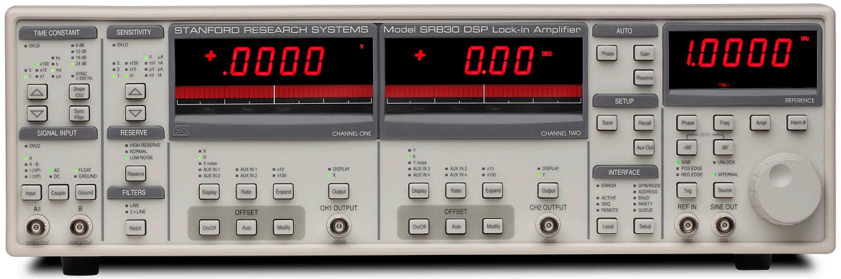
Frequency Domain Analysis of Linear Circuits using Synchronous Detection

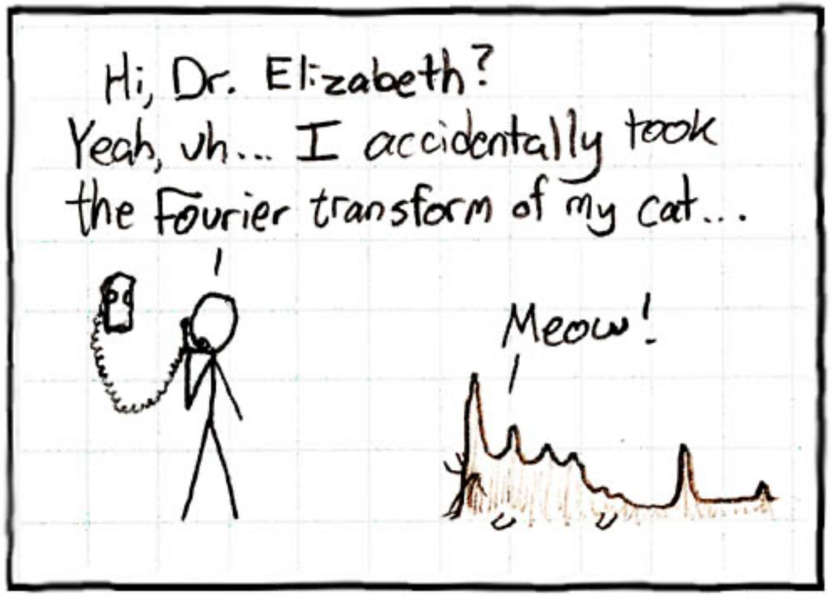


Professor Jeff Filippini

Physics 401

Spring 2020

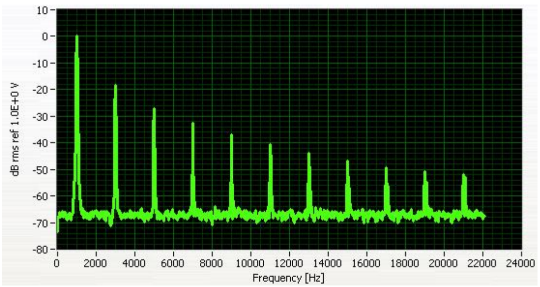
ILLINOIS



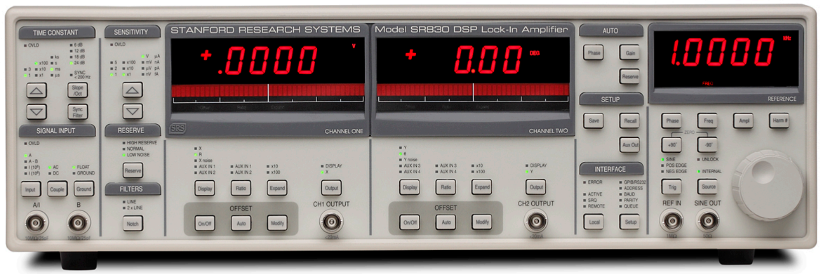
XKCD #26

Key Topics of this Lab

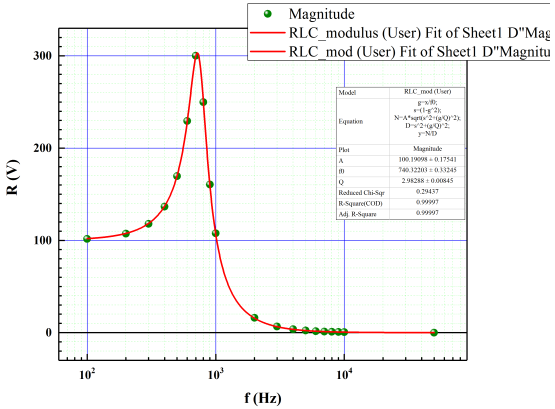
1. The Fourier Transform and its Uses



2. Lock-In Amplifiers



3. Data Analysis



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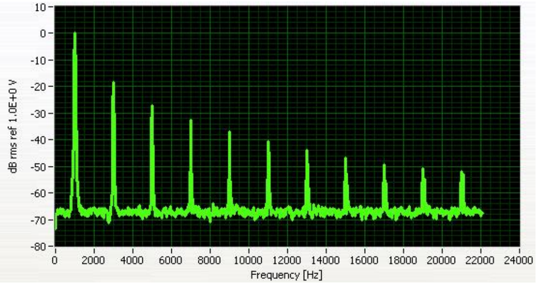

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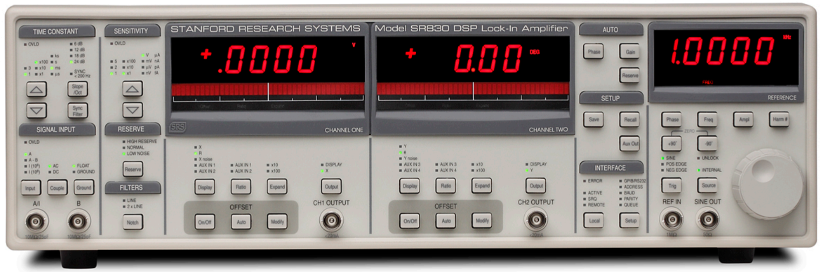

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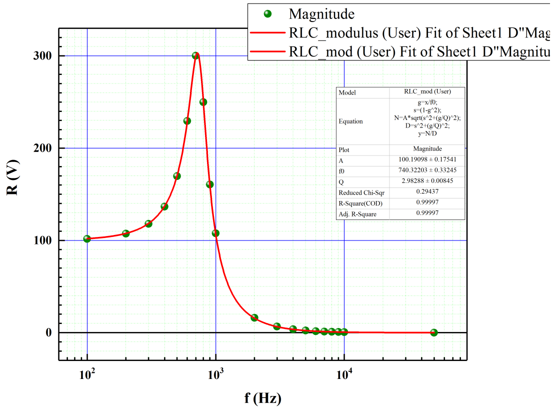
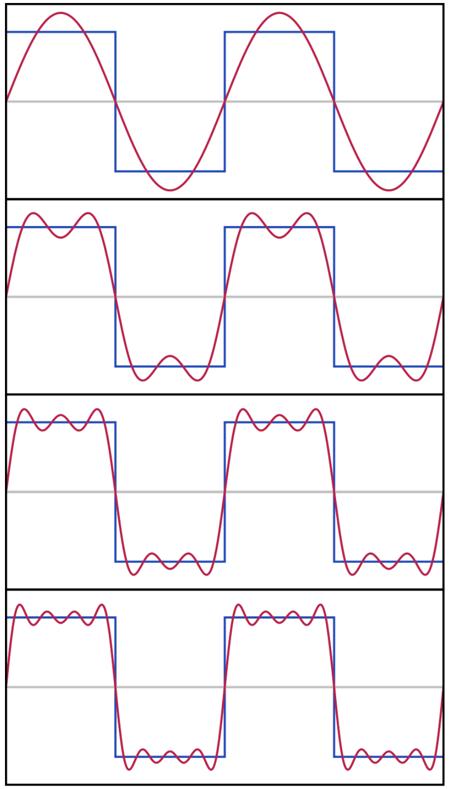


Fourier Series

Circa 1807, while struggling to solve the heat equation, Jean Baptiste Joseph Fourier described how any* real-valued waveform can be described uniquely** as a sum of sinusoidal waves (web visualization)

Jean Baptiste Joseph Fourier (1768 – 1830)



$$A \sin \omega t \quad (\omega \equiv 2\pi f)$$

$$+ \frac{A}{3} \sin 3\omega t$$

$$+ \frac{A}{5} \sin 5\omega t$$

$$+ \frac{A}{7} \sin 7\omega t + \ldots$$

** square-integrable, periodic, …*

*** sin, cos a **basis** for function space*

Fourier Transform

Extend this scheme elegantly to any* complex function by:

1. Replacing sin and cos with complex exponentials ($e^{i\omega t} = \cos \omega t + i \sin \omega t$)
2. Allow for complex coefficients (incorporates phases)
3. Replacing sums with integrals

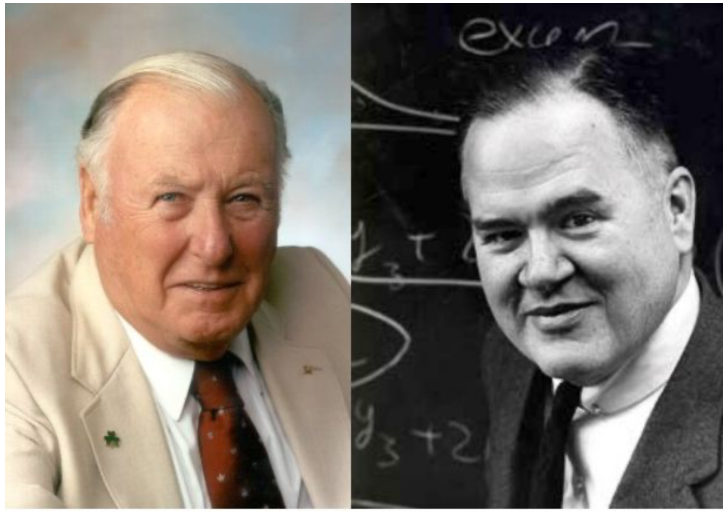
** square-integrable, periodic, …*

The Discrete Fourier Transform

Suppose that our data is a **regularly-sampled** time series, $h_n \equiv h(t_n) = h(n\,\Delta t)$. Then the right analog of the FT is the Discrete Fourier Transform:

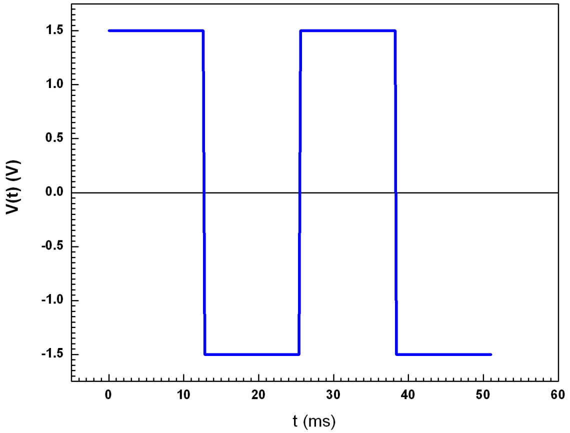
$$H_k = H(f_k) = H(k\,\delta f) = \sum_{n=0}^{N-1} h_n e^{-2\pi i n k/N} \qquad \Delta f = \frac{1}{N\,\Delta t}$$

In 1965 J.W. Cooley and J. Tukey* showed how to compute this extremely efficiently if N is a power of 2 (or at least has few prime factors). This is the **Fast Fourier Transform (FFT)**, and is approx. the only Fourier transform anyone computes

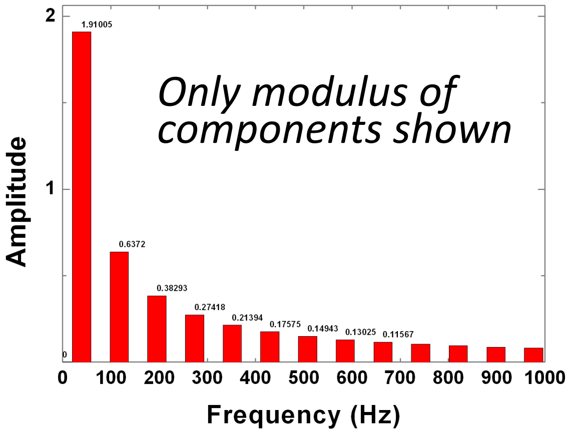


James William Cooley (1926-)

John Wilder Tukey (1915-2000)



Time domain



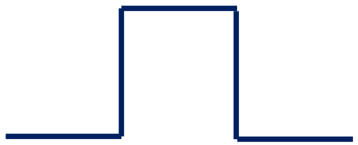
Frequency domain

* Also Carl Friedrich Gauss (1805)

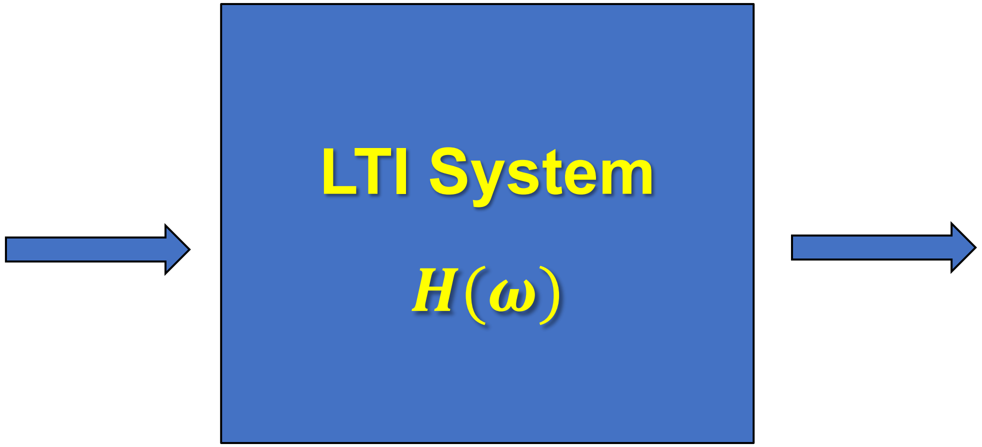
Fourier Analysis of Signals and Systems

Fourier analysis is very useful for working with Linear Time-Invariant (LTI) systems, because differential equations become algebraic ones

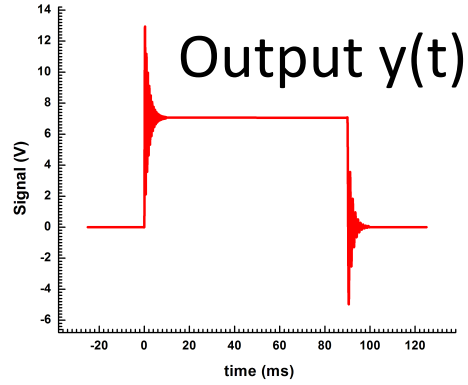
Input x(t)



$$X_1 e^{-i\omega_1 t} + X_2 e^{-i\omega_2 t} + X_3 e^{-i\omega_3 t} + \ldots$$



Output y(t)

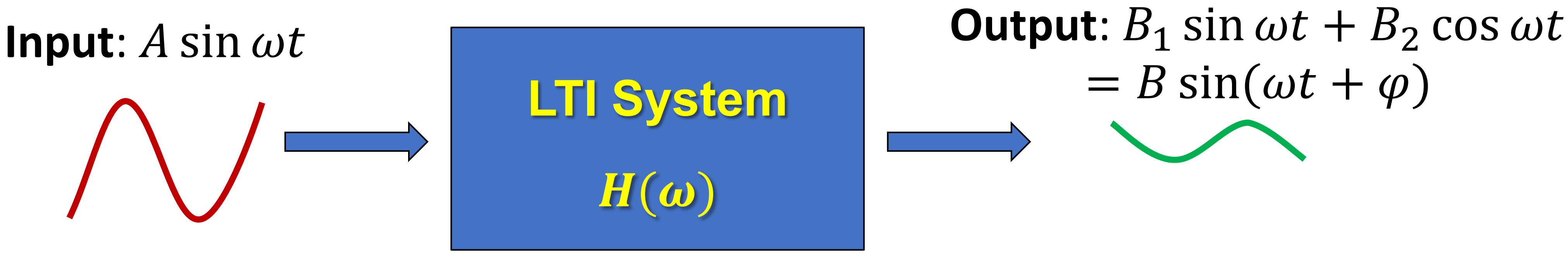


$$H_1 X_1 e^{-i\omega_1 t} + H_2 X_2 e^{-i\omega_2 t} + H_3 X_3 e^{-i\omega_3 t} + \ldots$$

The (complex) **transfer function H(f)** fully describes the response of an LTI system, and can be applied (in Fourier space!) to any desired input.

Use for diff. eq., filters, control systems, signal processing, circuits (complex impedance), …

Frequency Domain Spectroscopy



Apply a **sine wave input** to the system under study and measure the response.

For a **linear** system the response will be at the same frequency, but possibly amplified/attenuated (B/A) and phase-shifted (φ).

Vary the driving frequency to measure the **transfer function** $H(\omega)$ directly

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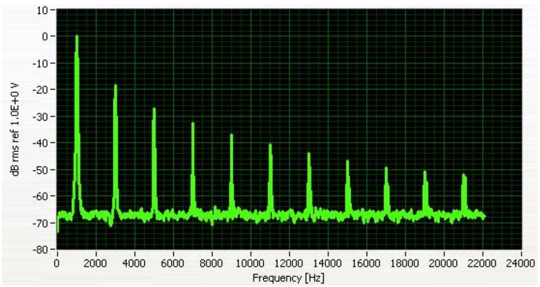
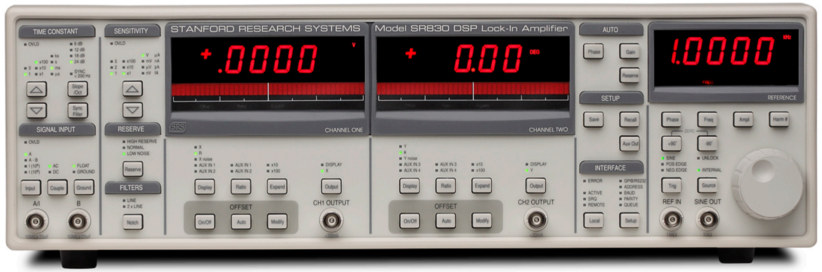


2. **Lock-In Amplifiers**



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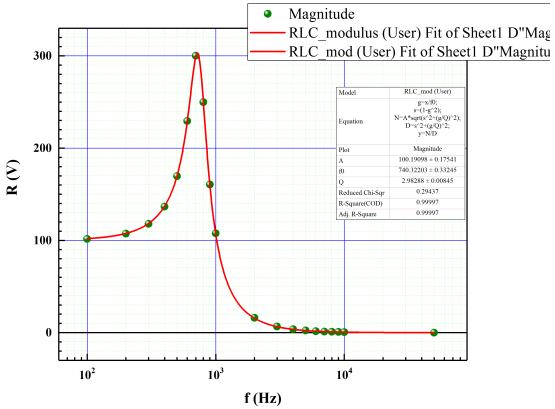
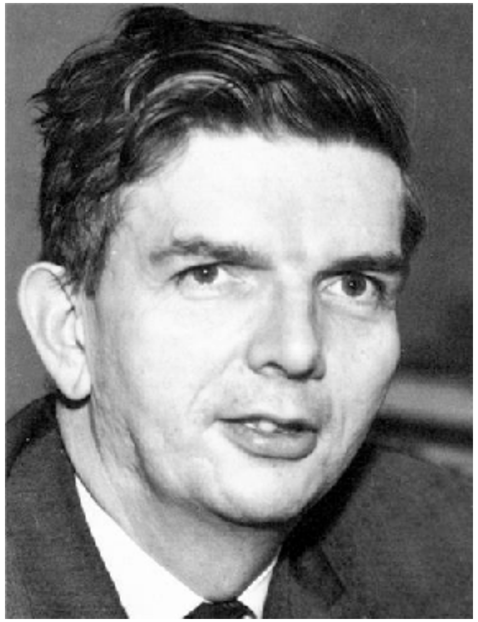
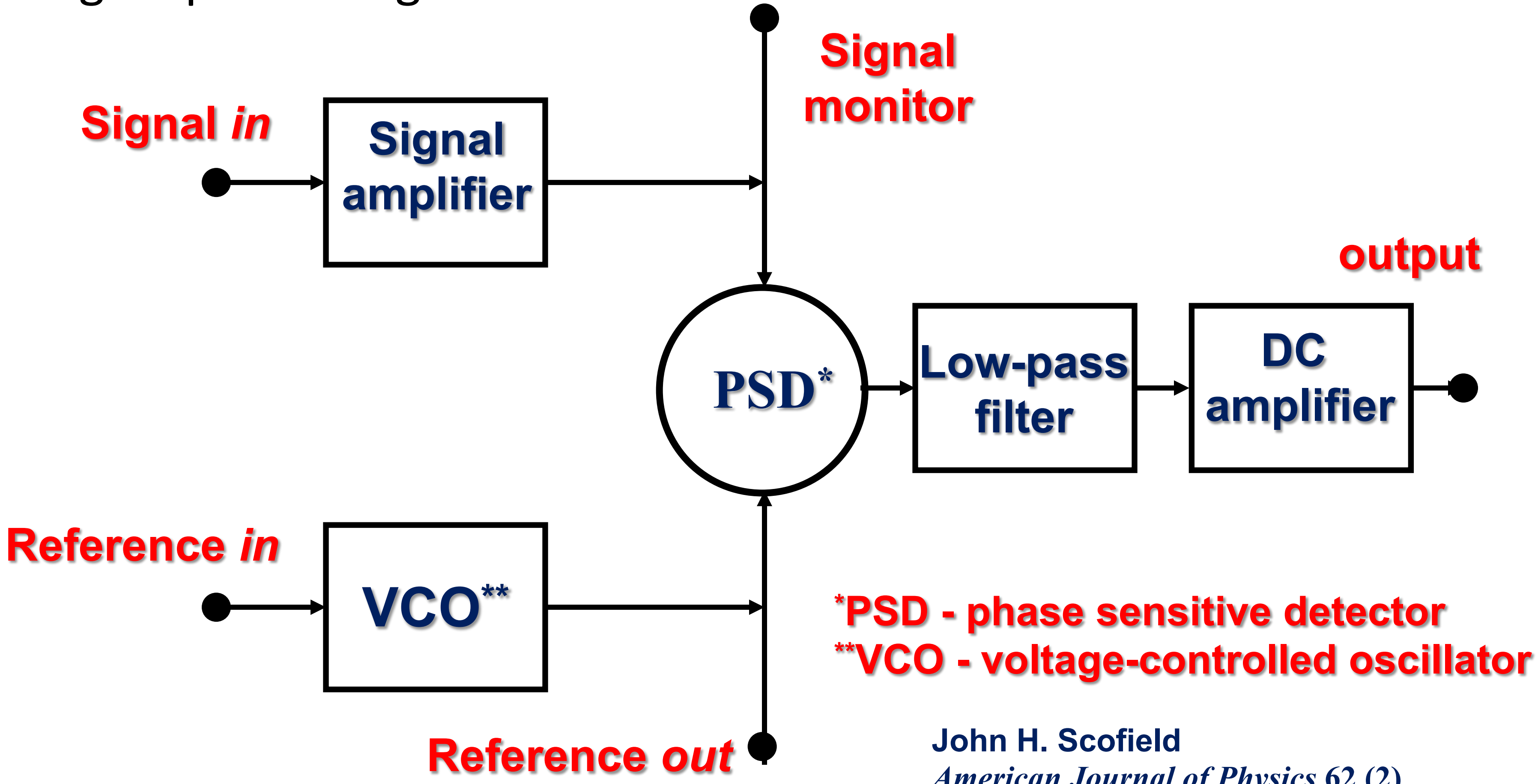


The Lock-In Amplifier

Immensely powerful and widely-applicable tool, implemented in hardware or (increasingly) digital signal processing



Modern lock-in amps credited to
Bob Dicke (1916-1997)
Princeton astronomer

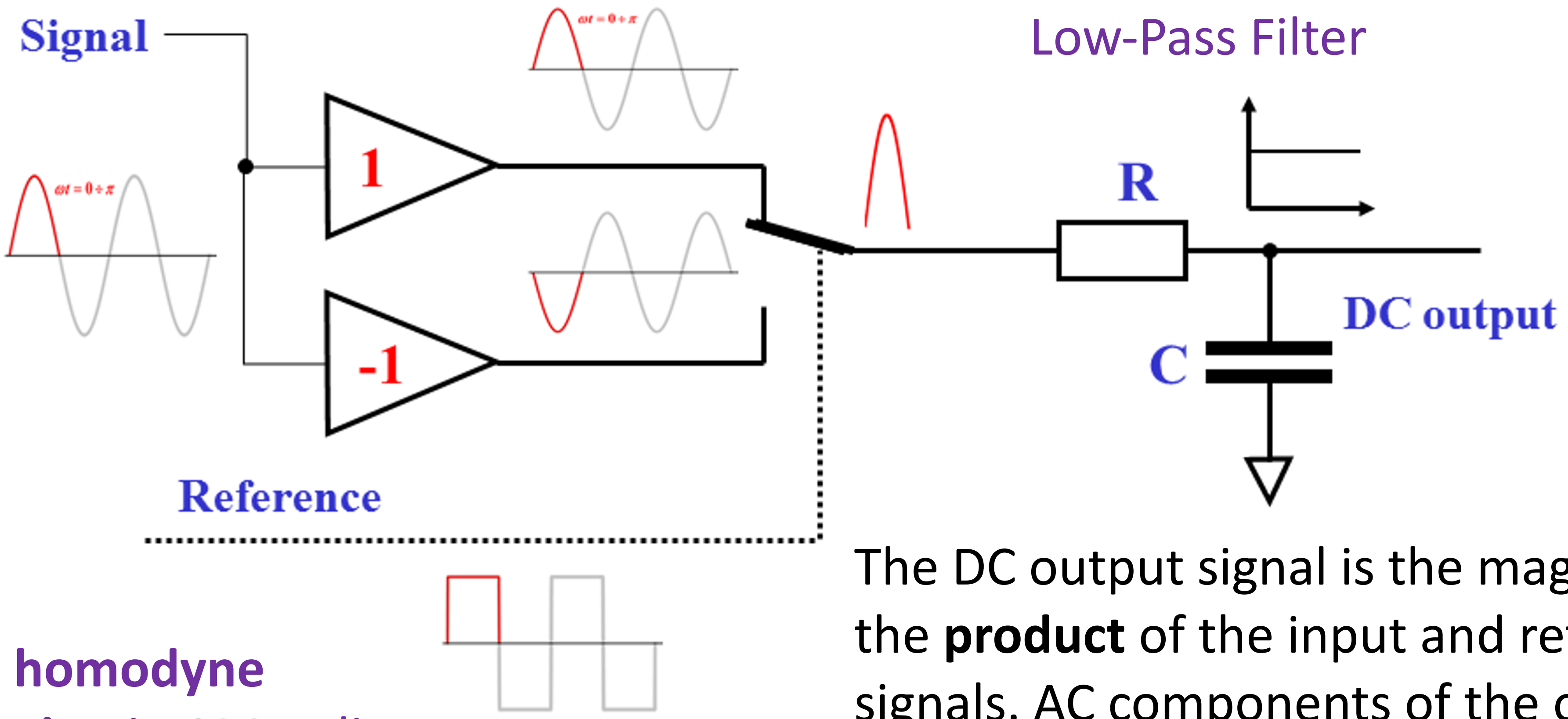


John H. Scofield
American Journal of Physics 62 (2) 129-133 (Feb. 1994).

The Lock-In Amplifier: How It Works

Multiplier (“Mixer”)

Low-Pass Filter

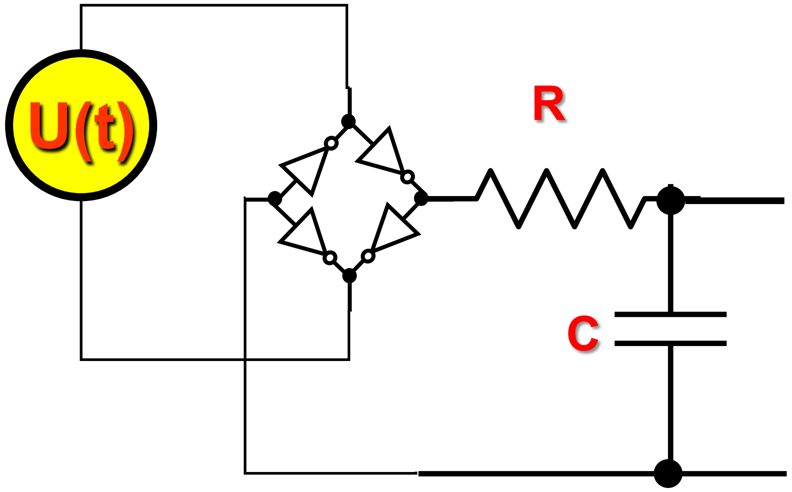


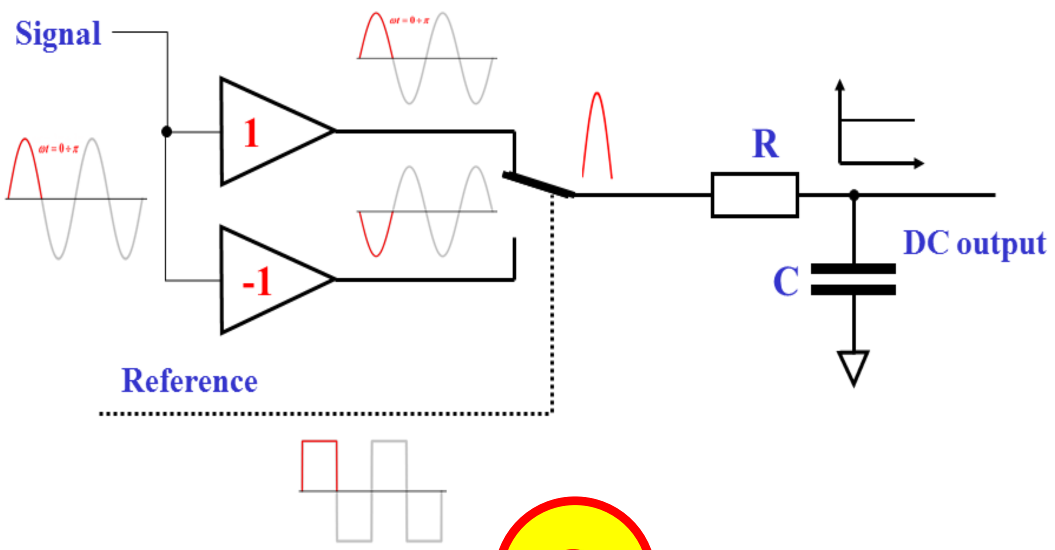
Amplitude of single Fourier component

Similar to **homodyne demodulation** in AM radio receivers

The DC output signal is the magnitude of the **product** of the input and reference signals. AC components of the output signal are filtered out by the **low-pass filter**, with time constant τ (*here τ=RC*)

The Lock-In Amplifier: How It Works





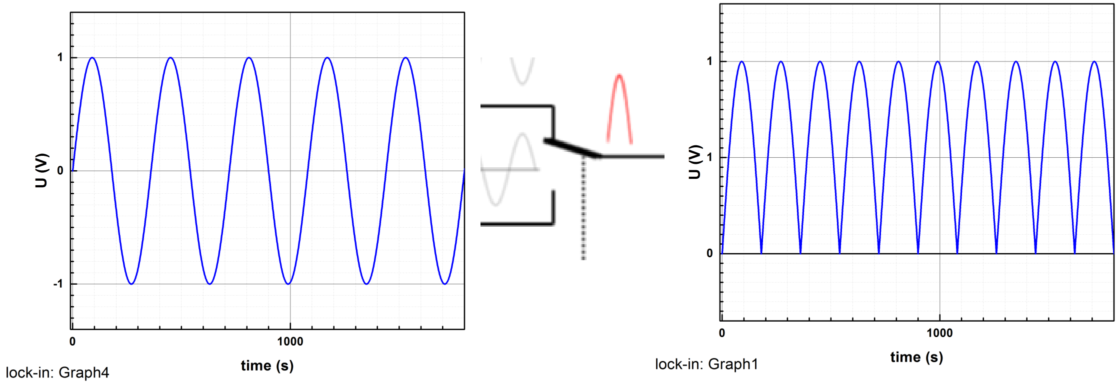
Simple implementation

1

2

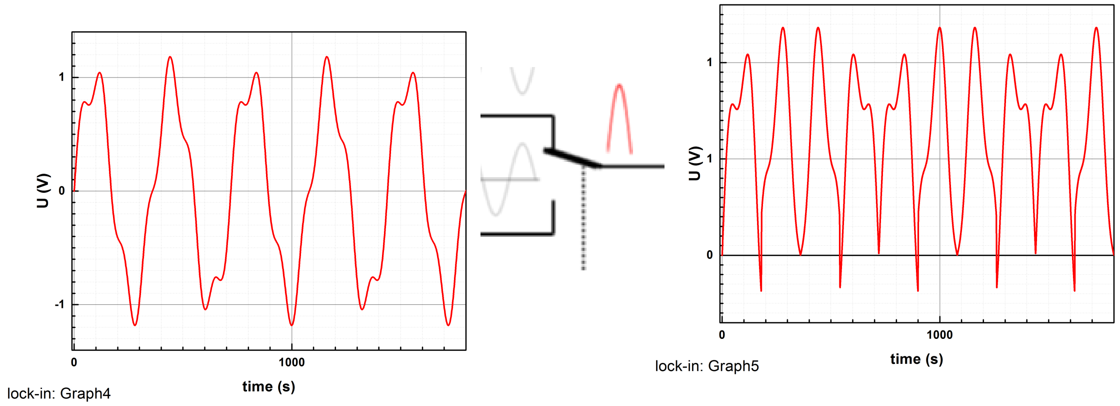
Why Synchronous Detection?

“Clean” sine wave – no noise



U_{DC} = 0.63643 V

“Noisy” sine wave



U_{DC} = 0.63643 V

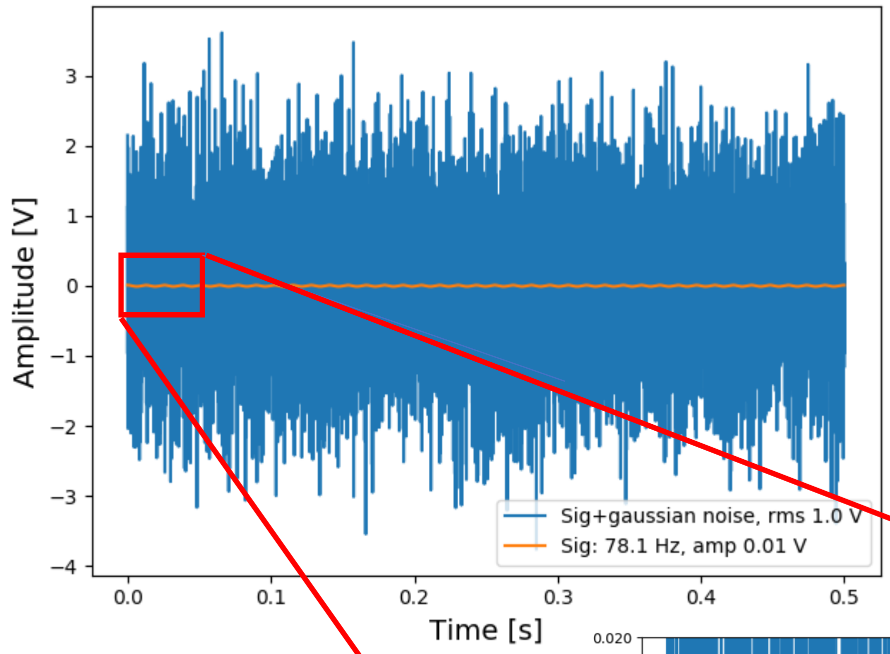
Because sine wave of different frequencies are **orthogonal**

$$\int_0^{2\pi} \sin nx \sin mx \, dx = \delta_{nm}$$

lock-ins can reject contamination *extremely* well

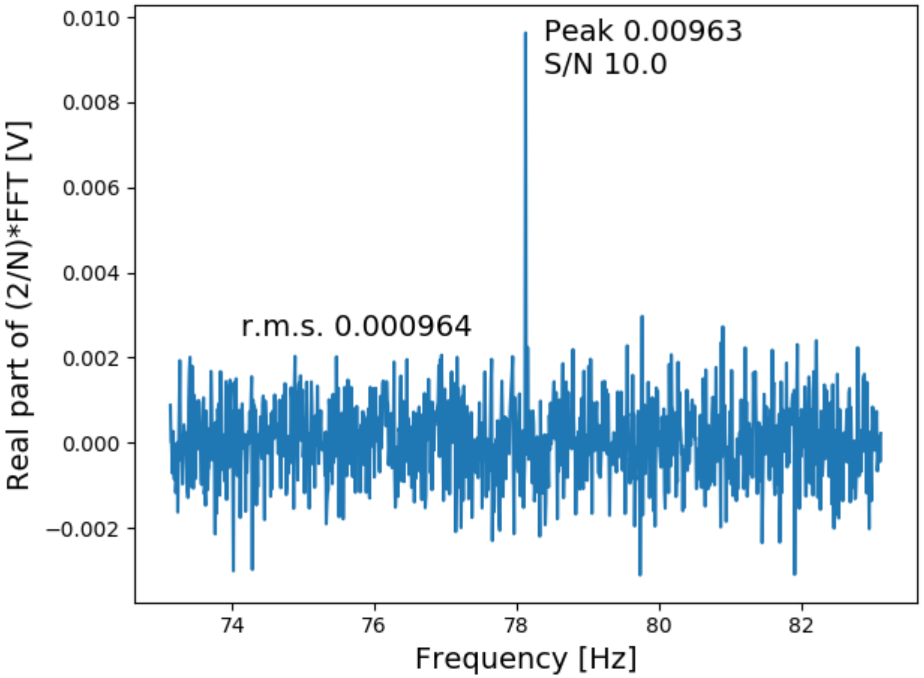
How well depends on integration time constant τ

Why Synchronous Detection?



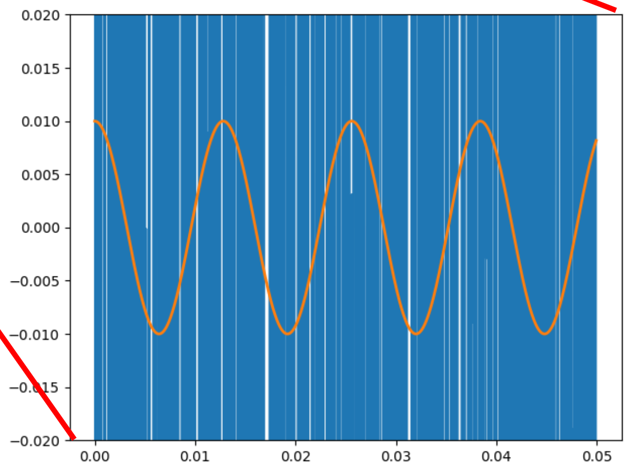
0.5s integration time

FFT



Signal amplitude
1/100th of noise rms

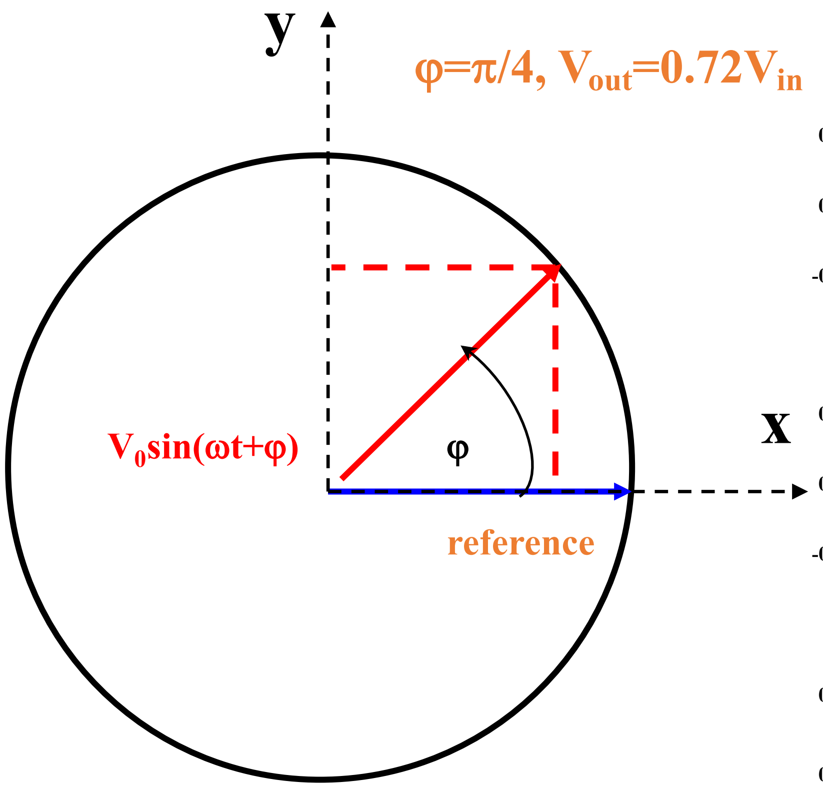
20 kHz sampling

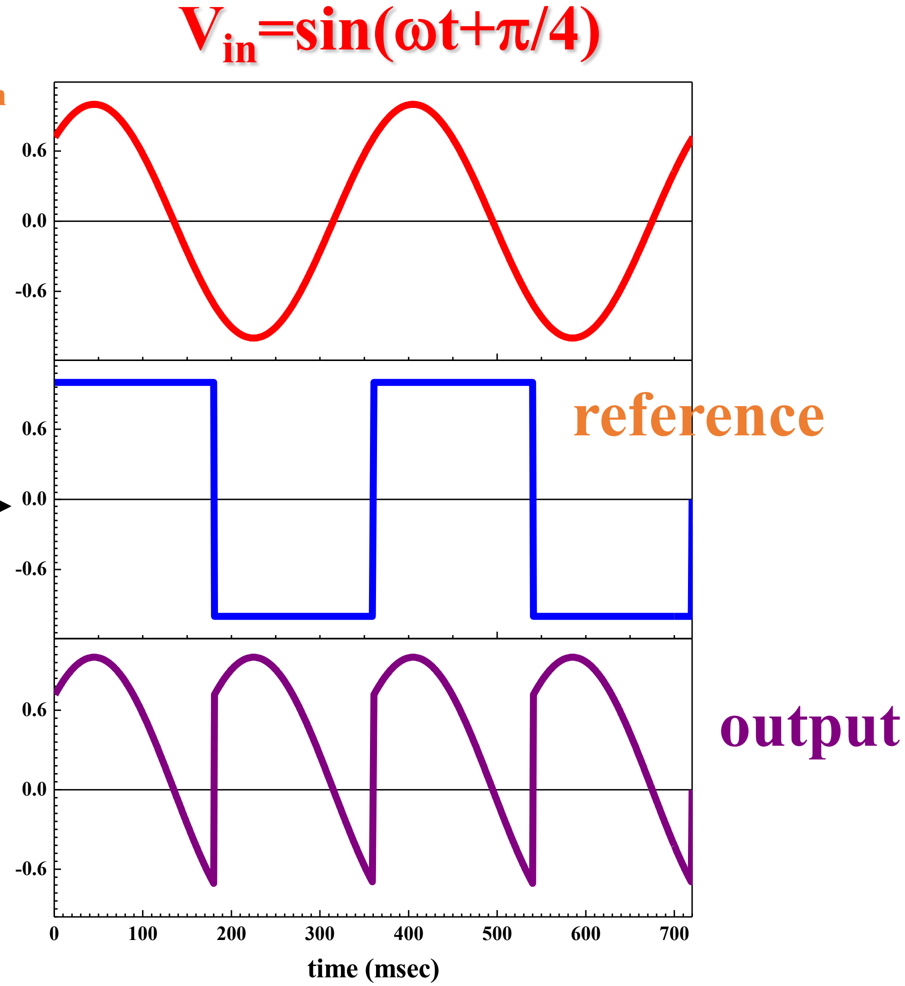


Moral: Synchronous detection can identify faint signals buried in overwhelming noise.

Precision measurements often take this form!

Lock-In Amplifier: Phase Shifts

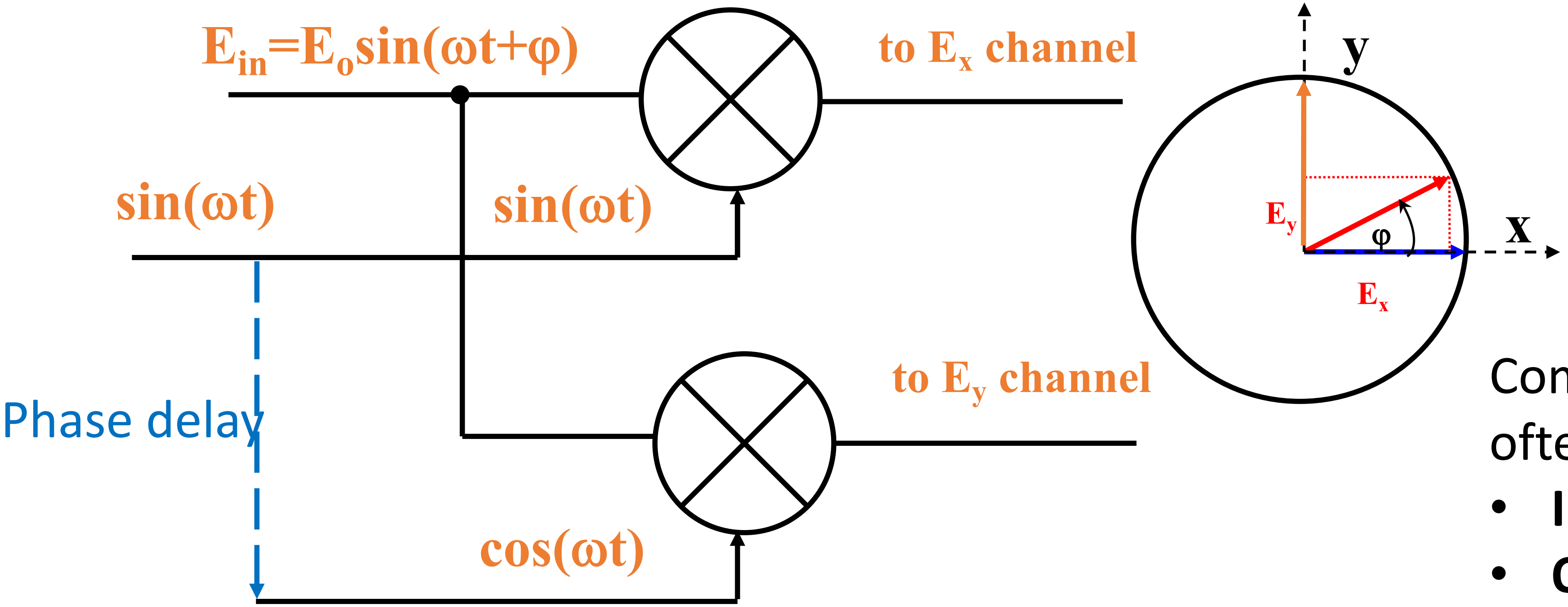




Dual-Channel Lock-In Amplifier

We can measure the **phase shift** of the output relative to the reference by demodulating with both the reference and a $\frac{\pi}{2}$ phase-shifted copy of the reference.

Yields two amplitudes (E_x/E_y, or I/Q), or equivalently an amplitude and phase shift

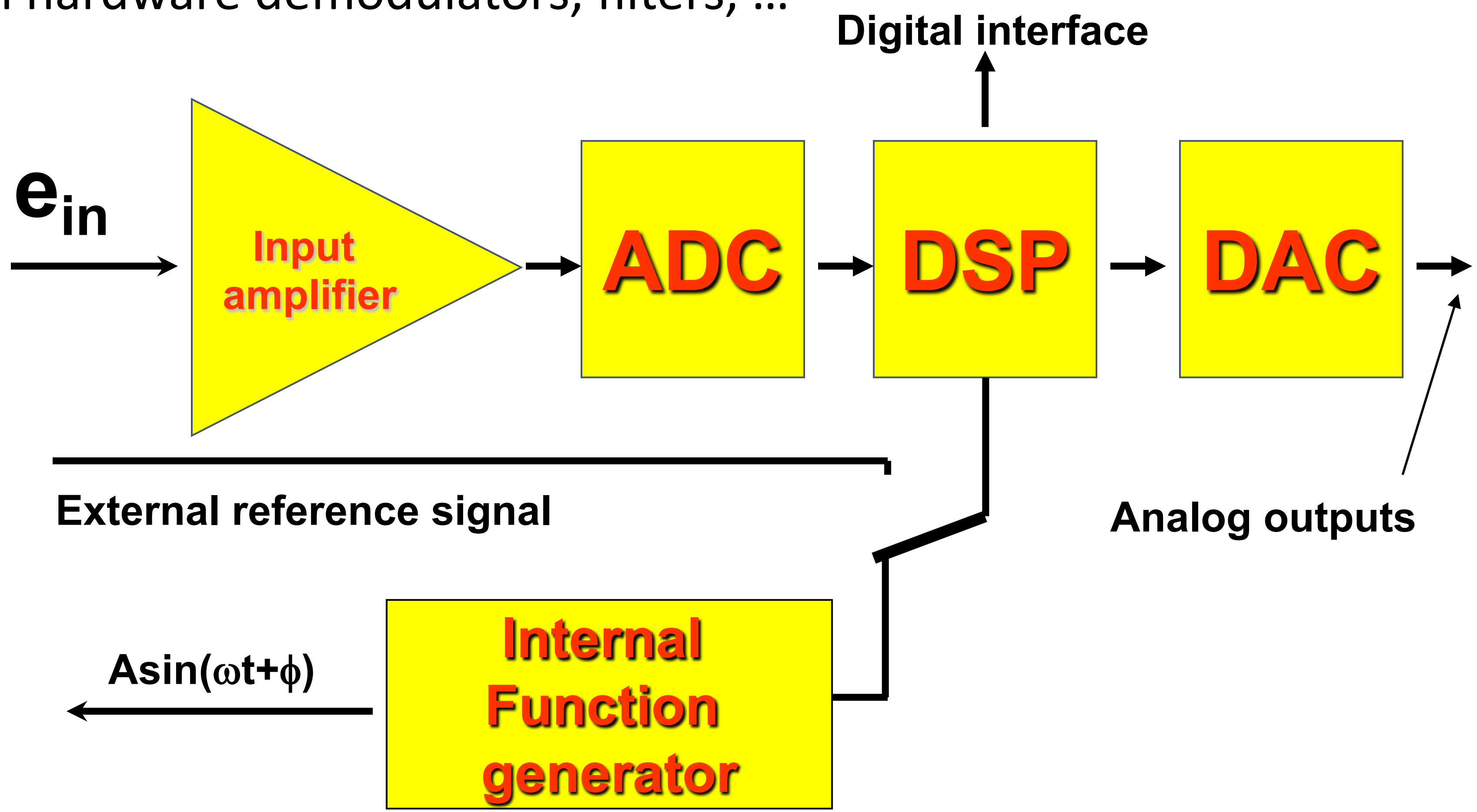


Components also often called

- **I**: in-phase
- **Q**: quadrature

Digital Lock-In Amplifier

Digitize everything and implement the above with **Digital Signal Processing (DSP)** rather than hardware demodulators, filters, …



SR830 Digital Lock-In Amplifier

Output filter time constant & order

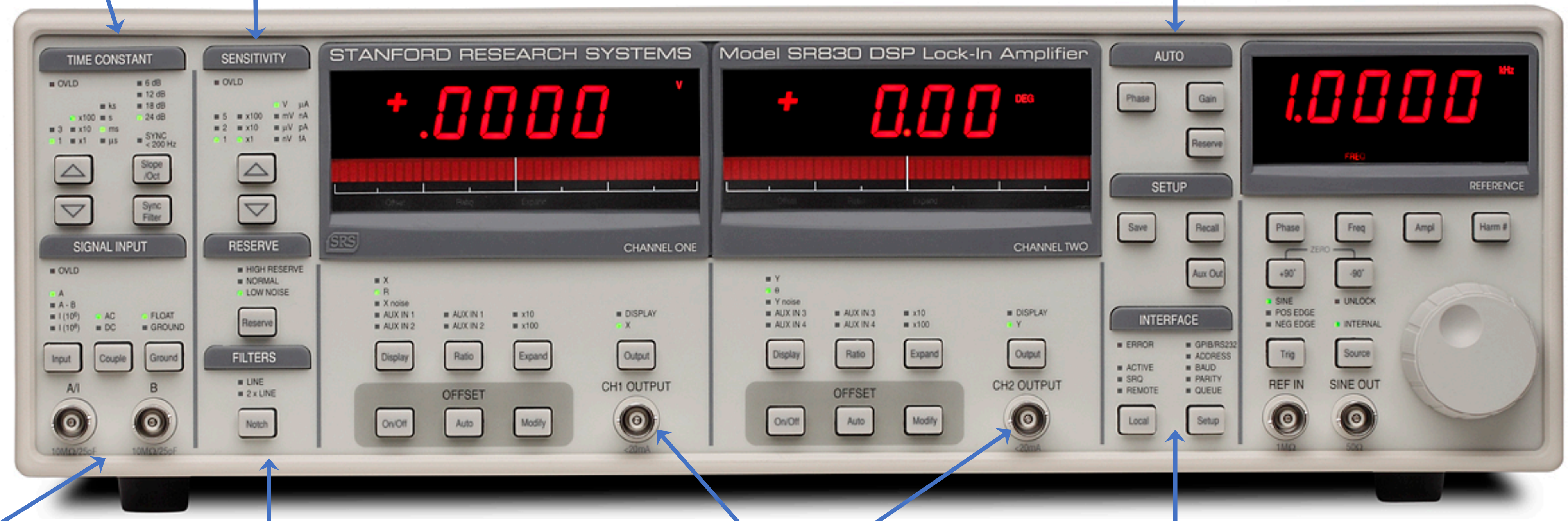
Sensitivity range

Channel #1

Channel #2

Auto settings

Function Generator



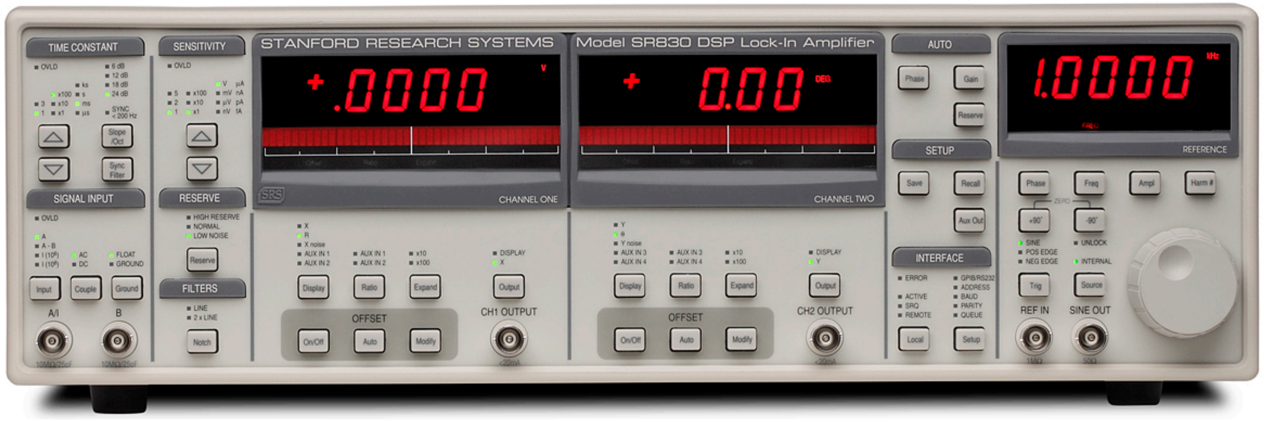
Analog inputs

Power line notch filter settings

Analog outputs

Interface settings

SR830 Digital Lock-In Amplifier



The SR830 manual includes a chapter dedicated to a general description of the lock-in amplifier concept

SR830 BASICS

WHAT IS A LOCK-IN AMPLIFIER?

Lock-in amplifiers are used to detect and measure very small AC signals - all the way down to a few nanovolts! Accurate measurements may be made even when the small signal is obscured by noise

experiment at the reference frequency. In the diagram below, the reference signal is a square wave at frequency ω_r. This might be the sync output from a function generator. If the sine output from

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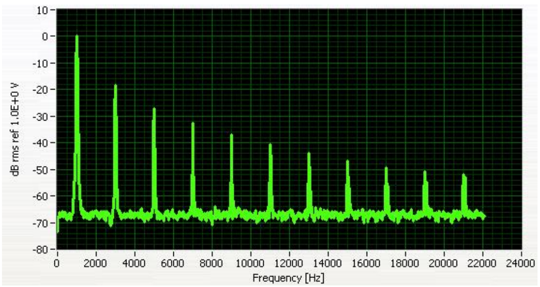

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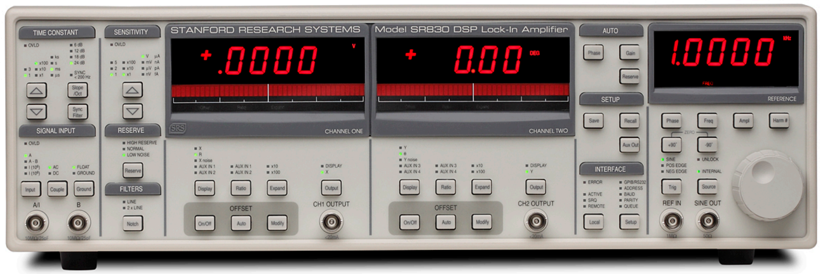

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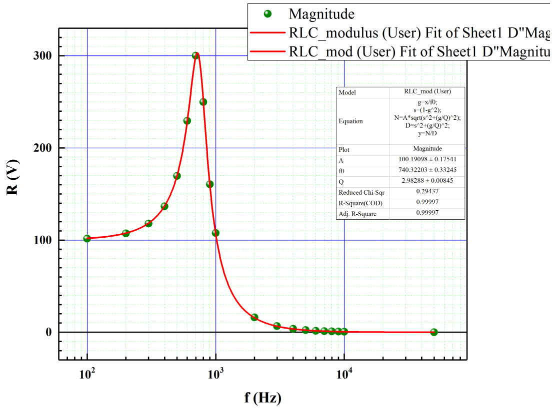
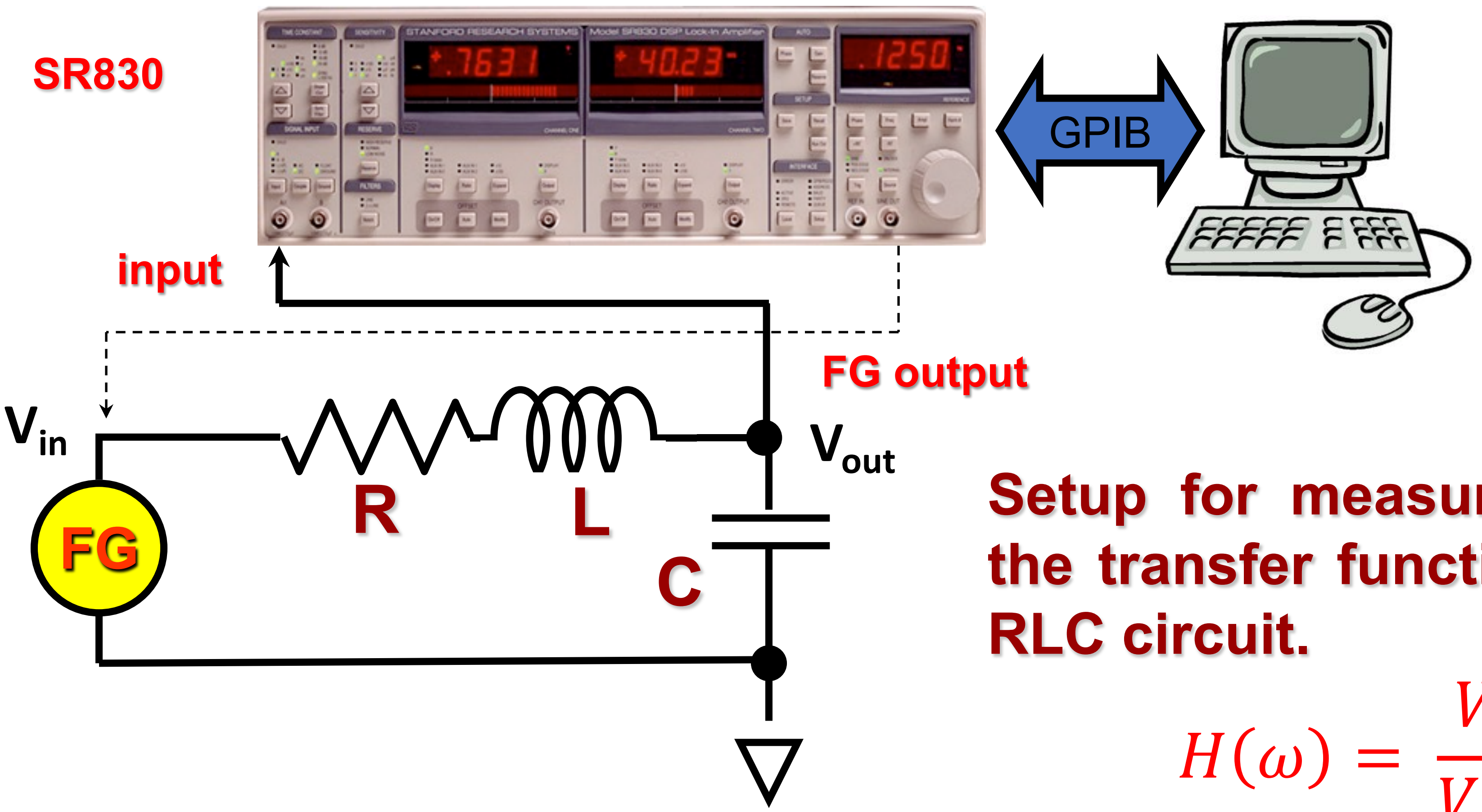


Experiment: Lock-In Measurement of the Transfer function of an RLC Circuit

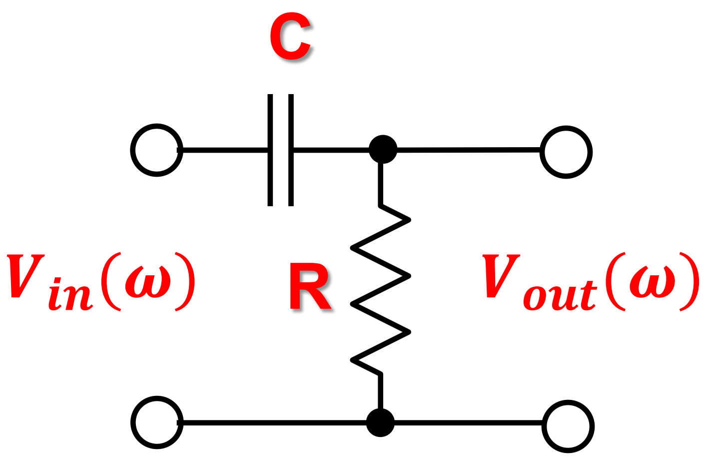


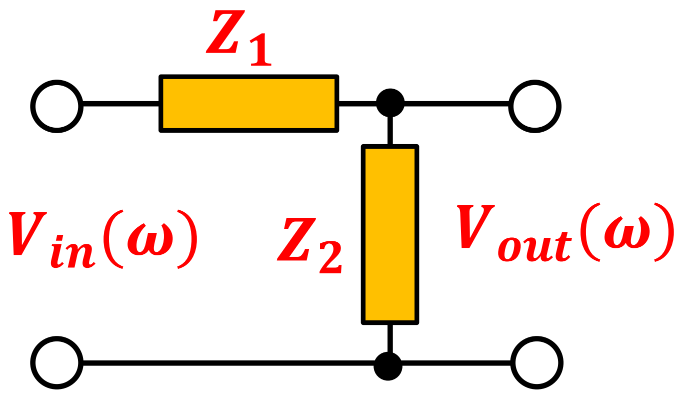
Setup for measurement of the transfer function of the RLC circuit.

$$H(\omega) = \frac{V_{in}}{V_{out}}$$

Calculating Frequency-Domain Response

Example 1. Simple high-pass filter





In the frequency domain, each linear component has a (complex) transfer function, called its **impedance**.

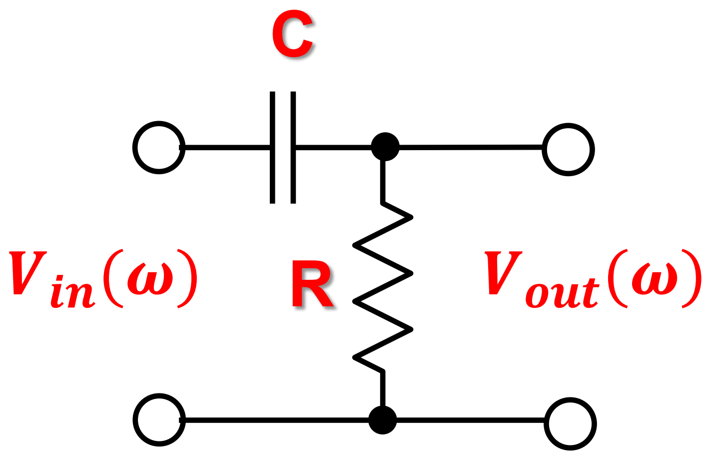
$$Z(\omega) \equiv \frac{V(\omega)}{I(\omega)}$$

Makes everything a voltage divider!

Applying Kirchhoff's Law to this simple network…

$$V_{out}(\omega) = H(\omega) * V_{in}(\omega) = \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)} V_{in}(\omega)$$

Assigning Impedances



$$Z(\omega) \equiv \frac{V(\omega)}{I(\omega)}$$

Inductor: $Ve^{i\omega t} = L\frac{d}{dt}(Ie^{i\omega t}) = i\omega L I e^{i\omega t}$

Capacitor: $CV = Q;\ C\frac{dV}{dt} = \frac{dQ}{dt} = I;\ i\omega C V e^{i\omega t} = Ie^{i\omega t}$

Ideal components

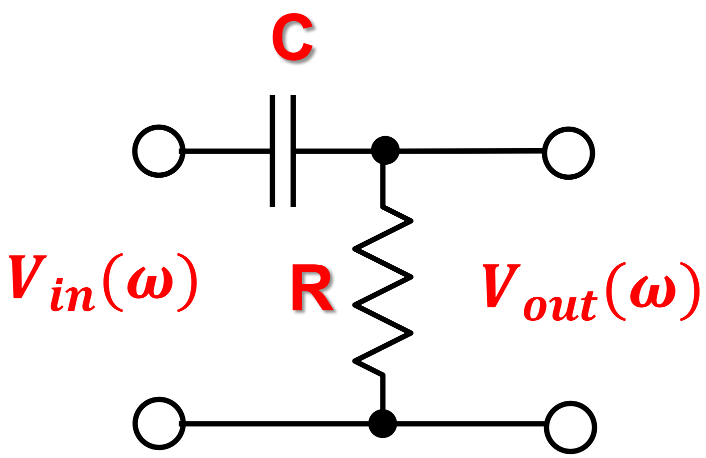
$$Z_R = R$$
$$Z_L = i\omega L$$
$$Z_C = \frac{1}{i\omega C} = -\frac{i}{\omega C}$$

More realistic…

$$Z_R = R + \cdots$$
$$Z_L = i\omega L + R_L$$
$$Z_C = \frac{1}{i\omega C + R_C^{-1}}$$

$$V_{out}(\omega) = H(\omega) * V_{in}(\omega) = \frac{Z_2(\omega)}{Z_1(\omega) + Z_2(\omega)} V_{in}(\omega)$$

High-Pass Filter Frequency-Domain Response



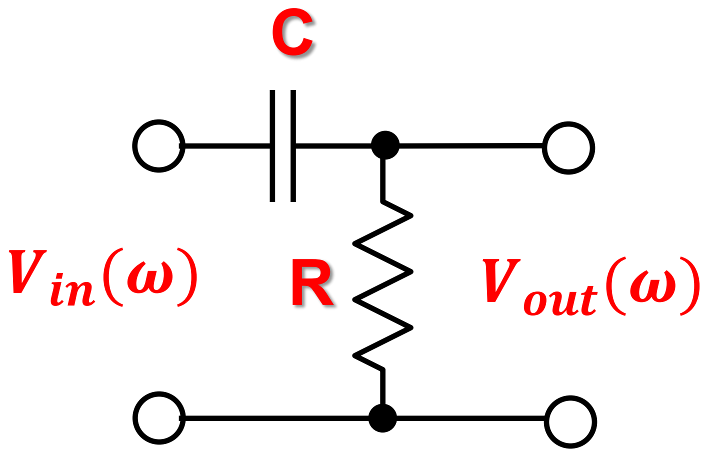
Convenient to define:

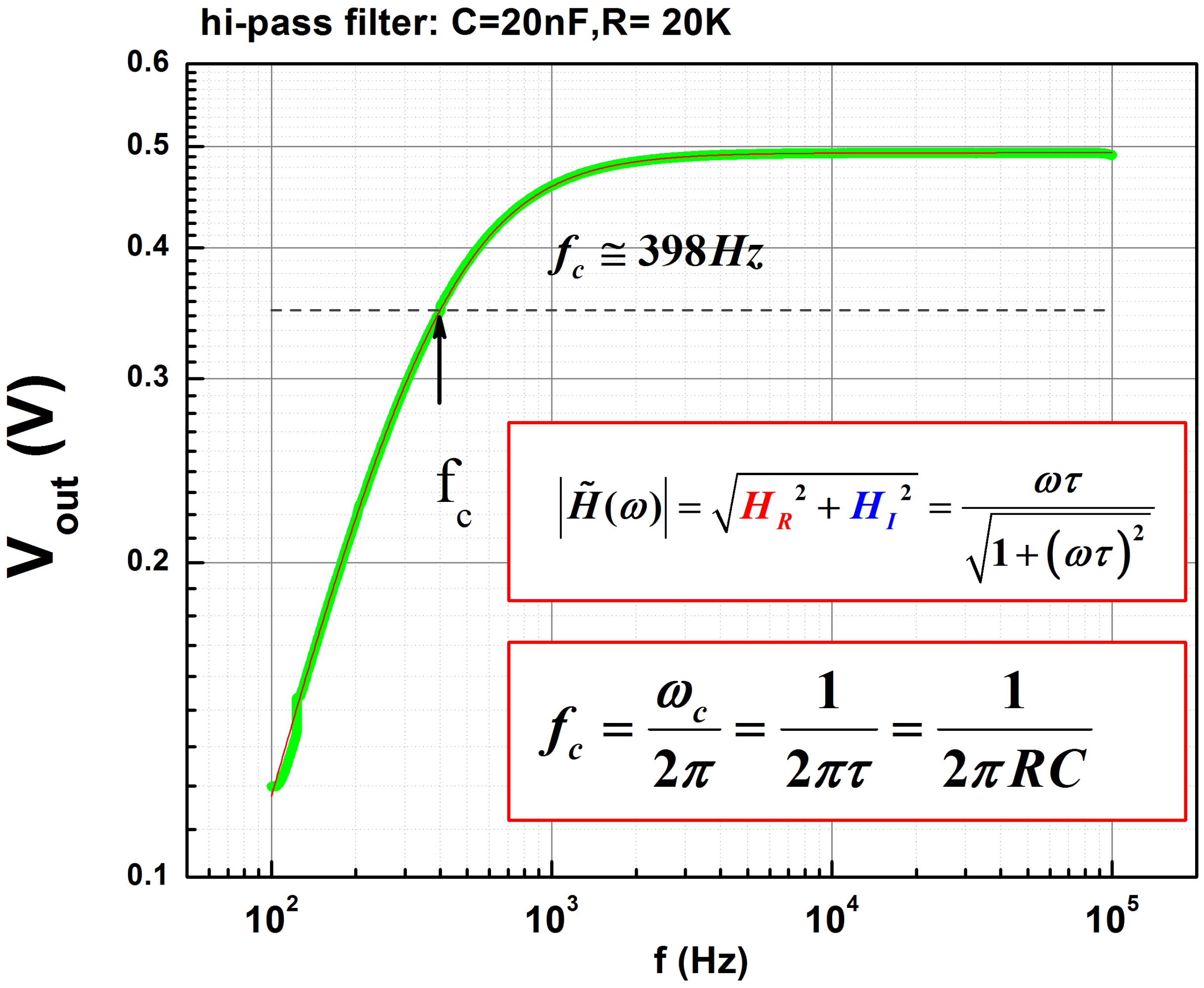
- **Time constant** $\tau = RC$ *[seconds]*
- **Cutoff frequency** $\omega_c = \frac{1}{\tau}$ *[rad/s]*

$$H(\omega) \equiv \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{R}{R + \frac{1}{i\omega C}} = \frac{i\omega RC}{1 + i\omega RC} = \frac{i\omega\tau}{1 + i\omega\tau} = \frac{\omega\tau}{1 + \omega^2\tau^2}(\omega\tau + i) = H_R(\omega) + iH_I(\omega)$$

$$|\boldsymbol{H}(\boldsymbol{\omega})| = \sqrt{H_R^2 + H_I^2} = \frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}}; \quad \boldsymbol{\Theta}(\boldsymbol{\omega}) = \tan^{-1}\left(\frac{H_I}{H_R}\right) = \tan^{-1}\left(\frac{1}{\omega\tau}\right)$$

High-Pass Filter Frequency-Domain Response





$$\left|\tilde{H}(\omega)\right| = \sqrt{H_R^2 + H_I^2} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}$$

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC}$$

Aside on Frequencies and Time Constants

Be careful of your factors of 2π!

- **Time constant** $\tau = RC$ *[seconds]*
- **Cutoff frequency** $\omega_c = \frac{1}{\tau}$ *[radians/s]*

  $f_{3dB} = \frac{\omega_c}{2\pi}$ *[cycles/s = Hz]*

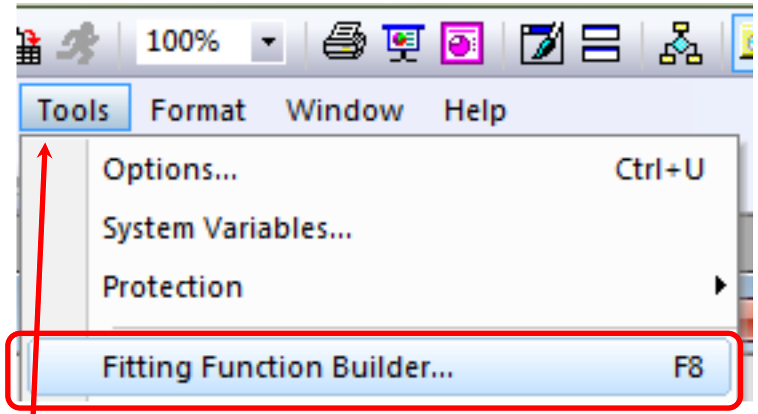
Your **formulas** will naturally use these

Your **measurements** will naturally use these

The **cutoff frequency** is when the **power** $(P \propto V^2)$ is down by a factor of 2 (*a.k.a.* -3 dB), so the **amplitude** has dropped by a factor of $\sqrt{2}$ (0.707 of peak).

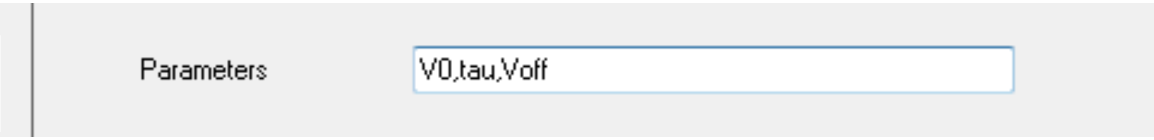
Note to remember: Time constants are the inverses of *omegas* (angular frequencies), not *frequencies*!

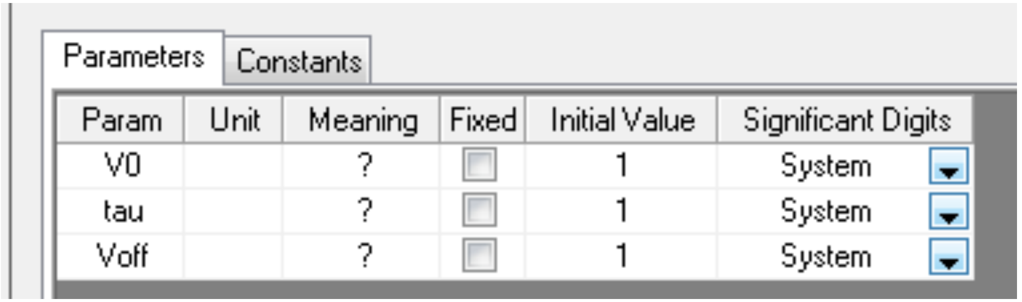
High-Pass Filter: Fitting



$$\left|\tilde{V}_{out}\right| = \left|\tilde{V}_{in}\right| * \left|\tilde{H}(\omega)\right| = V_0 * \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}}; \quad \tau = RC$$

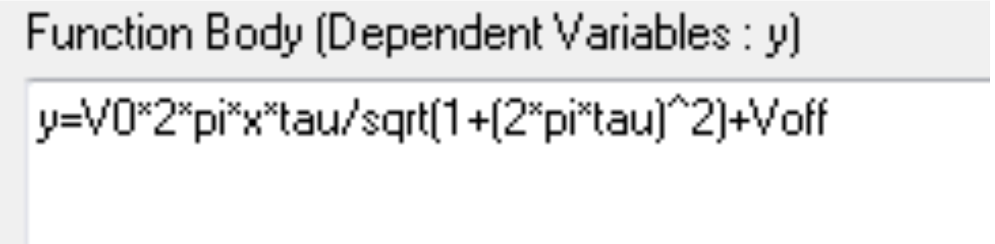
Fitting parameters: V_0, τ, Voff



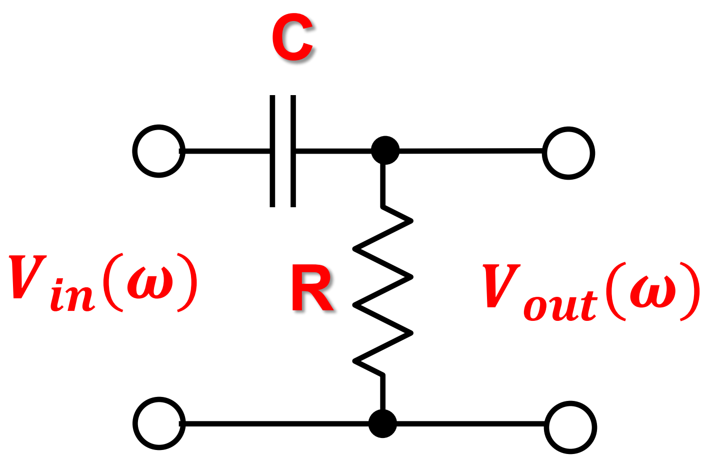


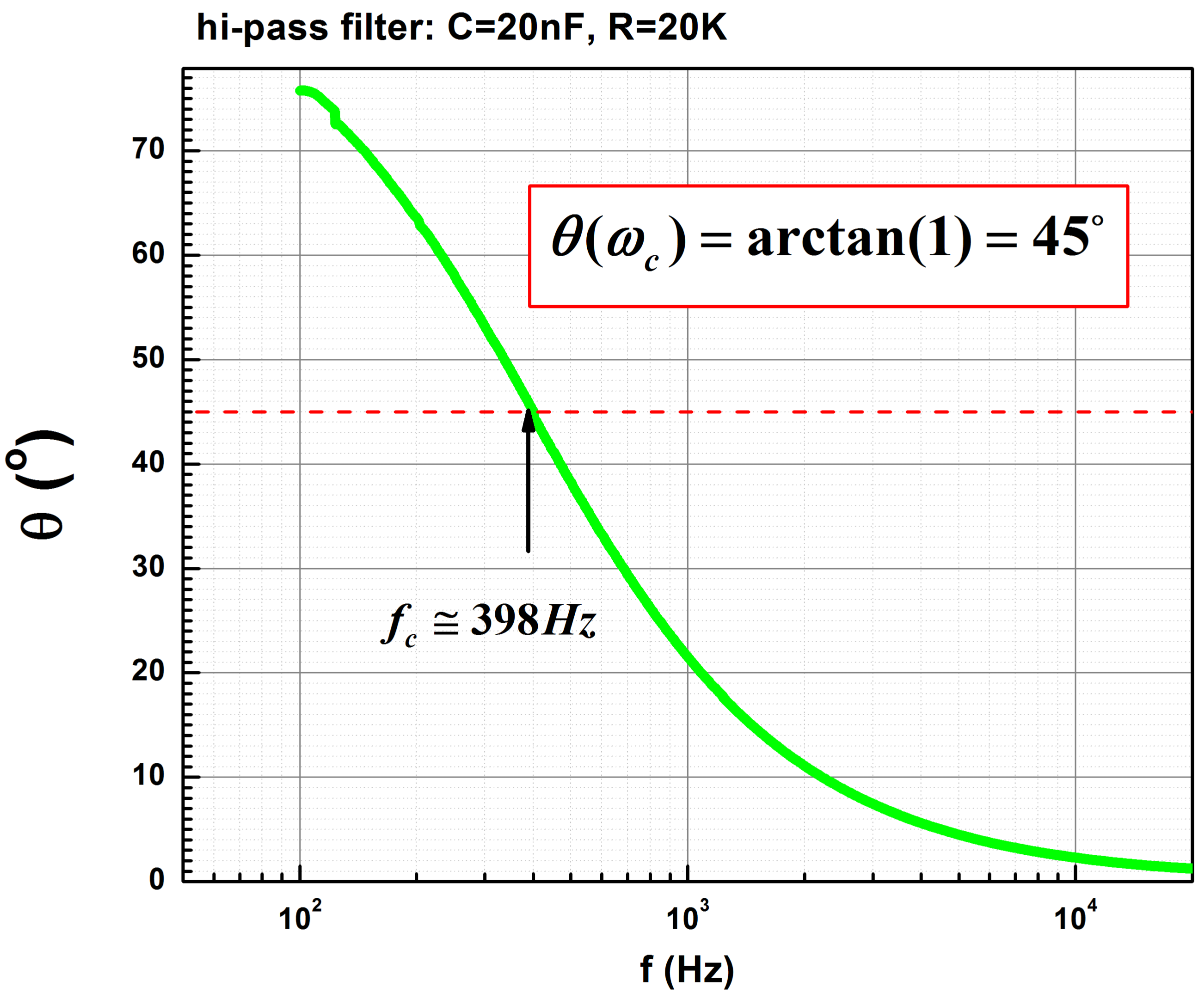
Param	Unit	Meaning	Fixed	Initial Value	Significant Digits
V0		?		1	System
tau		?		1	System
Voff		?		1	System

Fitting function

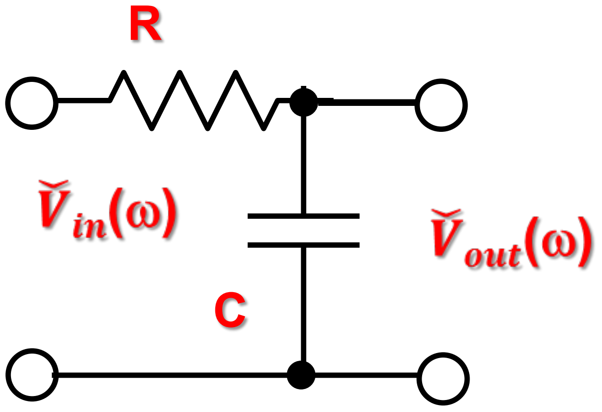


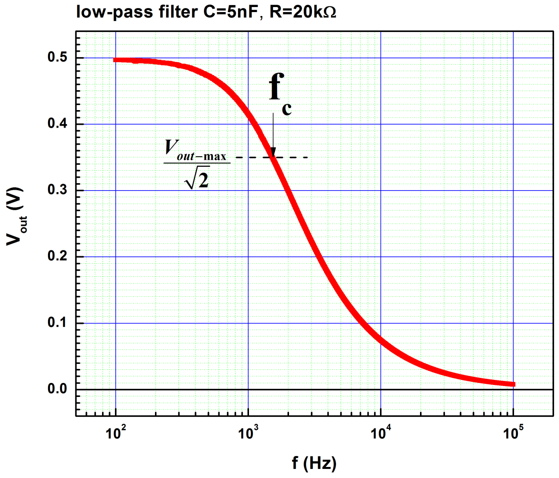
High-Pass Filter Frequency-Domain Response

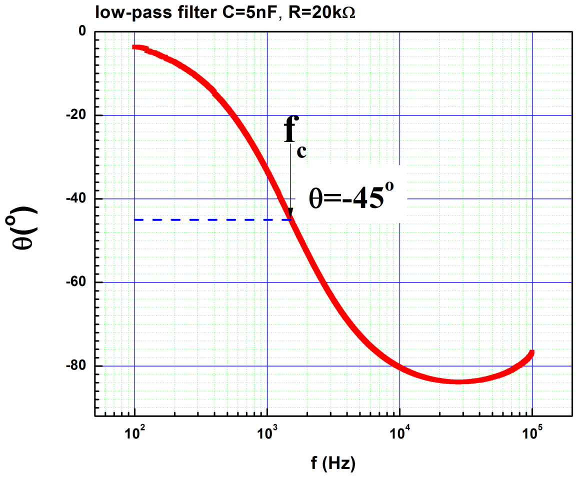




Low-Pass Filter Frequency-Domain Response





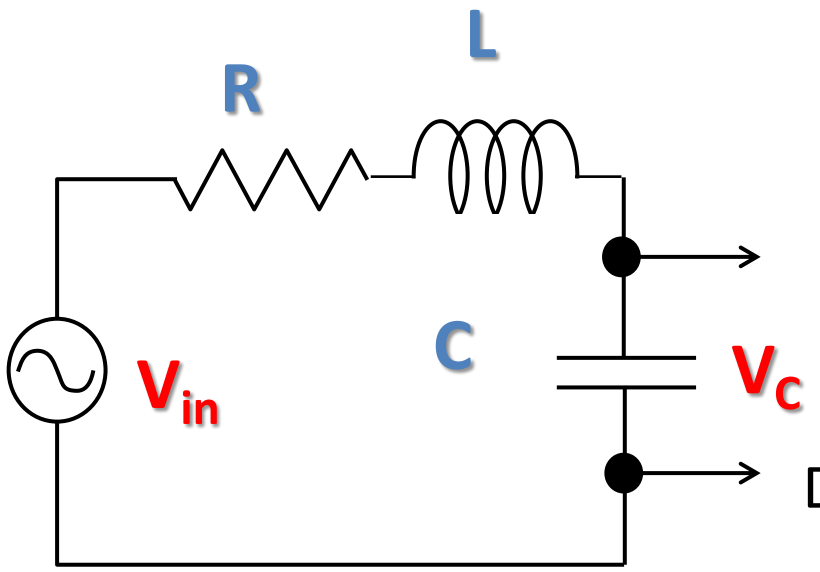


$$H_R(\omega) + iH_I(\omega)$$

$$H(\omega) \equiv \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{1 + i\omega RC} = \frac{1}{1 + i\omega\tau} = \frac{1 - i\omega\tau}{1 + \omega^2\tau^2}$$

$$|\boldsymbol{H(\omega)}| = \sqrt{H_R^2 + H_I^2} = \frac{1}{\sqrt{1+\omega^2\tau^2}}; \quad \boldsymbol{\Theta(\omega)} = \tan^{-1}\left(\frac{H_I}{H_R}\right) = -\tan^{-1}(\omega\tau)$$

RLC Circuit Frequency-Domain Response



$$H(\omega) = \frac{V_c(\omega)}{V_{in}(\omega)} = \frac{\frac{1}{i\omega C}}{R + i\omega L + \frac{1}{i\omega C}} = \frac{1}{1 + i\omega RC - \omega^2 LC}$$

Defining $\boldsymbol{\omega_0} \equiv \frac{1}{\sqrt{LC}}$; $\boldsymbol{\tau} \equiv RC$; $\boldsymbol{Q} \equiv \frac{1}{\omega_0 \tau}$ and rearranging…

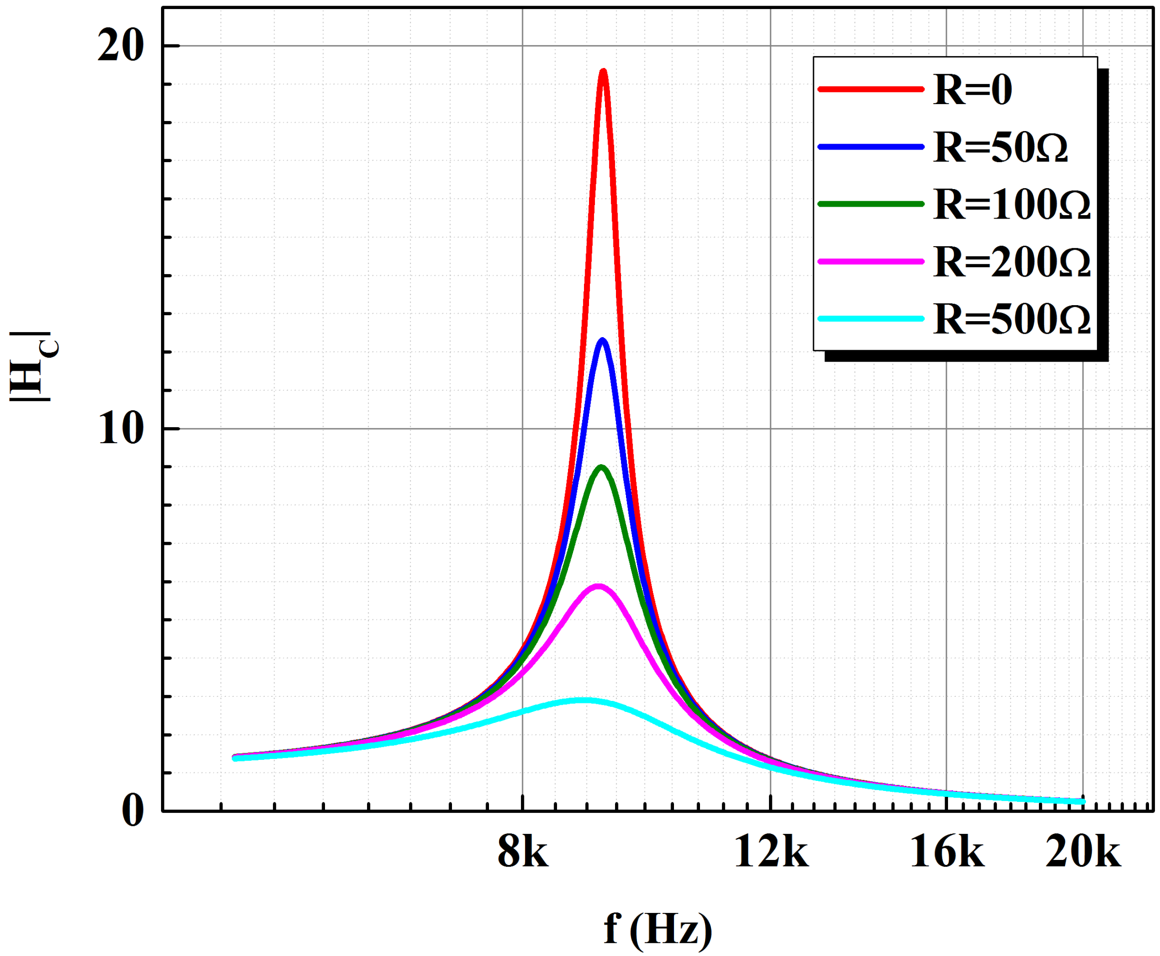
$$H(\omega) = \frac{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) - i\omega\tau}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + (\omega\tau)^2} = \frac{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) - \frac{i}{Q}\left(\frac{\omega}{\omega_0}\right)}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}$$

RLC Circuit Frequency-Domain Response

$$\boldsymbol{\omega_0} \equiv \frac{1}{\sqrt{LC}}; \quad \boldsymbol{\tau} \equiv RC; \quad \boldsymbol{Q} \equiv \frac{1}{\omega_0 \tau}$$

$$|H(\omega)| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

$$\Theta = \tan^{-1}\left(\frac{\left(\frac{\omega}{\omega_0}\right)}{Q\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$



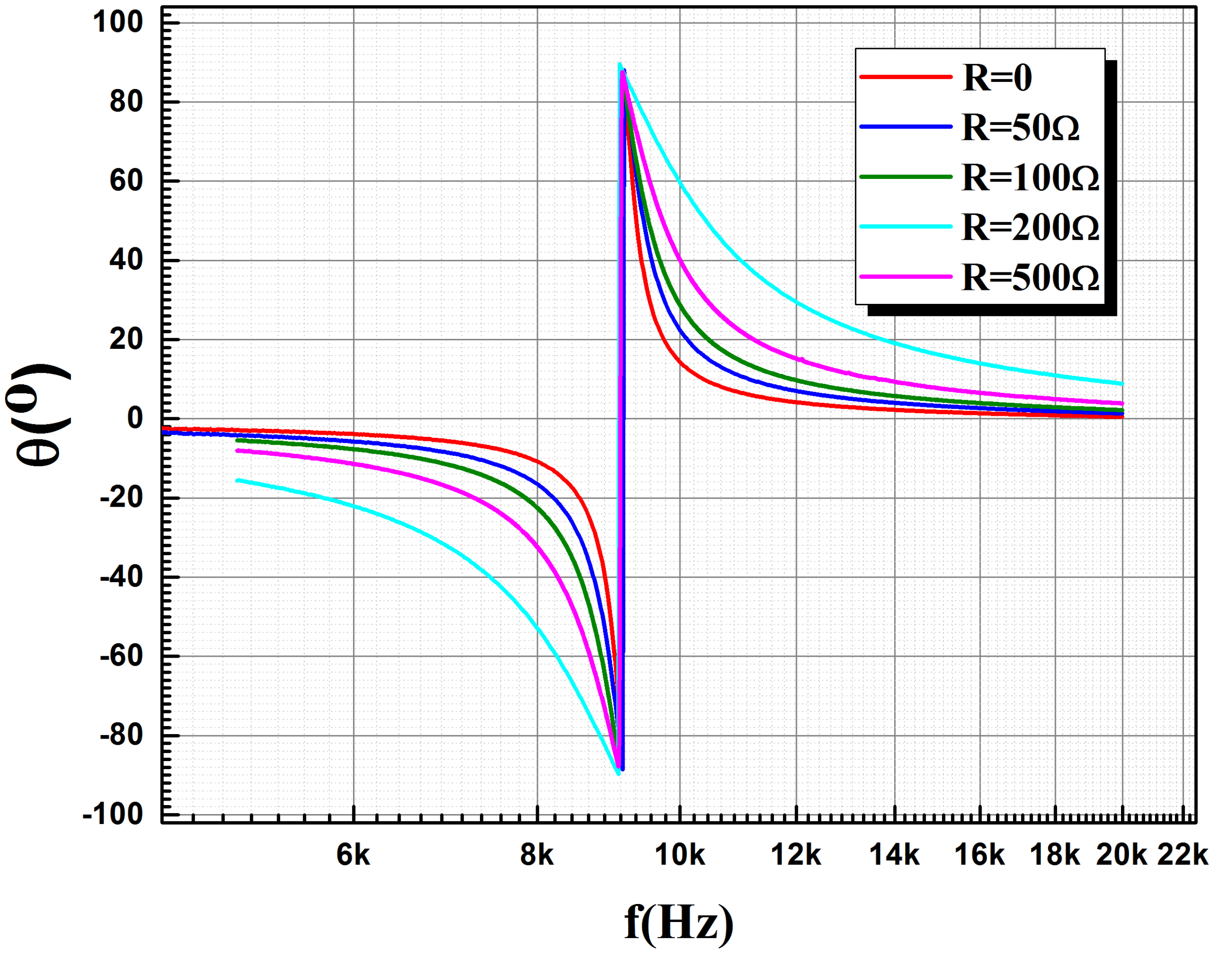
Resonance curves with varying **R** (and thus **Q**)

RLC Circuit Frequency-Domain Response

$$\boldsymbol{\omega_0} \equiv \frac{1}{\sqrt{LC}}; \quad \boldsymbol{\tau} \equiv RC; \quad \boldsymbol{Q} \equiv \frac{1}{\omega_0 \tau}$$

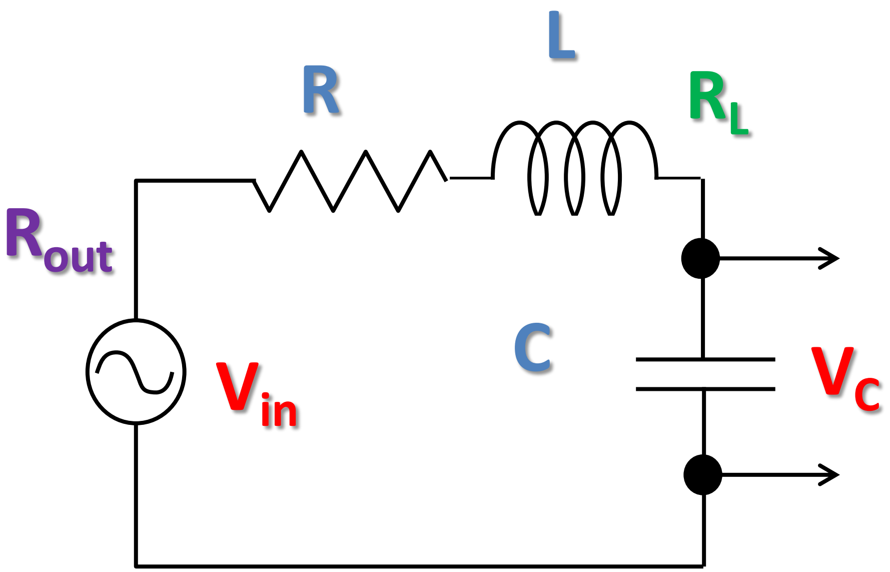
$$|H(\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

$$\Theta = \tan^{-1}\left(\frac{\left(\frac{\omega}{\omega_0}\right)}{Q\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$



Resonance curves with varying **R** (and thus **Q**)

Applying a Lock-In to Study RLC Response



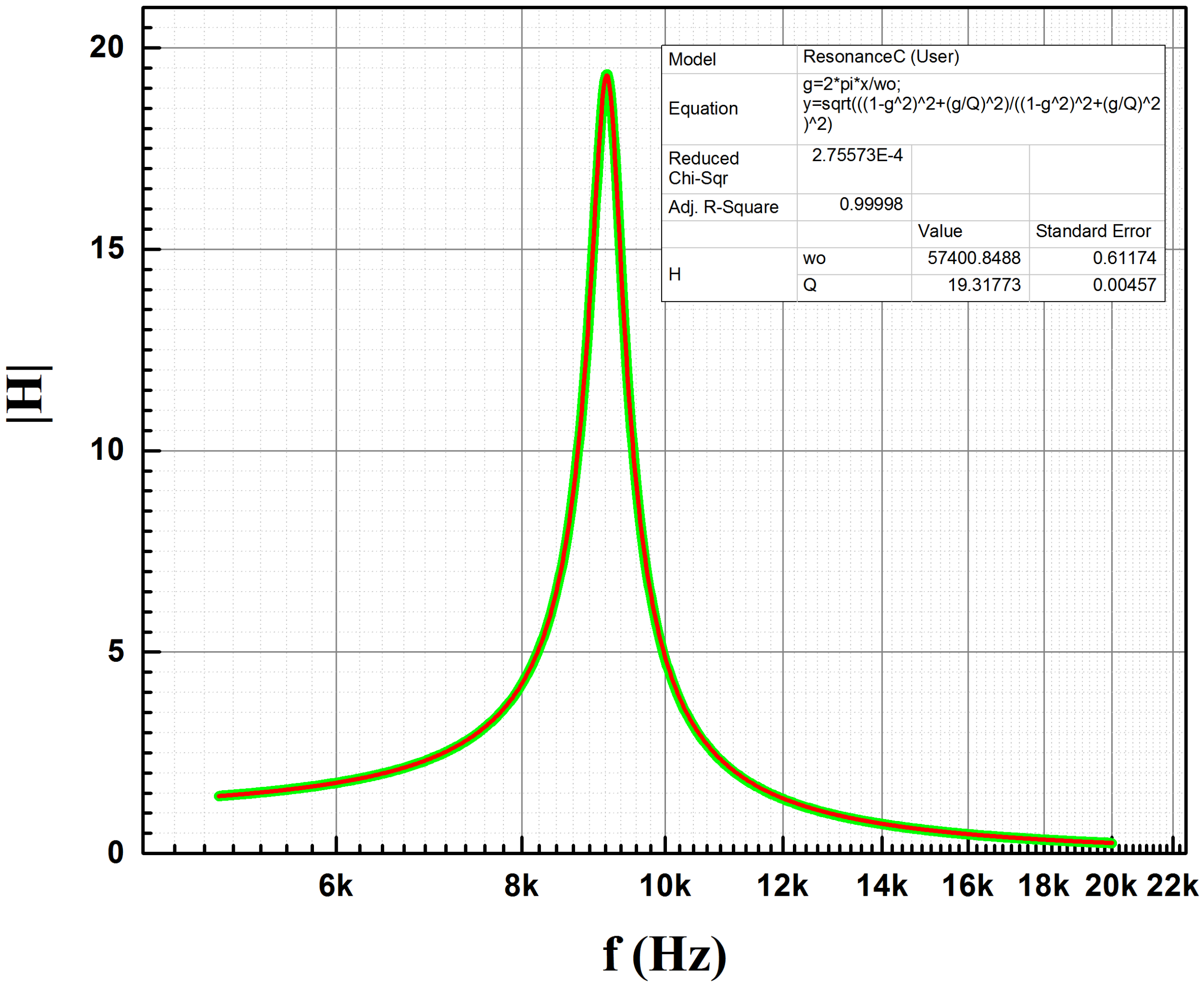
R=0; R_{out} = 50 Ω; R_L = 35.8 Ω

Actual damping resistance R is the sum of:

- **R**: the explicit resistor
- R_{out}: the function generator output impedance
- R_L: the resistance of the coil

Actual R from fitting parameters is ~88.8 ohms, not far from calculated 85.8 ohms

Fitting RLC Circuit Response



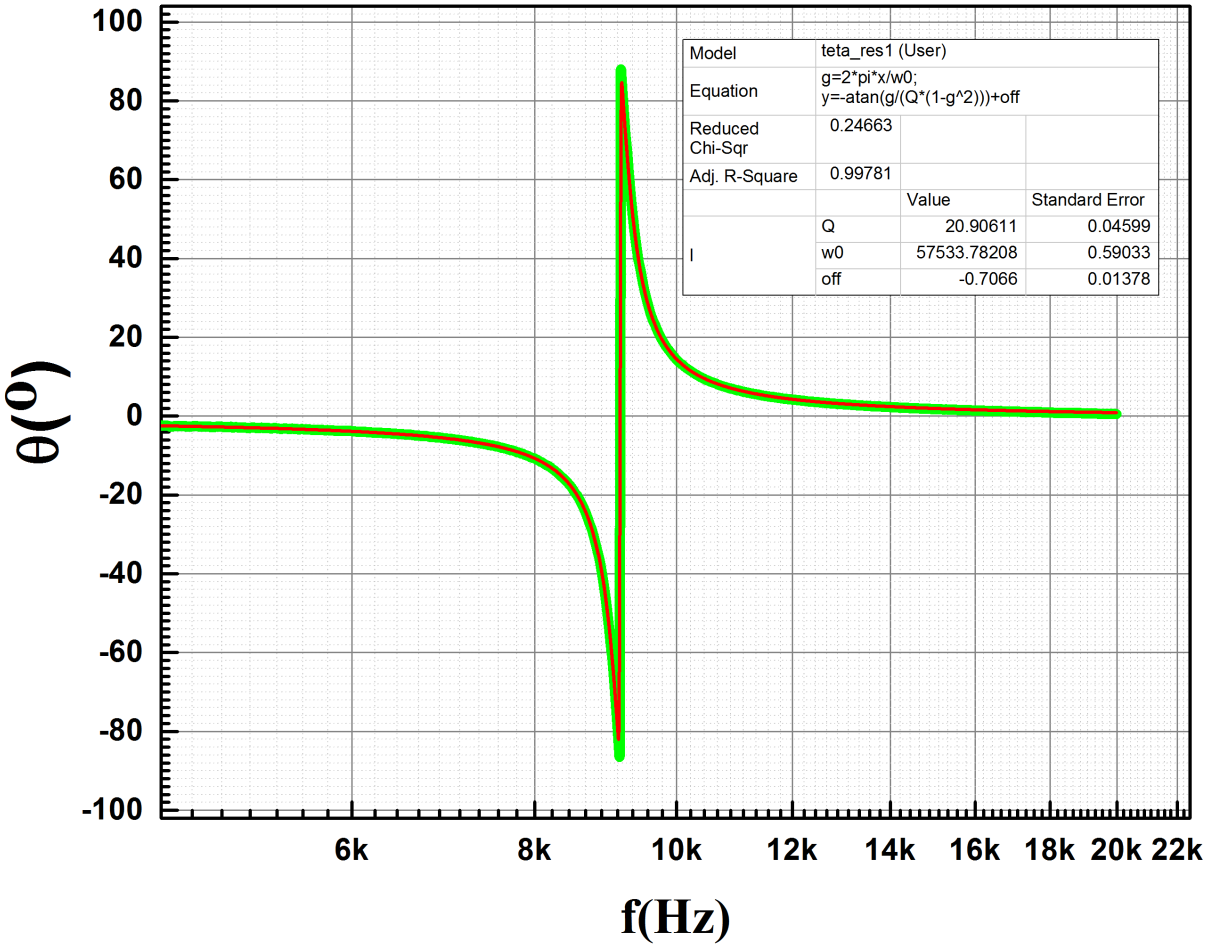
$$\boldsymbol{\omega_0} \equiv \frac{1}{\sqrt{LC}}; \quad \boldsymbol{\tau} \equiv RC; \quad \boldsymbol{Q} \equiv \frac{1}{\omega_0 \tau}$$

$$|H(\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

Fitting for amplitude $|\boldsymbol{H}|$

Fit parameters $\boldsymbol{\omega_0}$ and $\boldsymbol{Q}$

Fitting RLC Circuit Response



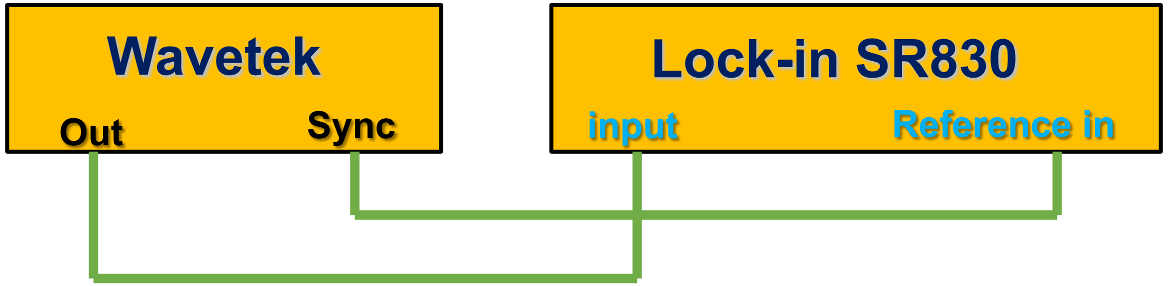
$$\boldsymbol{\omega_0} \equiv \frac{1}{\sqrt{LC}}; \quad \boldsymbol{\tau} \equiv RC; \quad \boldsymbol{Q} \equiv \frac{1}{\omega_0 \tau}$$

$$\Theta = \tan^{-1}\left(\frac{\left(\frac{\omega}{\omega_0}\right)}{Q\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$

Fitting for amplitude $|\boldsymbol{H}|$

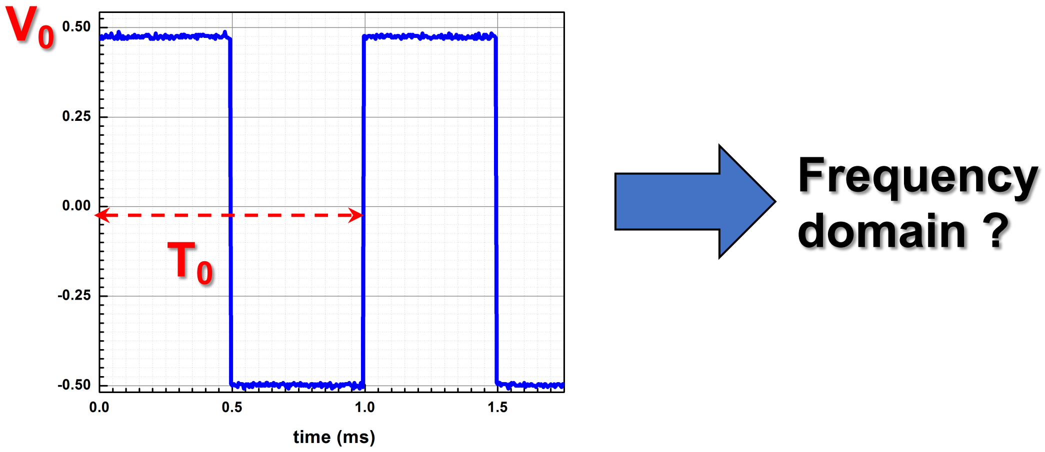
Fit parameters $\boldsymbol{\omega_0}$ and $\boldsymbol{Q}$

From Time Domain to Frequency Domain: Experiment



$$V(t) = \begin{cases} V_0: 0 < t \le \frac{T_0}{2} \\ -V_0: \frac{T_0}{2} < t \le T_0 \end{cases}$$

F(t) – periodic function $F(t)=F(t+T_0)$:



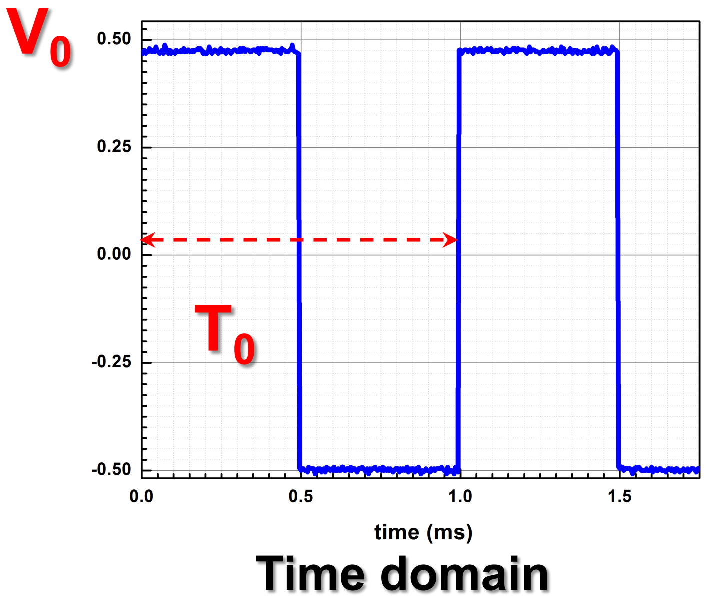
Time domain pattern

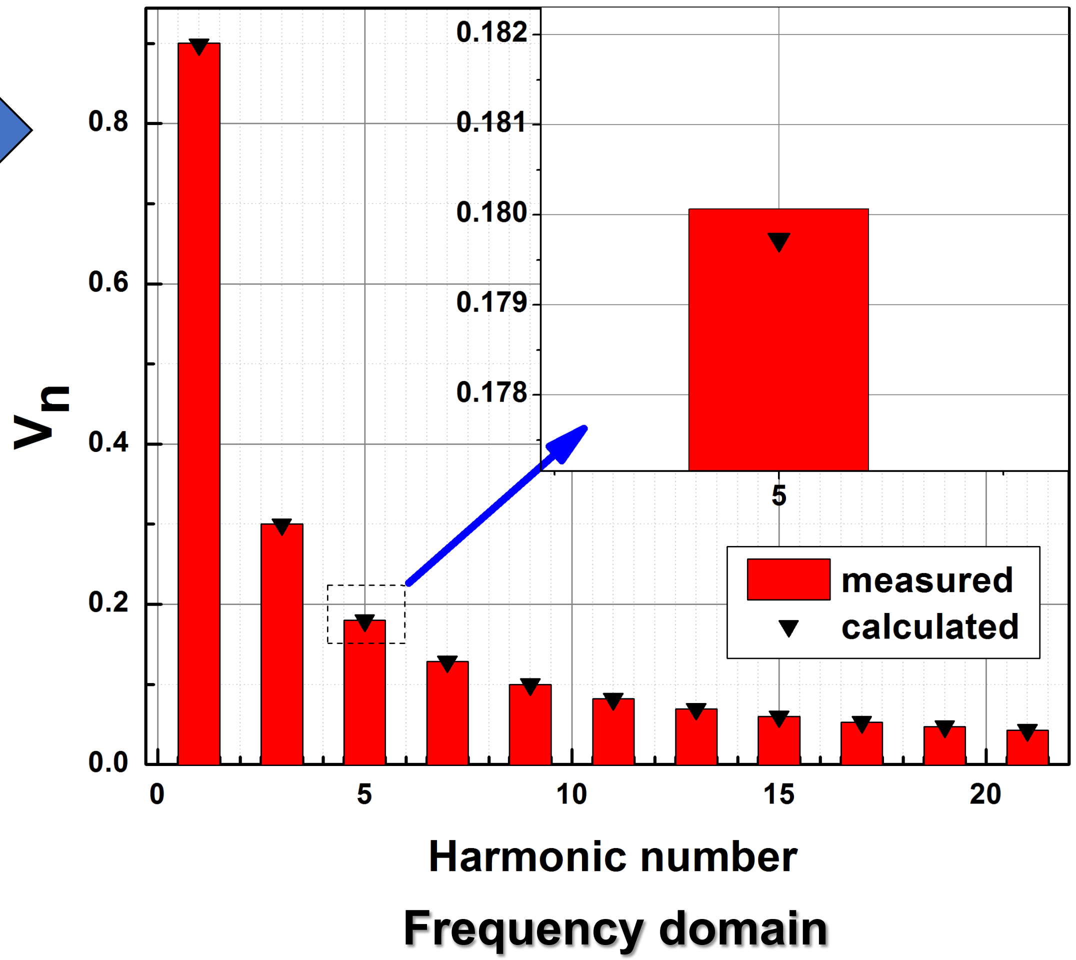
$$a_n = \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi n t}{T_0}\right)dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} F(t)\sin\left(\frac{2\pi n t}{T_0}\right)dt$$

$$a_0 = \frac{2}{T_0}\int_0^{T_0} F(t)\,dt$$

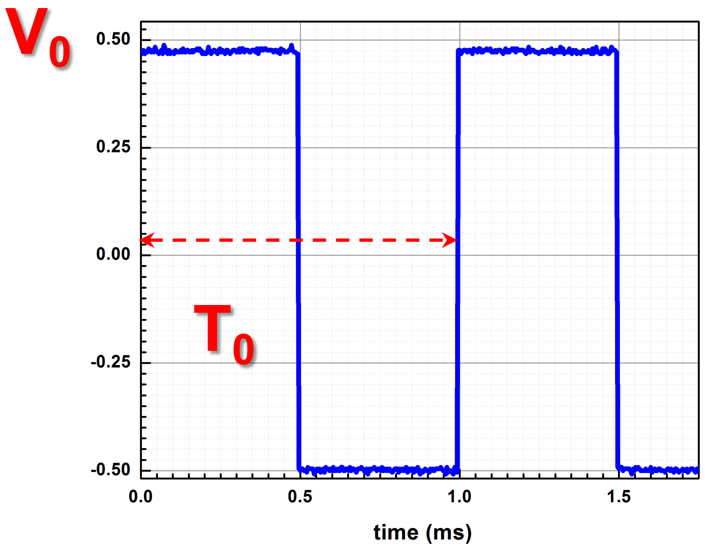
From Time Domain to Frequency Domain: Lock-In





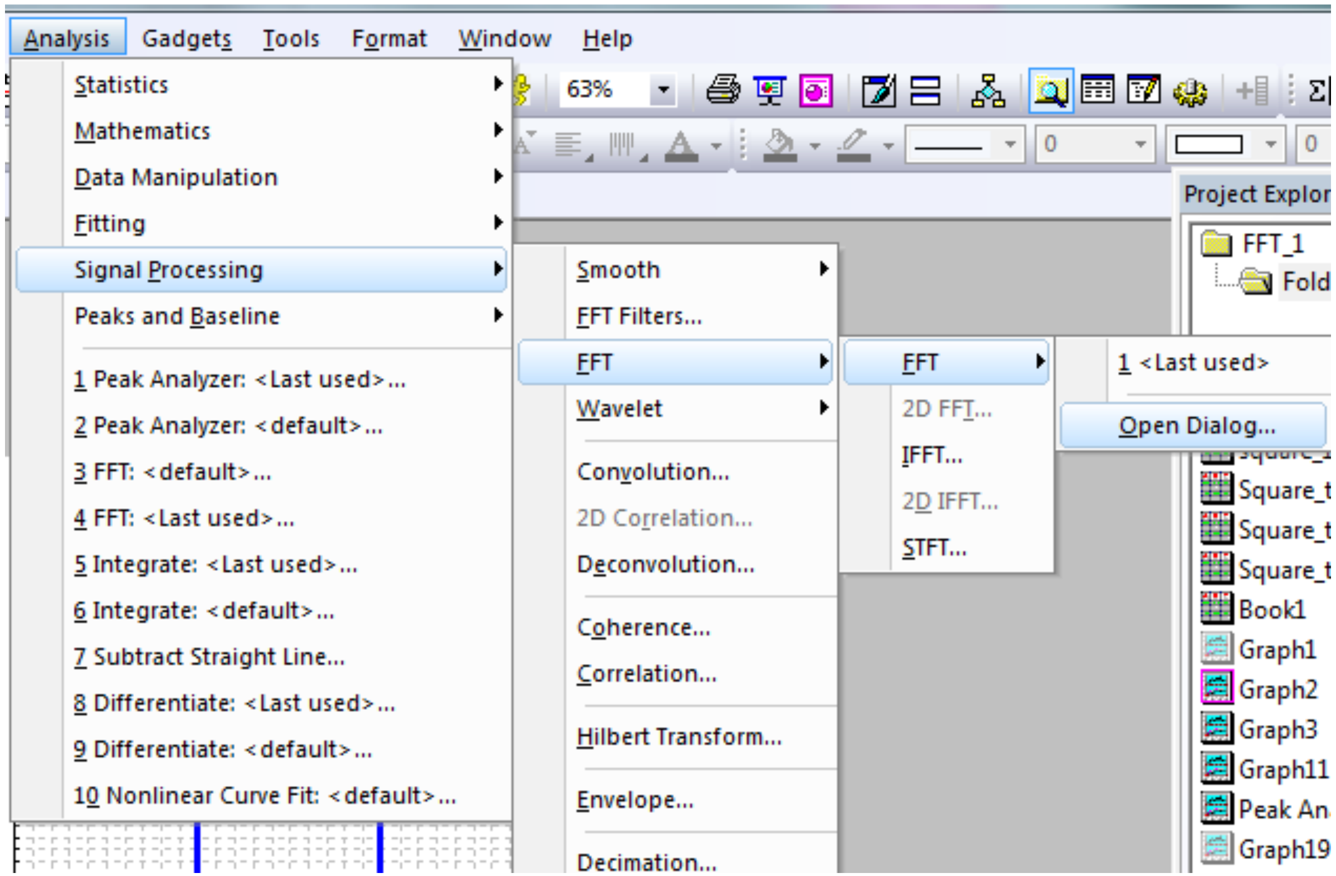
Spectrum measured by SR830 lock-in amplifier

From Time Domain to Frequency Domain: Origins

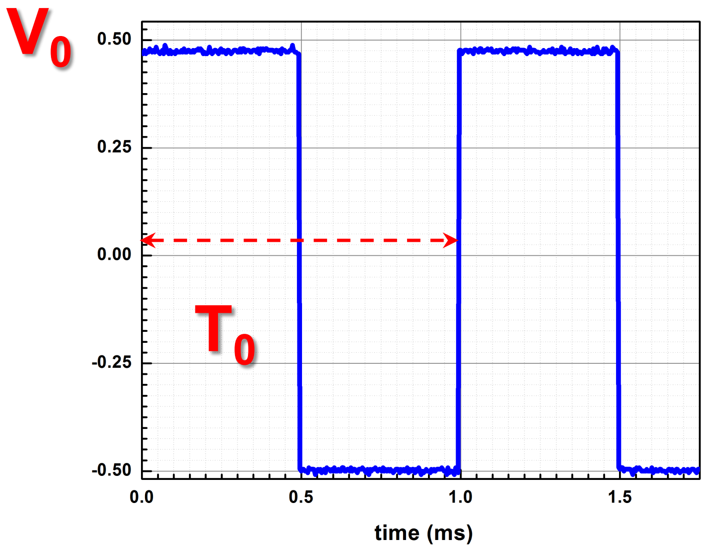


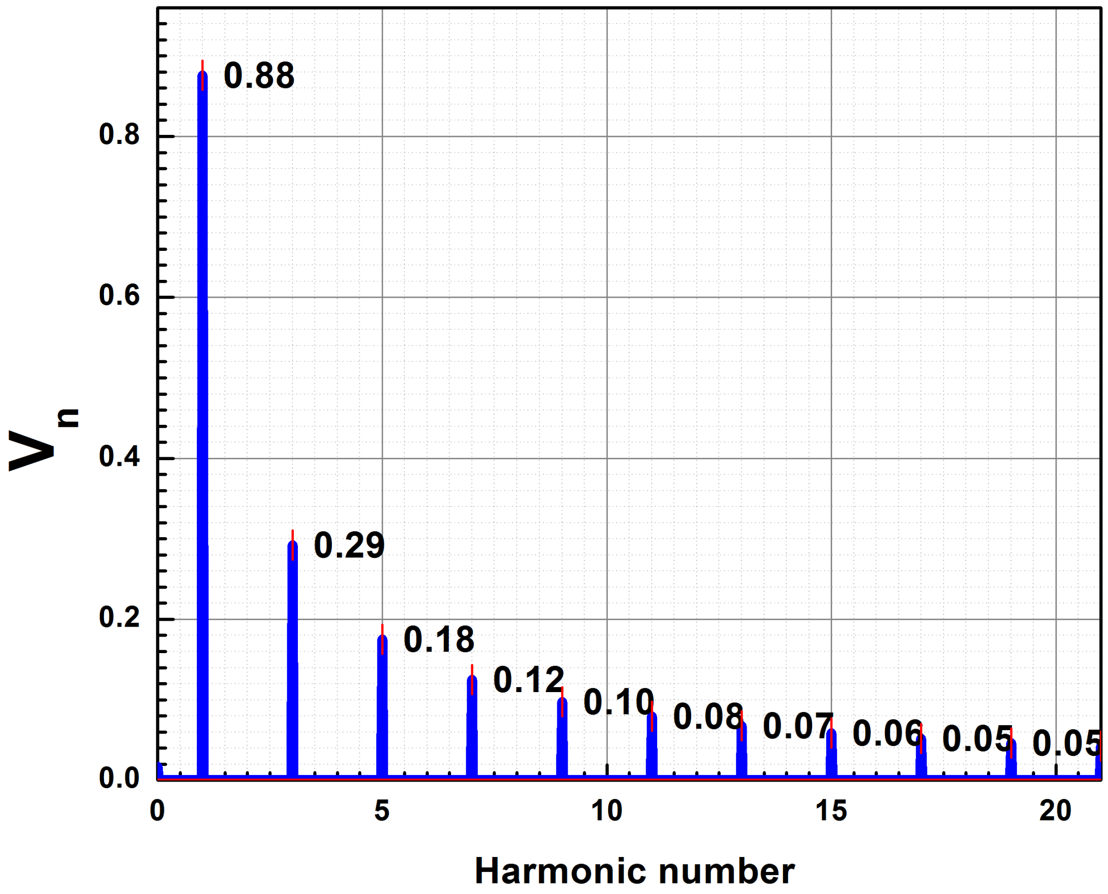
Time domain trace acquired using Tektronix scope

Origins can convert saved data file to frequency domain



From Time Domain to Frequency Domain: Origins

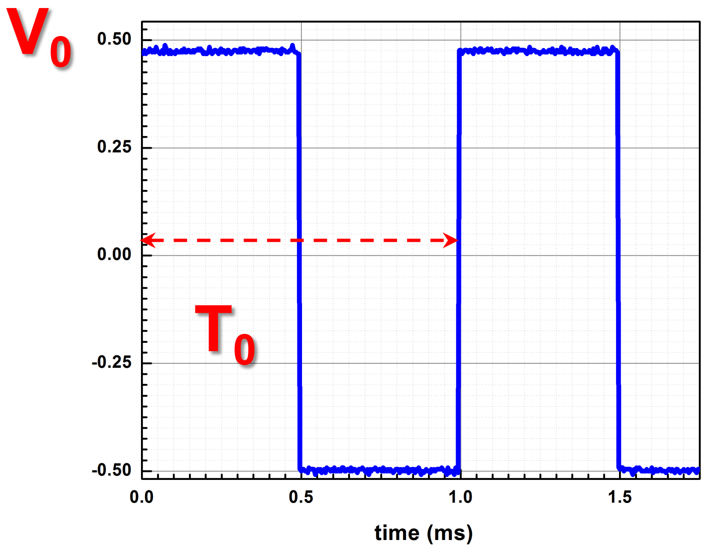




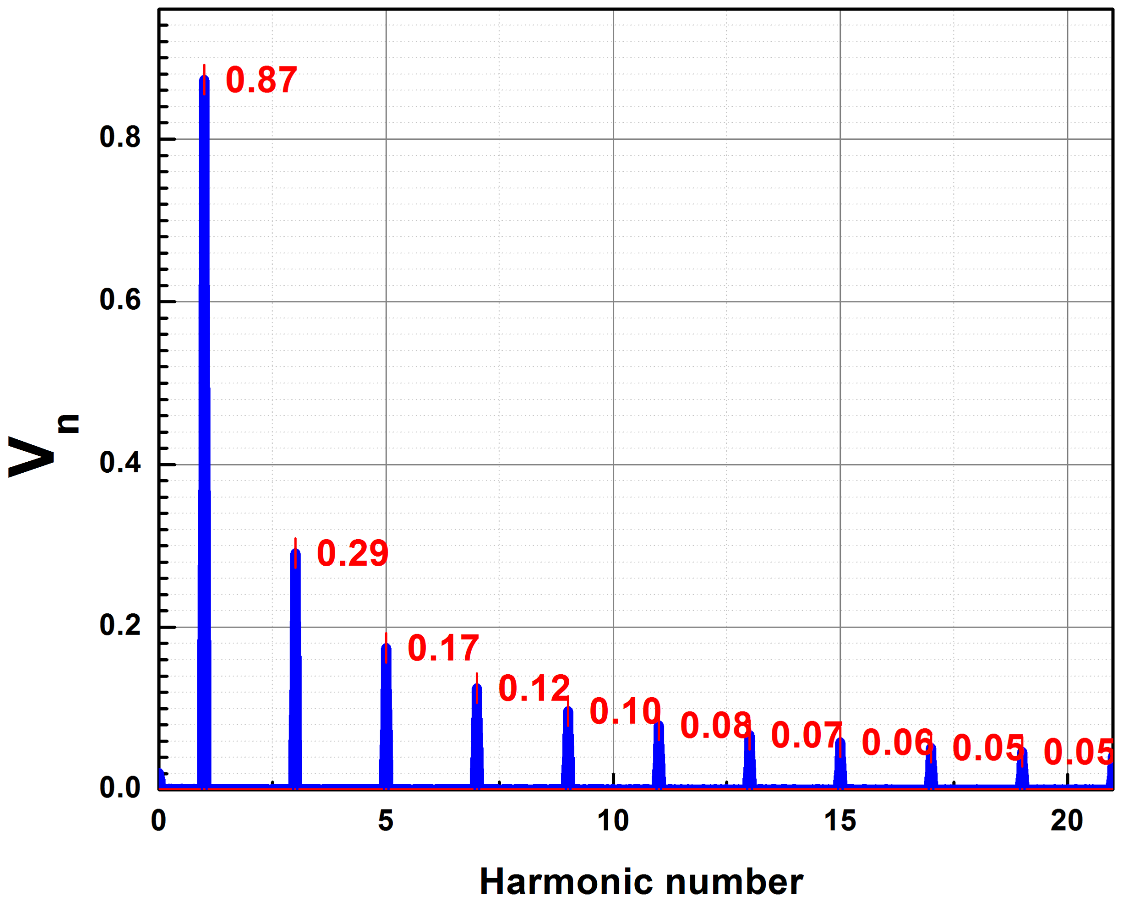
Time domain trace acquired using Tektronix scope

Spectrum calculated by **Origin**. Accuracy is limited by the modest resolution of the scope.

From Time Domain to Frequency Domain: Scope

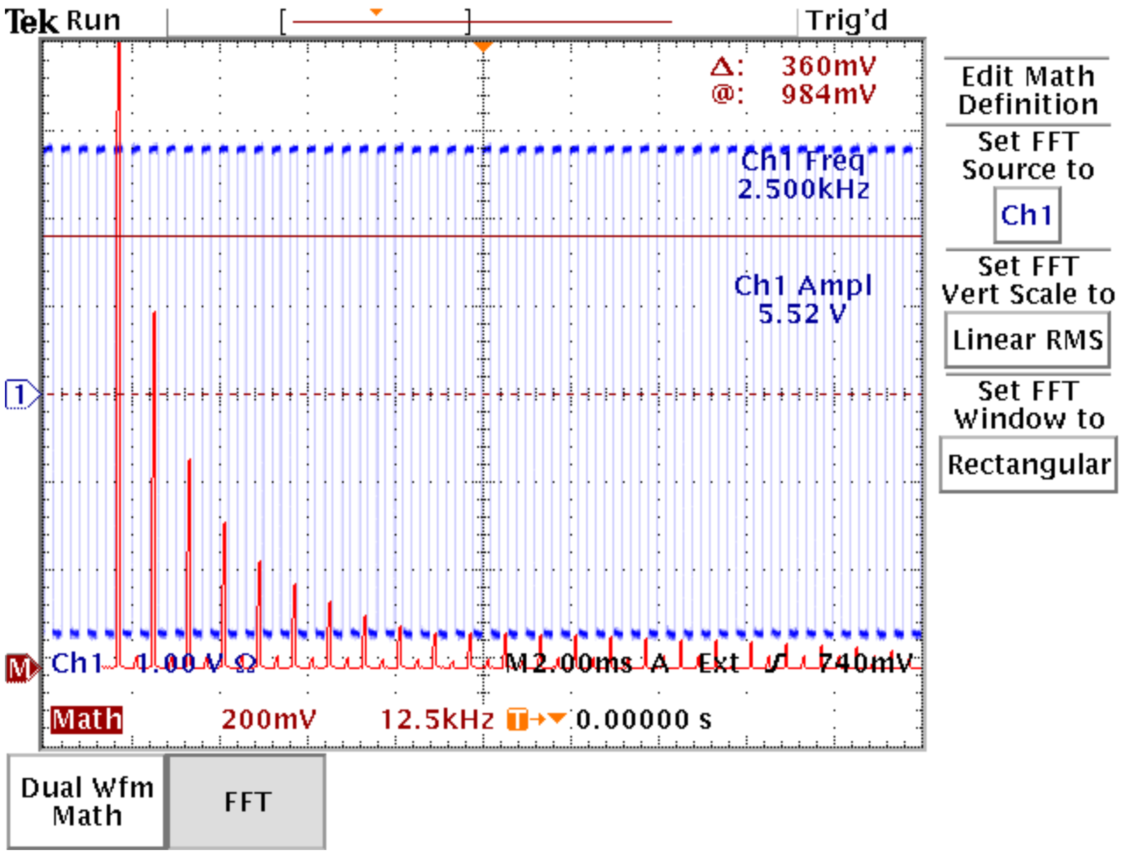


Time domain trace acquired using Tektronix scope

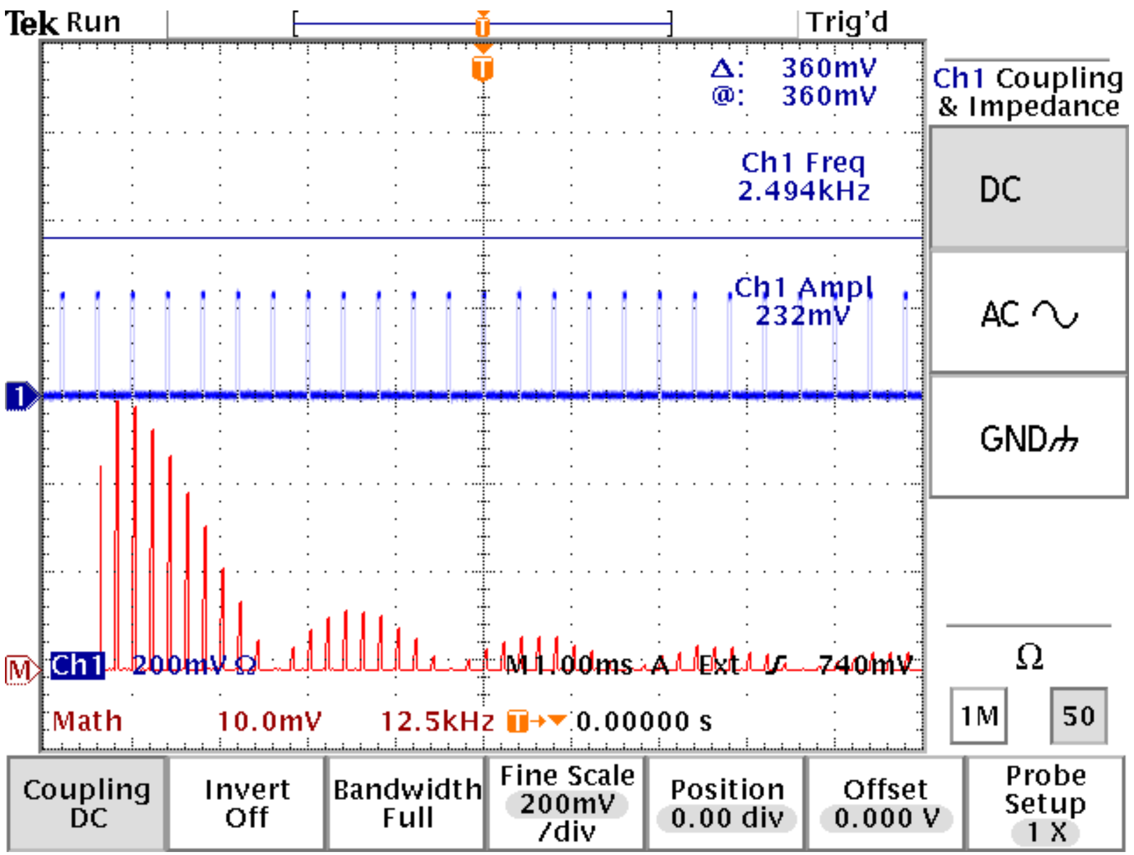


Spectrum calculated by the **Math options** of the **Tektronix scope**. Accuracy is limited by the modest resolution of the scope.

From Time Domain to Frequency Domain: Scope



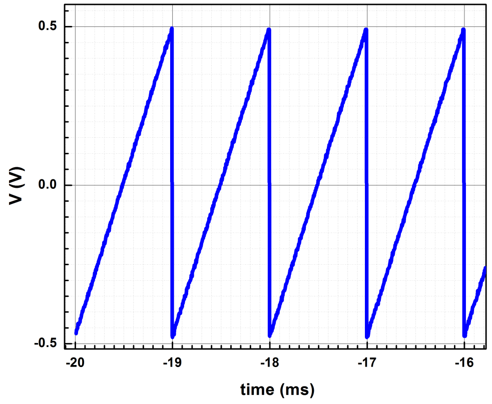
Spectrum of **square wave** from signal generator

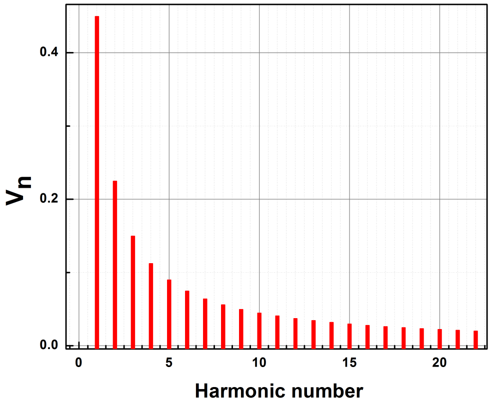


Spectrum of **pulse** from signal generator

From Time Domain to Frequency Domain: Lock-In

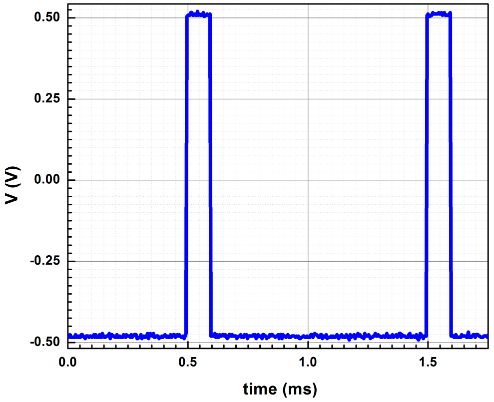
Examining different pulse shapes on the SR830

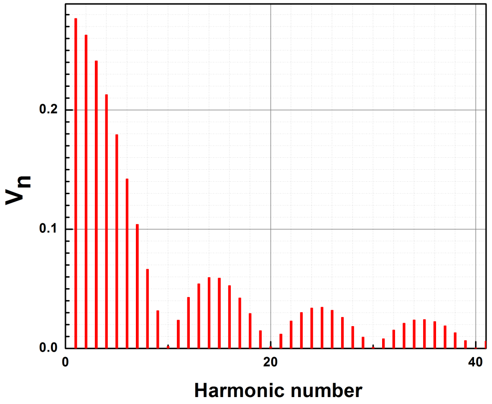




Ramp

Pulse





Appendix: References

1. John H. Scofield, “A Frequency-Domain Description of a Lock-in Amplifier” *American Journal of Physics* **62** (2), 129-133 (Feb. 1994). link

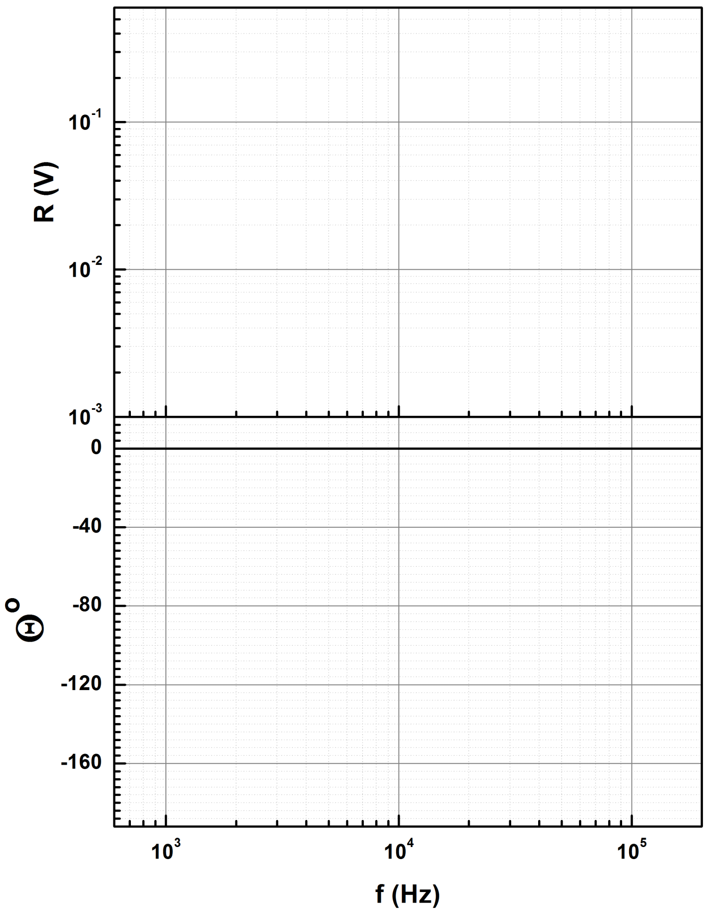
2. Steve Smith “The Scientist and Engineer's Guide to Digital Signal Processing” copyright ©1997-1998 by Steven W. Smith.

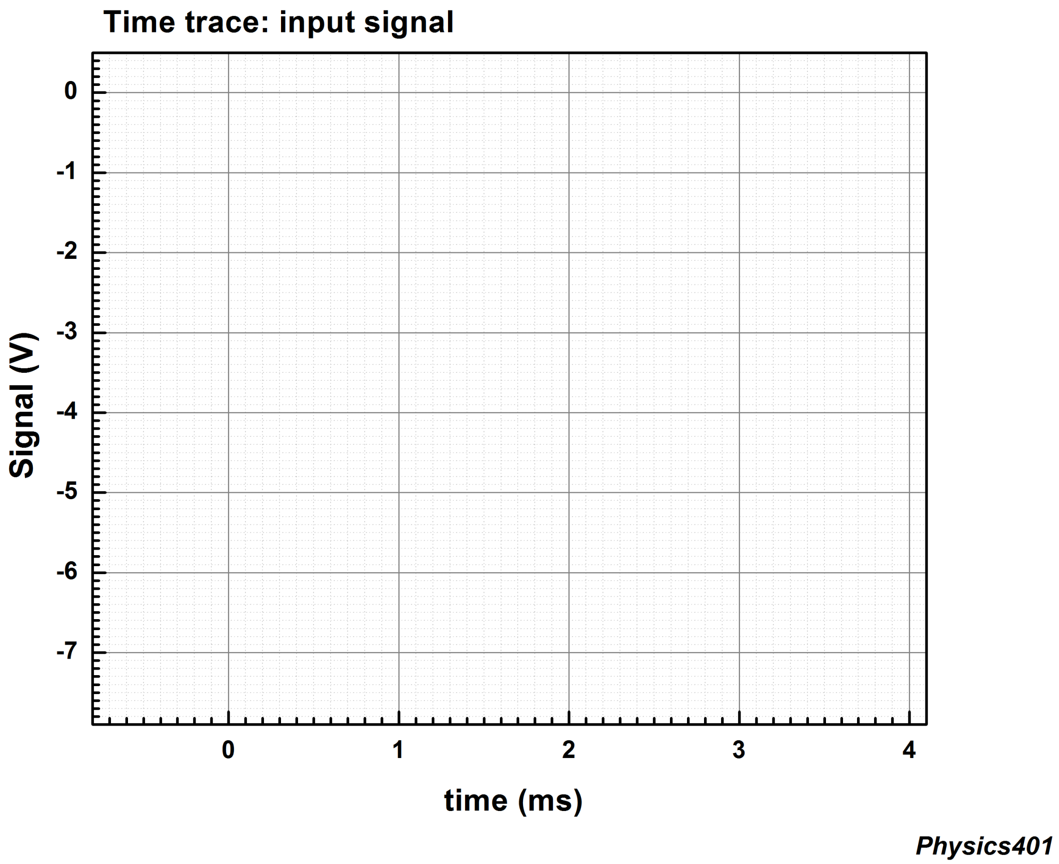
   For more information visit the book's website at: www.DSPguide.com

   You can also find an electronic copy of this book in:

\\engr-file-03\PHYINST\APL Courses\PHYCS401\Experiments\DSP and FFT

Appendix: Origin templates for this week's lab





Appendix: Using OriginPro for Fitting

Some recommendations for using the OriginPro nonlinear fitting option

You can find some examples of OriginPro projects and some recommendations of how to do the analysis in the following folder:

\\engr-file-03\PHYINST\APL Courses\PHYCS401\Students\3. Frequency Domain Experiment. Fitting

An example of Origin project

Slides with results of fitting for all studied in this Lab RLC circuits

Some recommendations how to write the fitting function and start the fitting process

Examples of fitting functions which can be used for fitting in Origin