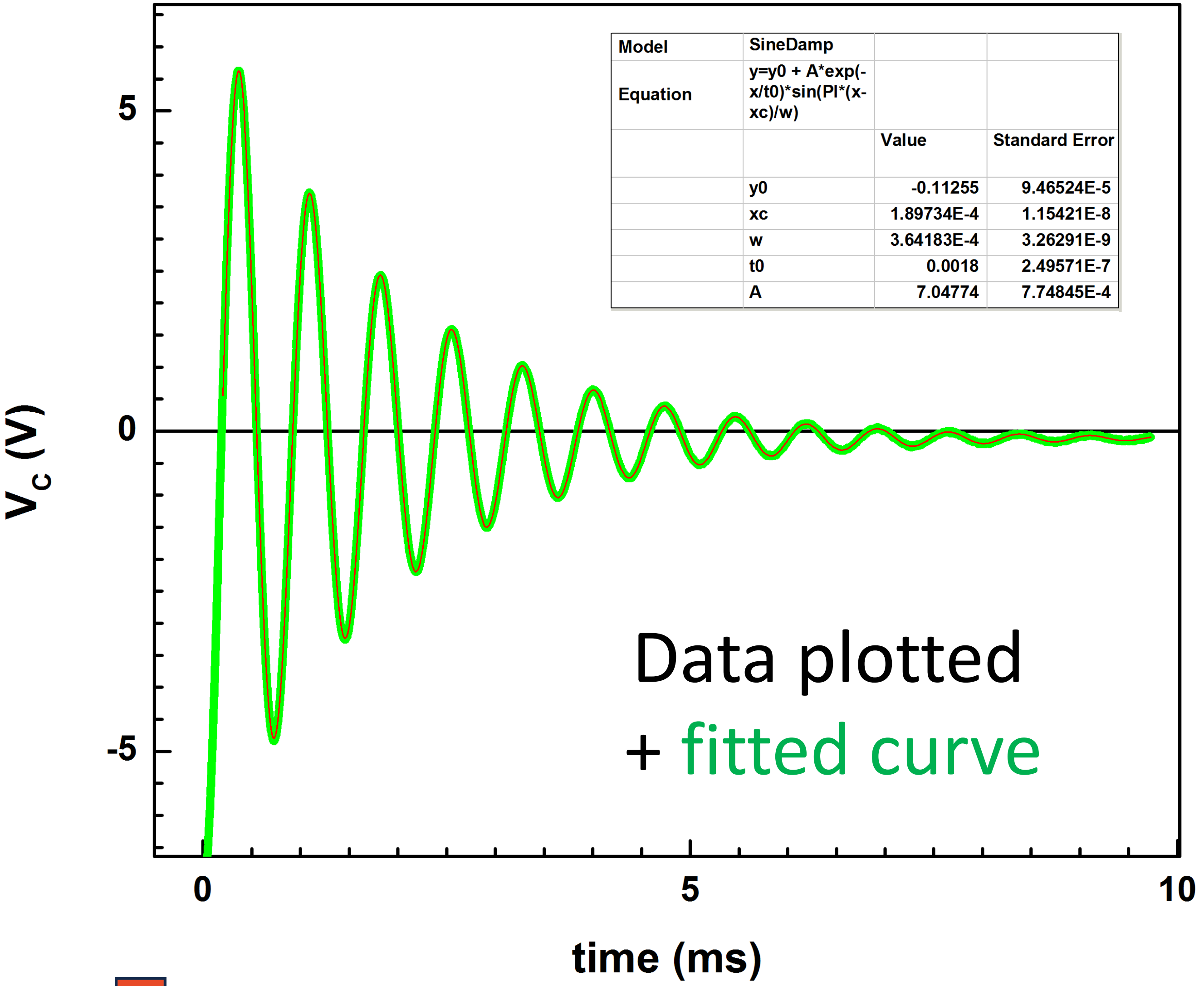
$$U_C(t) = \frac{q(t)}{C}$$

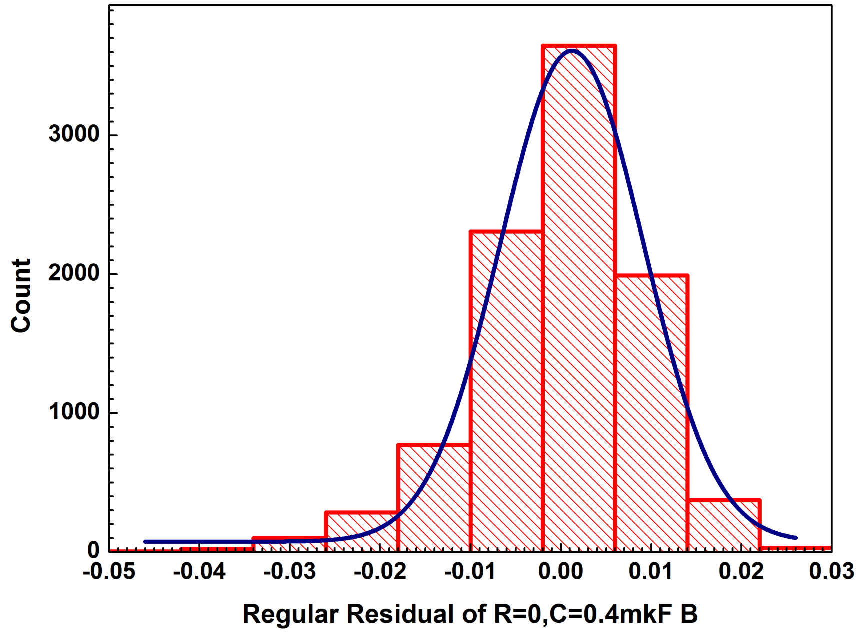
- Use Origin standard function
  - Category: Waveform
  - Function: SineDamp
- Fit parameters: y_0, A, t_0, x_c, w
- From fit we can obtain:

$$\boldsymbol{a} = \frac{1}{t_0}; \quad \boldsymbol{T} = \frac{1}{f} = 2w$$

Analysis Using Origin: Evaluating the Fit

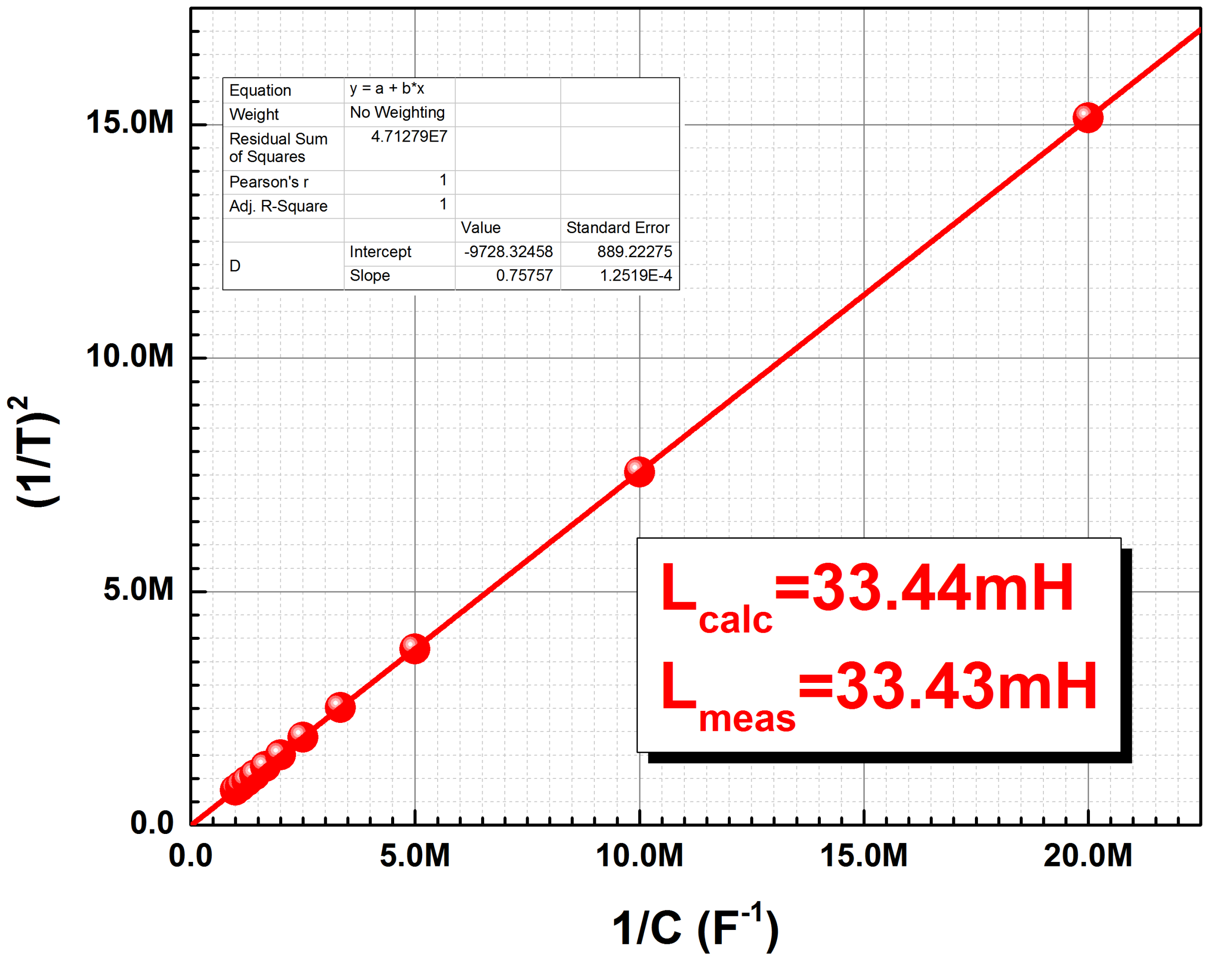
$$q(t) = Ae^{-at}\sin(\omega t + \varphi)$$





Residuals (data – fit): metric for quality of fit

Analysis Using Origin: Interpreting Results



$$q(t) = Ae^{-at}\sin(\omega t + \varphi)$$

$$f^2 = \left(\frac{1}{T}\right)^2 = \left(\frac{1}{2\pi}\right)^2\left(\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2\right)$$

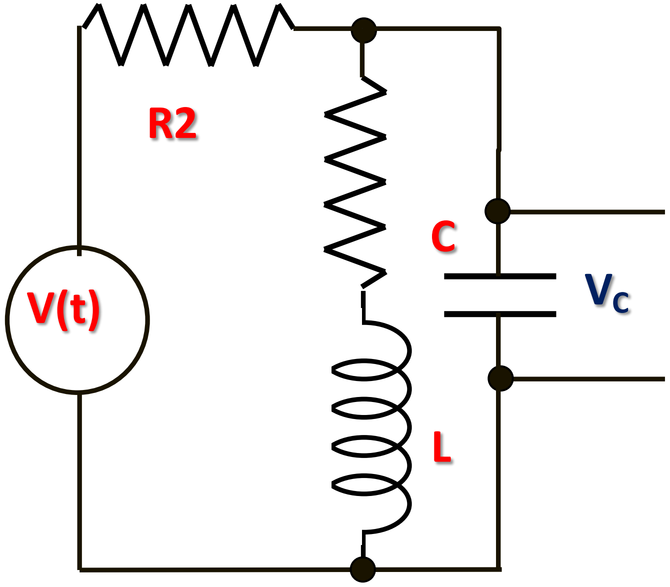
Final Results

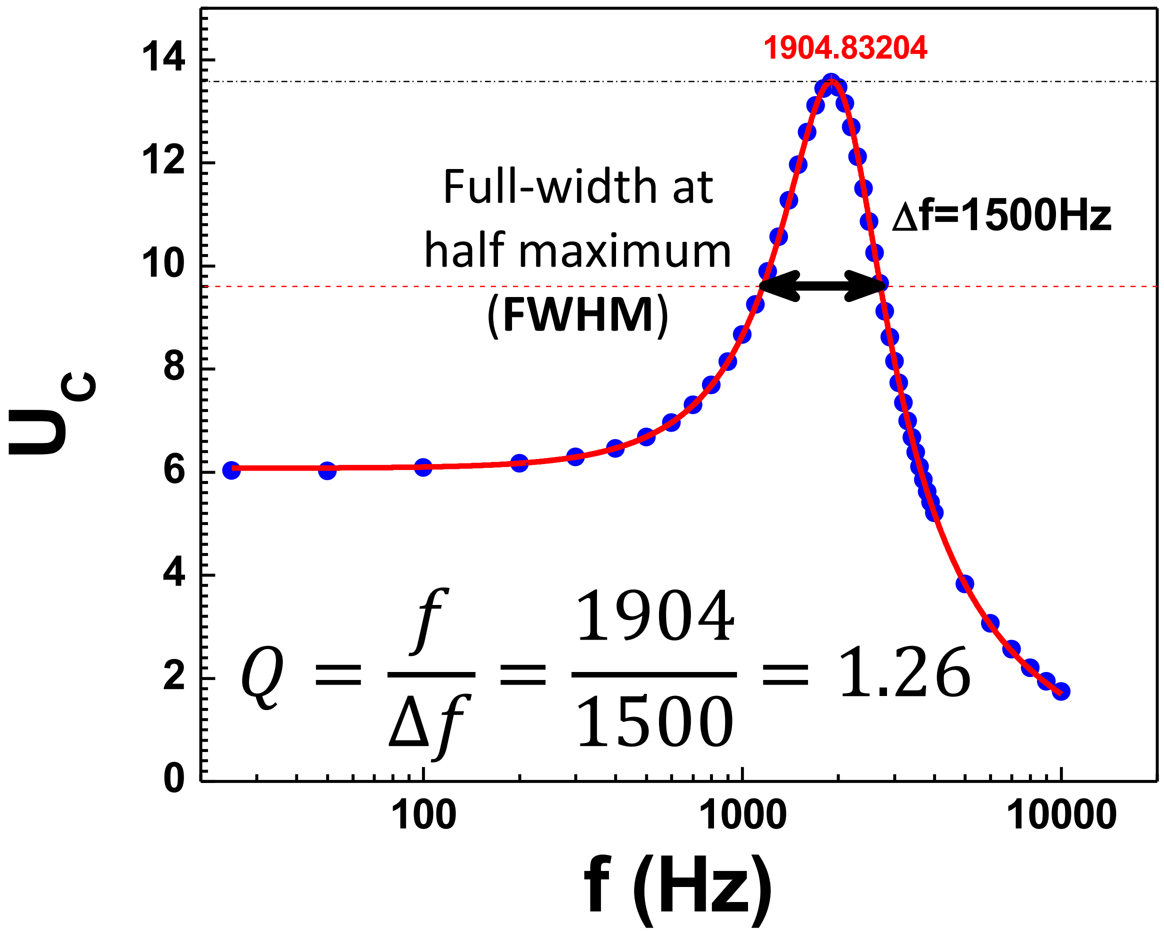
Not the fit itself, but constraints on the physical model parameters!

Foreshadowing: Resonance in RLC Circuits

Parallel config:

Energy can "slosh" from C to L and back

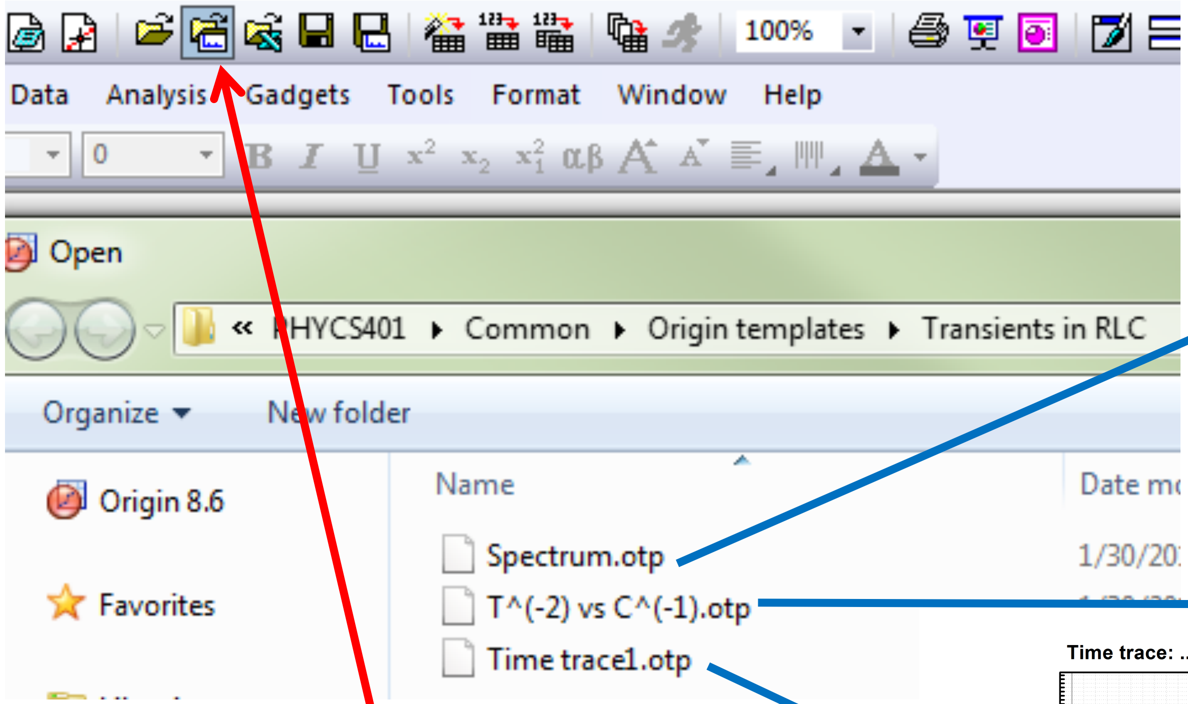


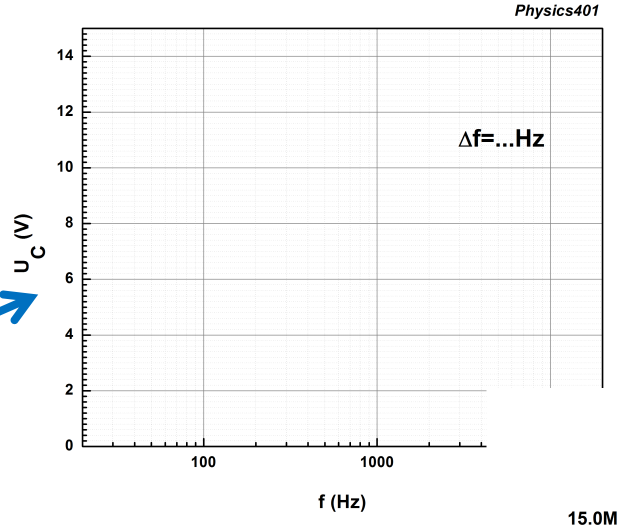


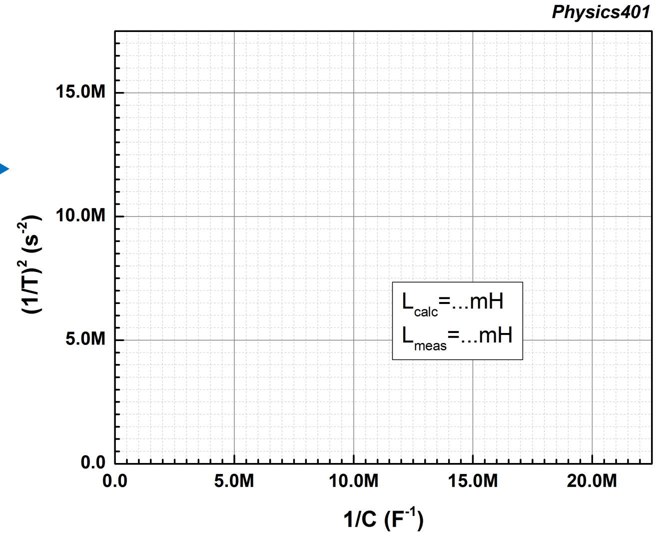
$$Q = \frac{f}{\Delta f} = \frac{1904}{1500} = 1.26$$

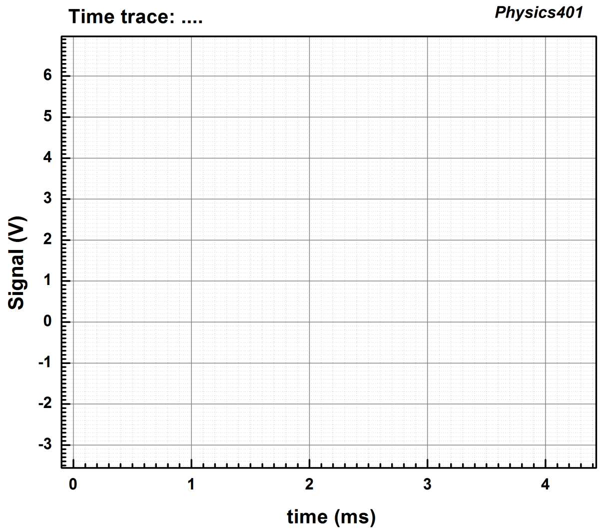
Resonance: Amplified response when a system is driven at (or near) one of its "natural" frequencies

Origin Templates for This Week's Lab



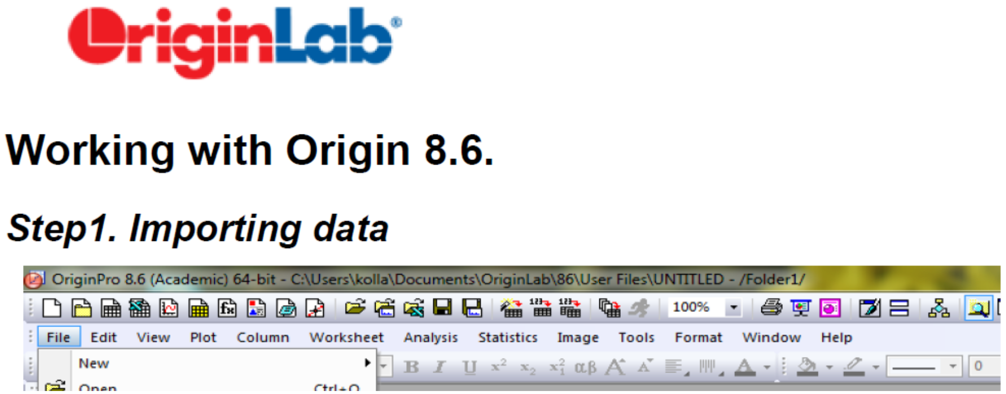






Open Template button

Origin Manuals



Short, simple manual covering only basic operations with Origin (*linked from* [*P401 webpage*])

Don't forget about Origin help!

Video tutorial library on company website