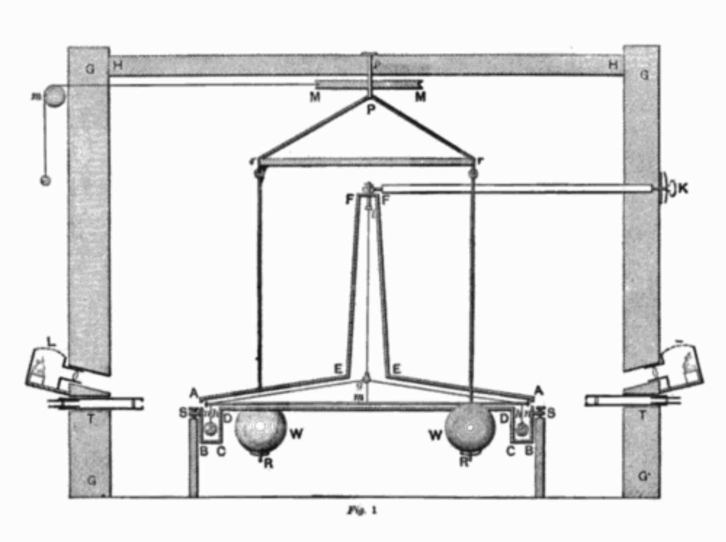
Torsional Oscillator

Episode I: Transient Response



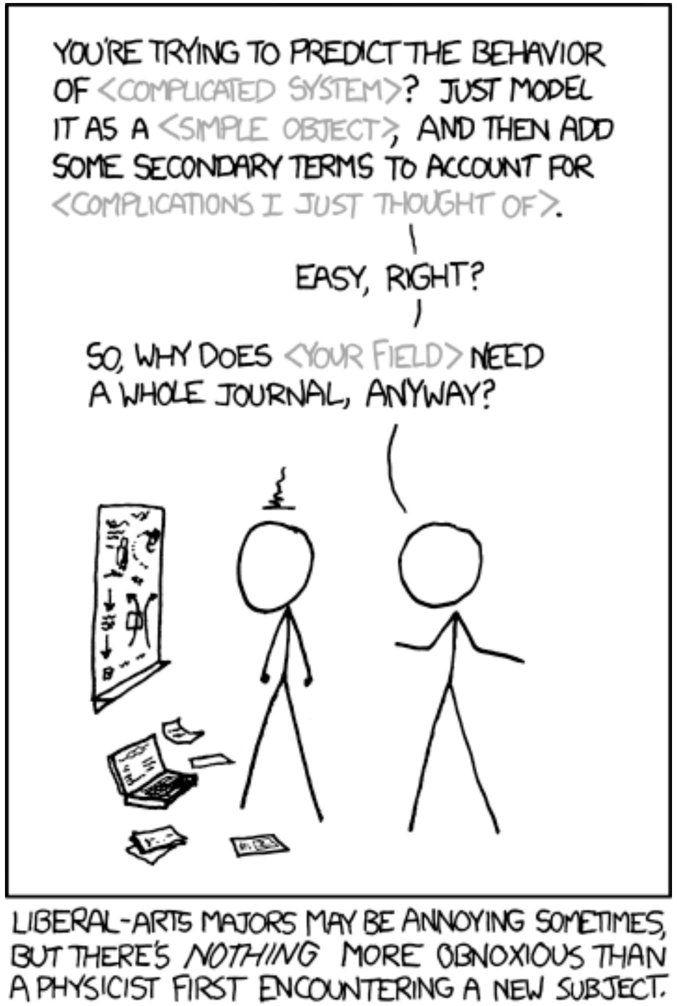
[Cavendish Experiment]

Professor Jeff Filippini

Physics 401

Spring 2020

ILLINOIS



[XKCD #793]

Transients in a Torsional Oscillator

1. Reminder: Electrical RLC Circuits
2. Torsional Oscillator
3. Damping Mechanisms
4. Data Analysis

Appendix: Last notes on oil drop data analysis

Transients in an RLC Circuit

$$V_L + V_R + V_C = V(t)$$

Inertia — Restoring

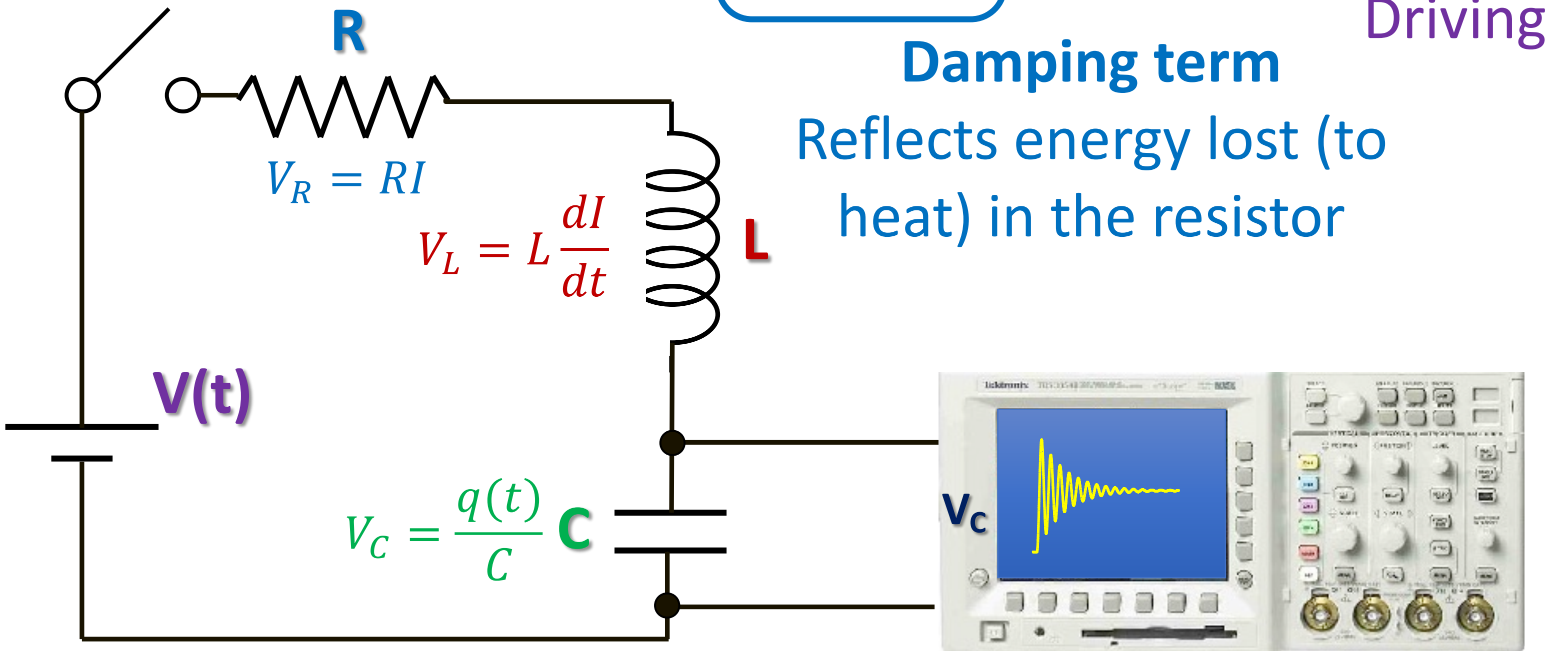
$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = V(\text{t})$$

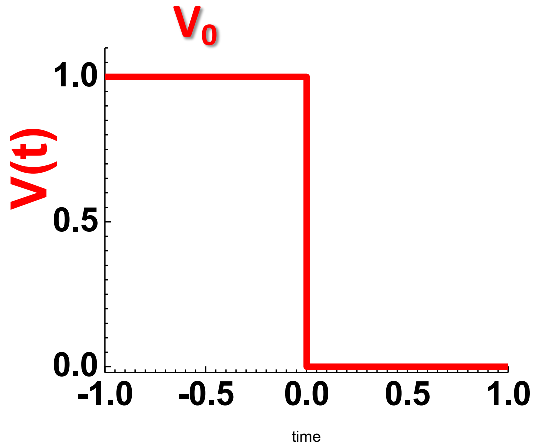
Driving

Damping term

Reflects energy lost (to heat) in the resistor

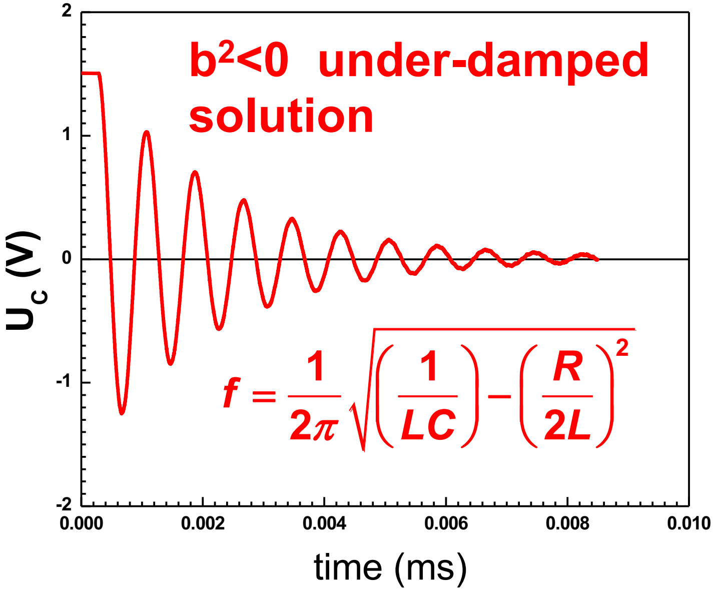
This week: $V(t>0)=0$
Time-domain transients
(*like in Lab 2*)
Driven oscillations next week!

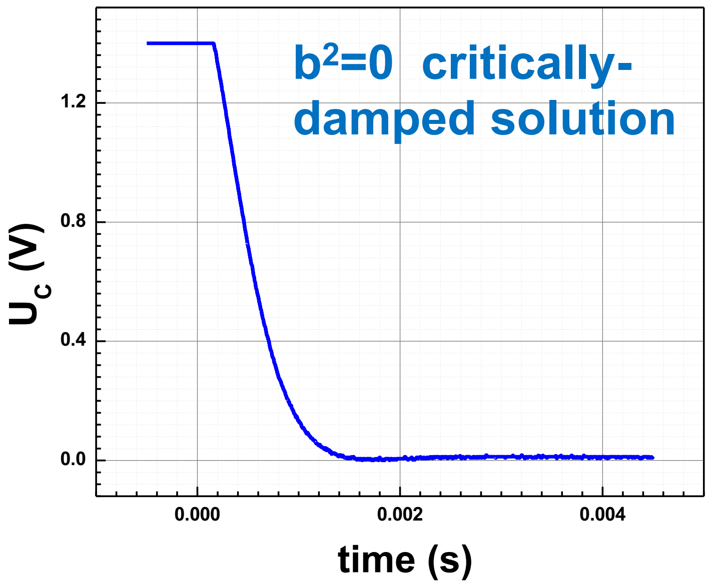


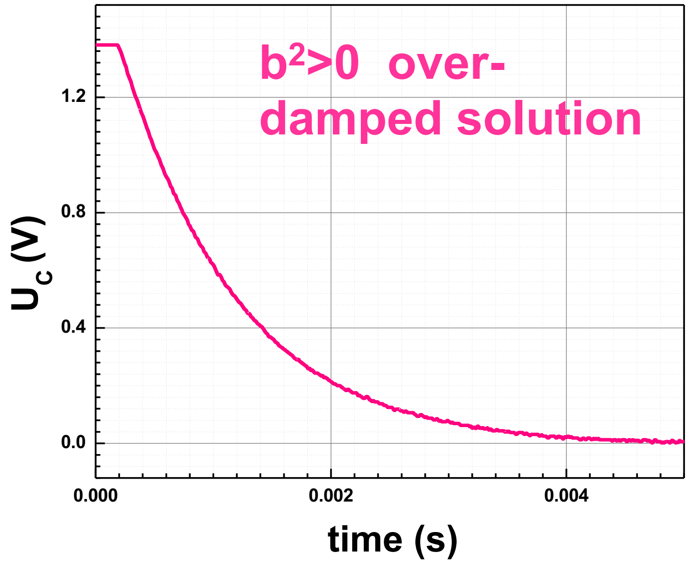


RLC Circuit: Three Damping Regimes

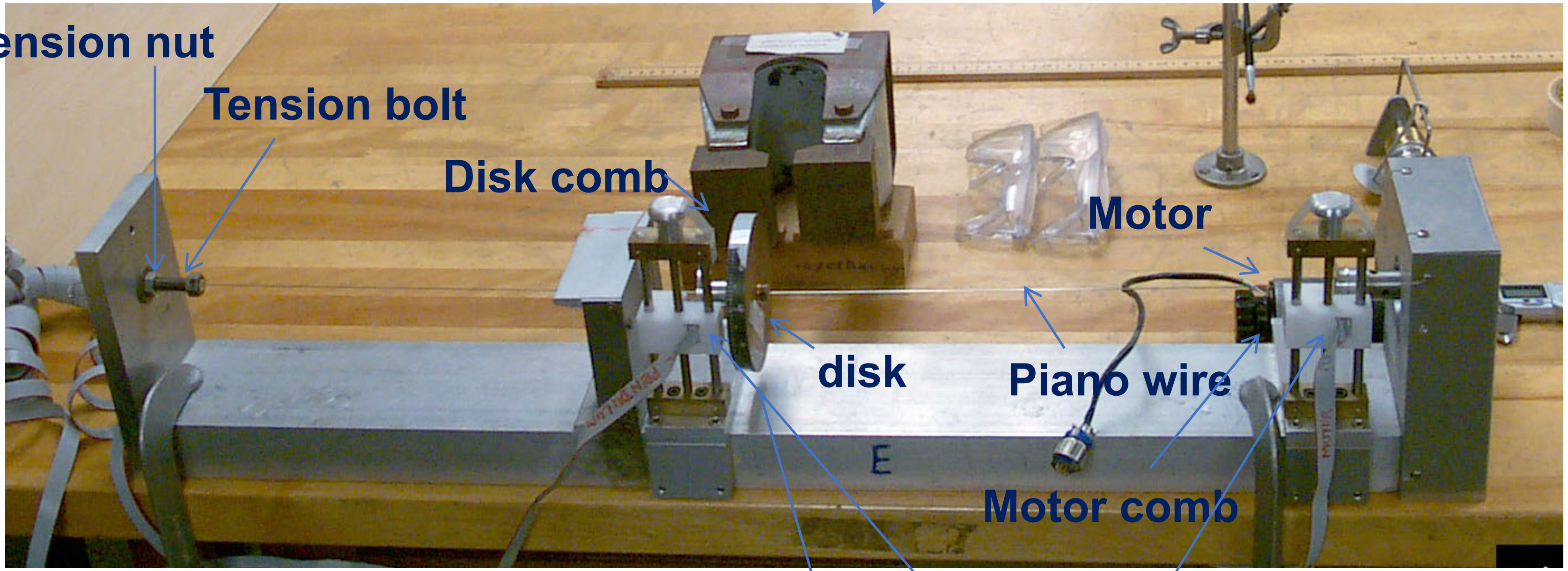
$$a = \frac{R}{2L} \quad , \quad b = \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$





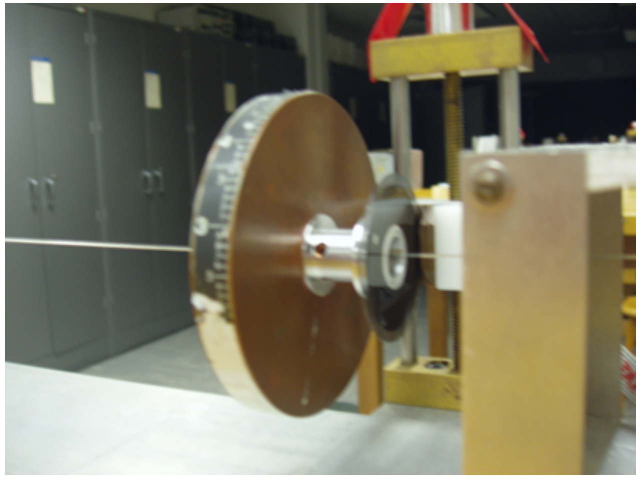


Introducing the Torsional Oscillator

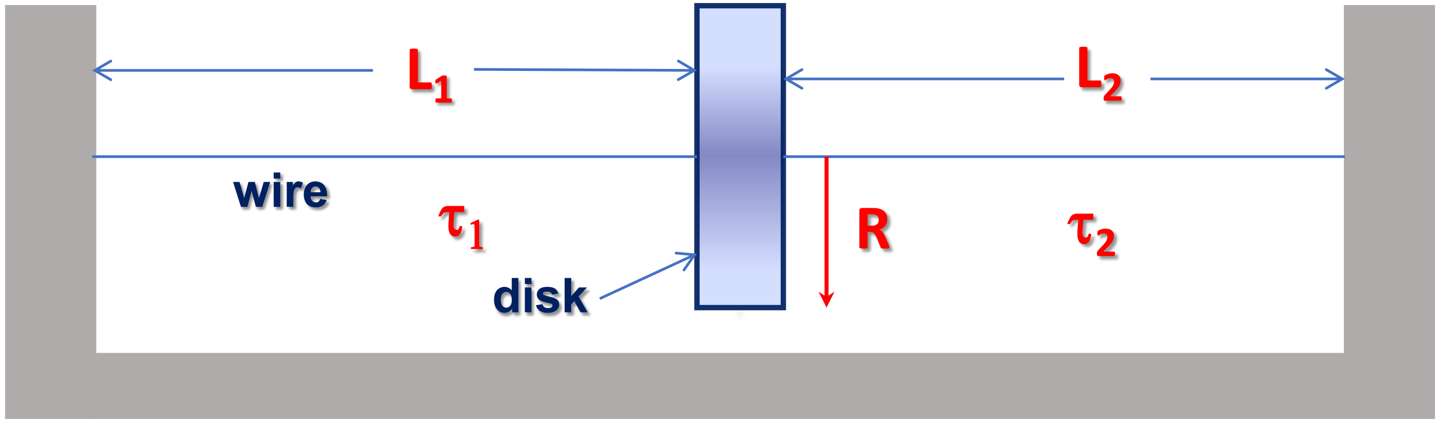


Reminder: **Moment of inertia** for disk of mass M, radius R:

$$I = \frac{1}{2} M R^2$$



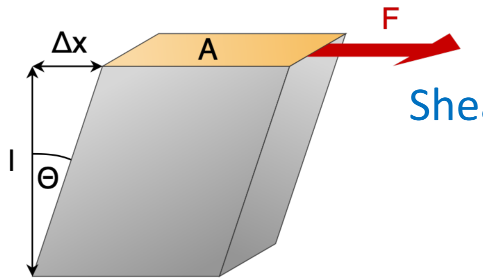
Introducing the Torsional Oscillator



Wires 1 and 2 resist twisting, exerting torques τ_1 and τ_2 on a disk of mass M

$$\tau = \tau_1 + \tau_2 = -K_1\theta - K_2\theta = -K\theta$$

A typical shear modulus for steel is 8.3×10^{10} N/m²



Wikipedia

$$\text{Shear modulus} = \frac{shear\ stress}{shear\ strain}$$

$$G = \frac{F/A}{\Delta x/L}$$

$$K_1 = \frac{\pi G r^4}{2L_1}$$

K* : *torsional spring constant
θ* : *angular deflection of the disk
r* : *radius of the wires
L_i*: *length of wire i
G*: *shear modulus of the wire

$$K = K_1 + K_2 = \frac{\pi}{2} G r^4 \left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$

Torsional Pendulums in Scientific History

Charles-Augustin de Coulomb
1736-1806

Measuring the electrostatic force

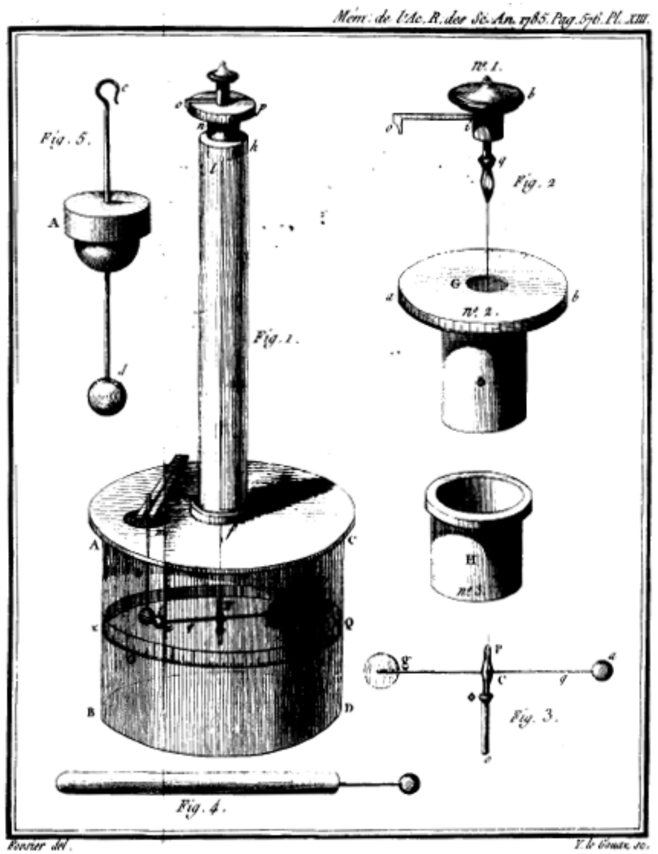
$$\vec{\tau}_1 + \vec{\tau}_2 = 0$$

$$K\,\theta = F\,L$$

… where ***F*** is the electrostatic force, and ***L*** is the length of the balance beam

Coulomb's Law

$$F = k_e \frac{q_1 q_2}{r^2}; \quad k_e = \frac{1}{4\pi\varepsilon_0}$$



Coulomb's torsion balance
[Wikipedia]

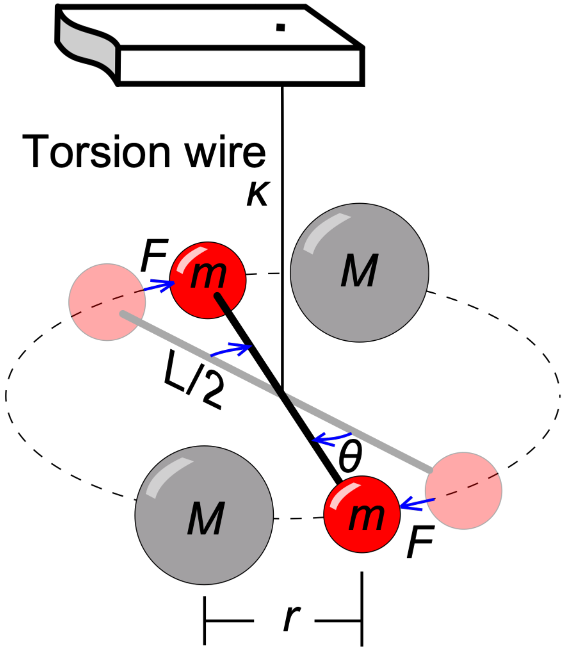
Torsional Pendulums in Scientific History

Henry Cavendish
1731-1810

Measuring the **gravitational** force

$$\vec{\tau}_1 + \vec{\tau}_2 = 0$$
$$K\,\theta = F\,L$$

… where ***F*** is the gravitational force,
and ***L*** is the length of the balance beam



Cavendish torsion balance
Wikipedia

Gravitational Law

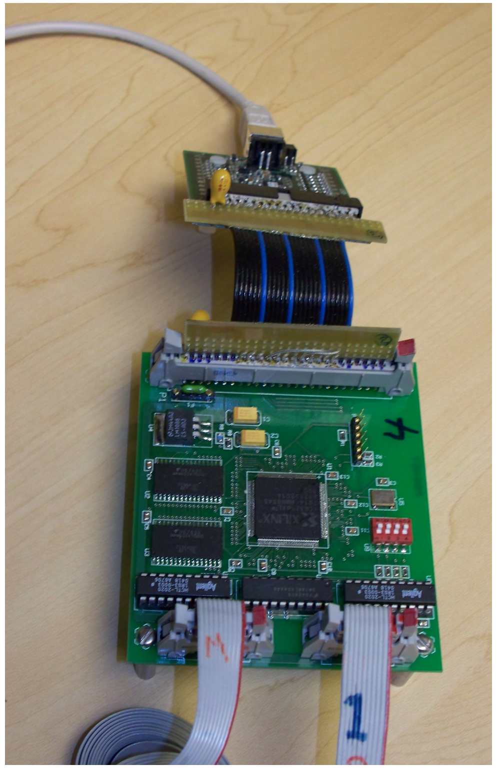
$$F = G\frac{m\,M}{r^2}$$

Cavendish: $G = 6.74 \times 10^{-11}\, m^3 kg^{-1} s^{-2}$

Modern: $G = 6.67430(15) \times 10^{-11}\, m^3 kg^{-1} s^{-2}$

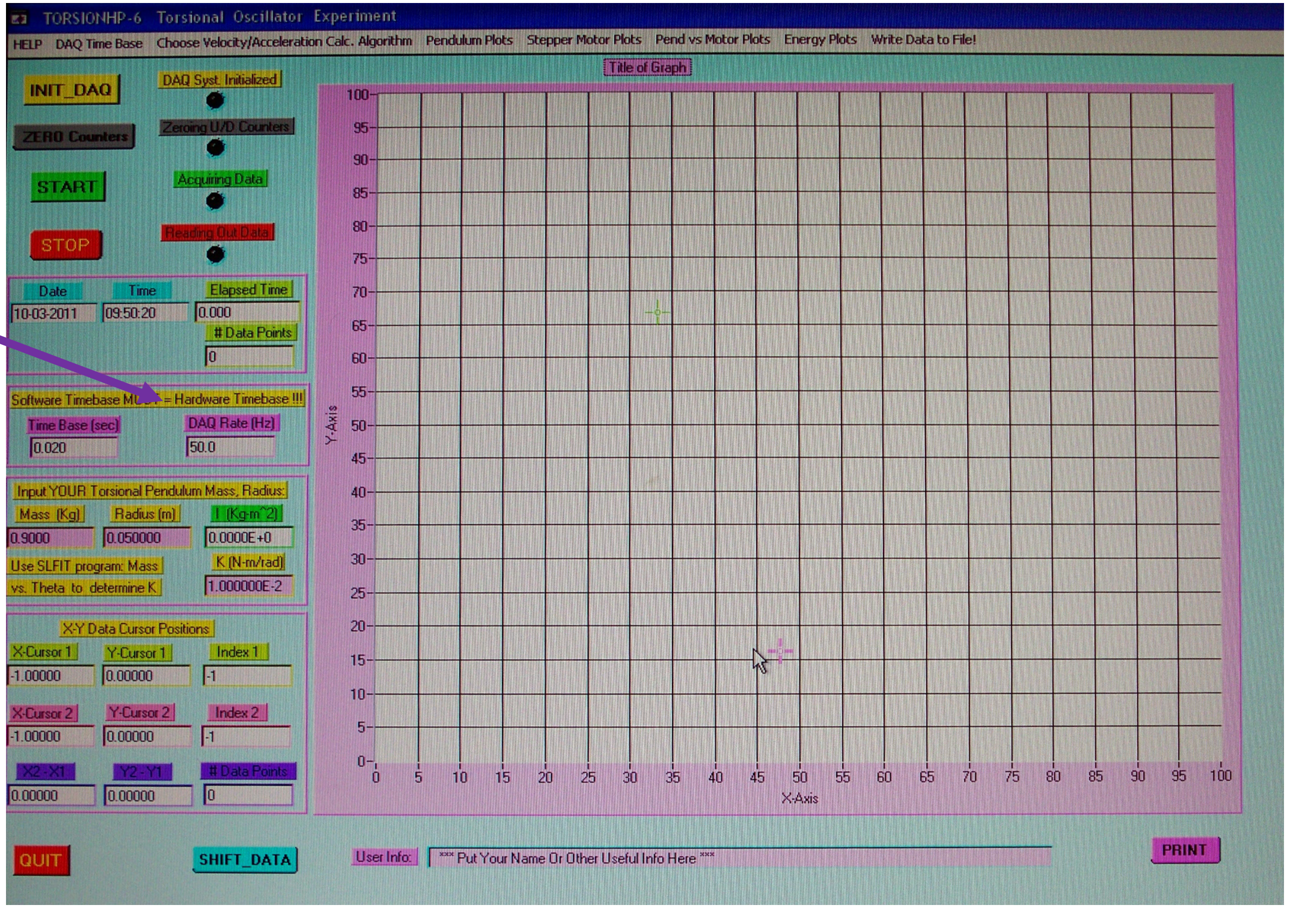
For modern variants, see e.g. the Eöt-Wash group

Data Acquisition



Interface card

DAQ rate (Hz)

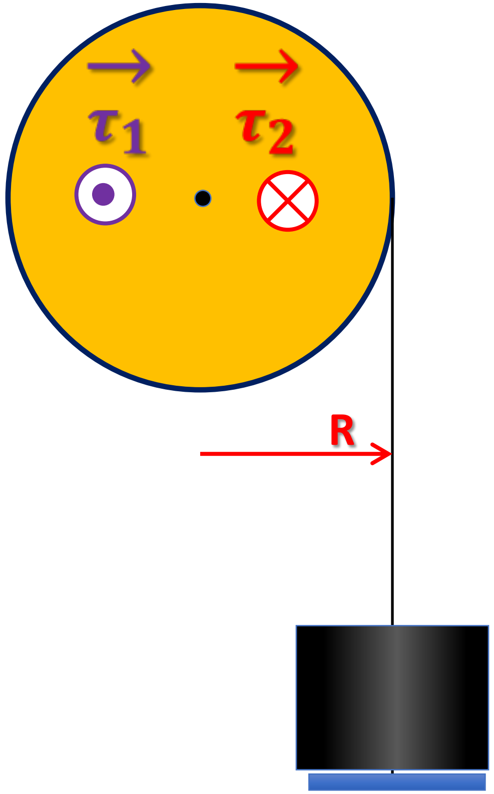


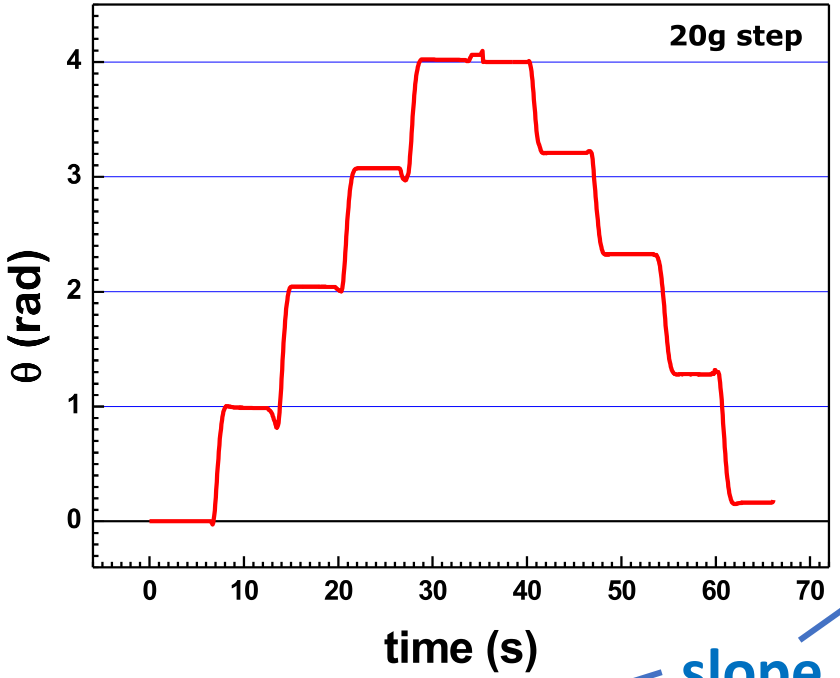
Note: Program can only store **10,000** data points!

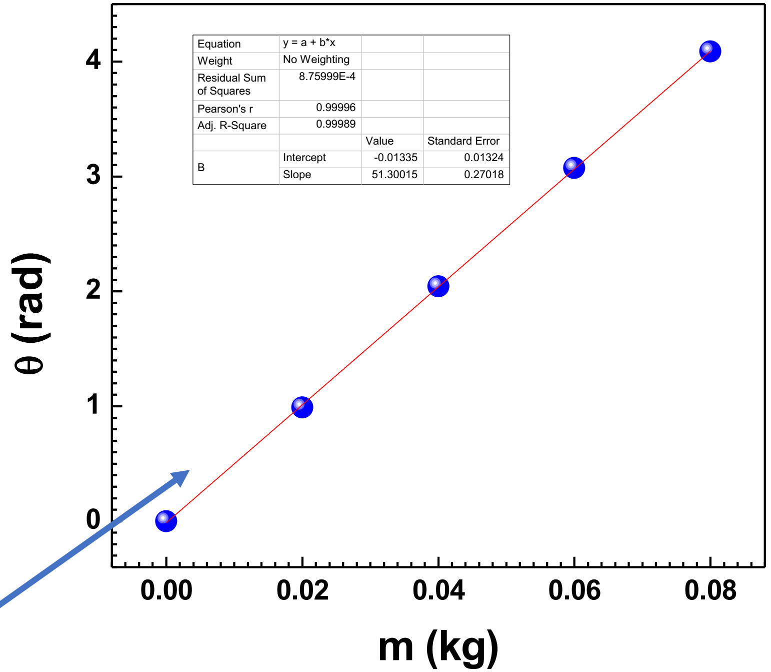
Sampling at **50 Hz**, that's a maximum collection time of **200 s**

Measuring the Torsional Spring Constant

$$\vec{\tau}_1 + \vec{\tau}_2 = 0 \quad \Longrightarrow \quad K\theta = mgR$$







slope

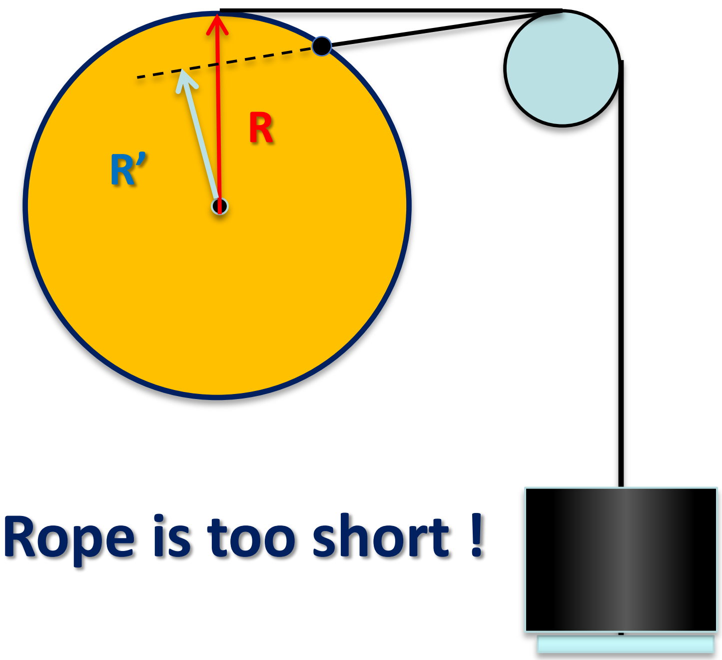
$$\theta = \frac{g\,R}{K}\,m \quad \Longrightarrow \quad K = \frac{g\,R}{slope}$$

g = 9.81 m/s^2

Slope = 51.3 rad/kg

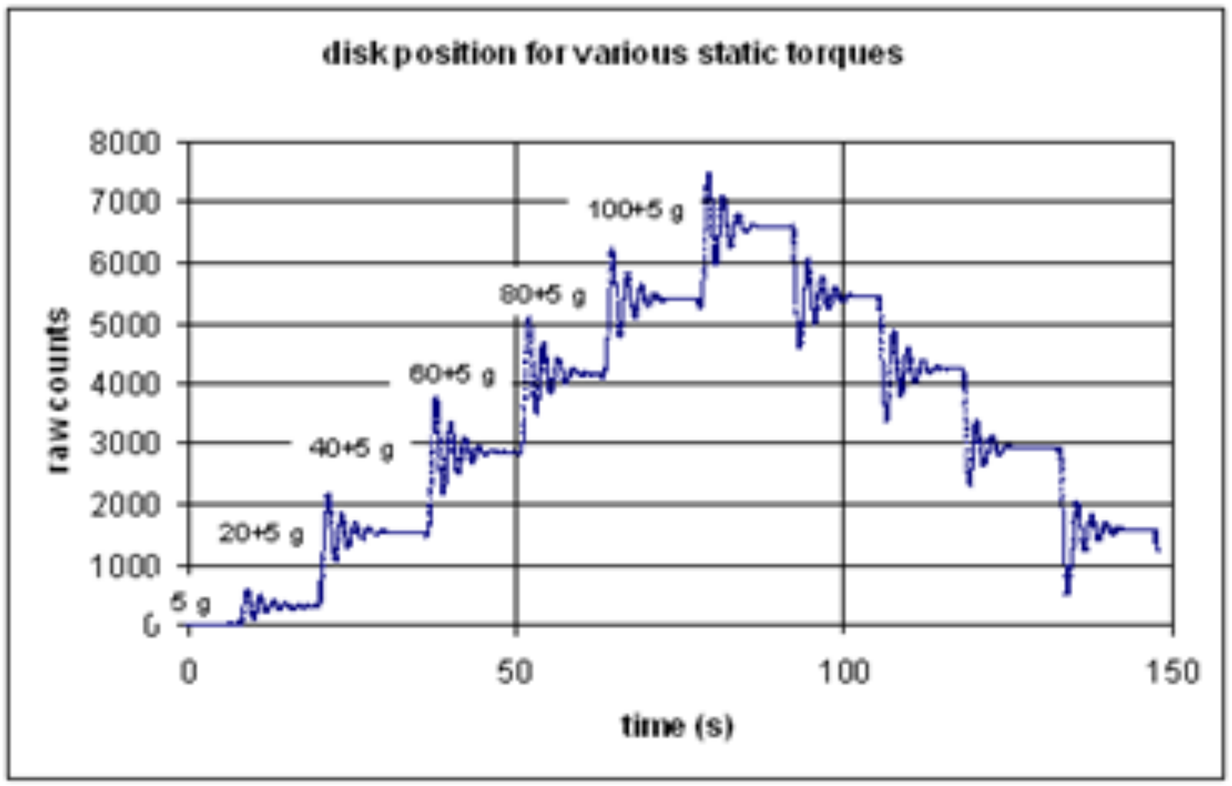
K = 0.00971 N-m/rad

Measuring Spring Constant: Possible Problems



Rope is too short !

$$\boldsymbol{\tau} = \boldsymbol{R} \times \boldsymbol{F}$$



Avoid overdamping of the pendulum motion, and any extra sources of friction.

Torsional Oscillator: "No Damping"

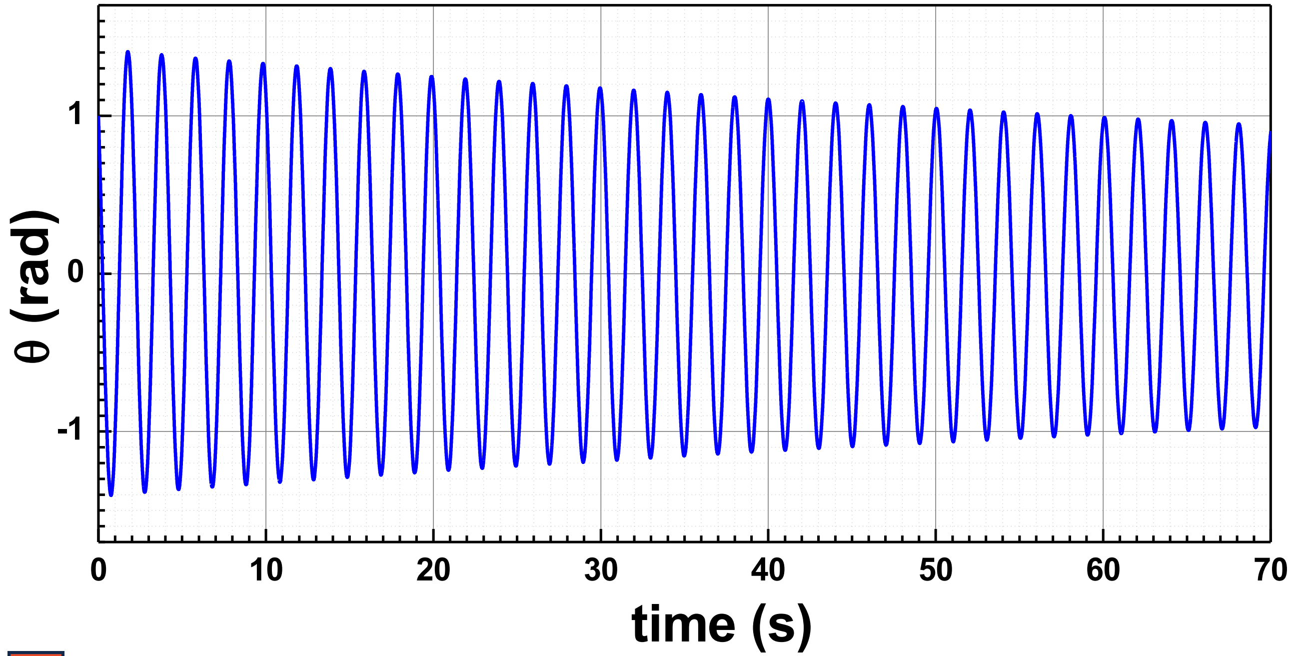
$$\tau = \tau_1 + \tau_2 = -K_1\theta - K_2\theta = -K\theta$$

$$K_1 = \frac{\pi\, G\, r^4}{2\, L_1}; \quad K = K_1 + K_2 = \frac{\pi\, G\, r^4}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$

Without dissipation:

$$I\,\frac{d^2\theta}{dt^2} = -K\,\theta$$

Solution: $\theta = \theta_0 \sin(\omega_0 t + \varphi)$ with $\omega_0 = \sqrt{\frac{K}{I}}$



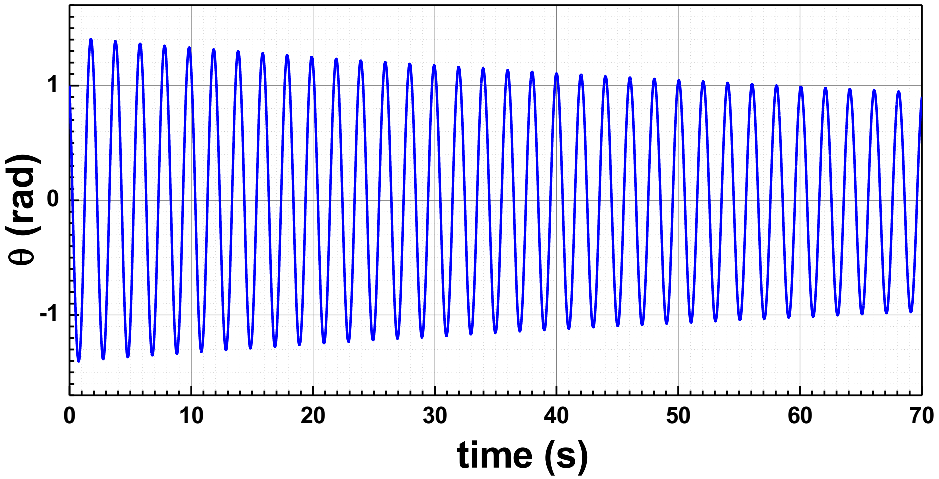
So if we know ***I*** and measure $\boldsymbol{\omega_0}$, we can calculate ***K***.

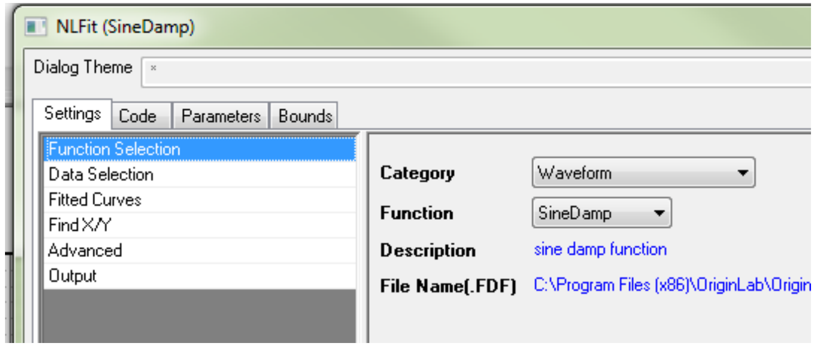
We can estimate $\boldsymbol{\omega_0}$ by measuring the period of **θ(t)**… but nonlinear fitting works better!

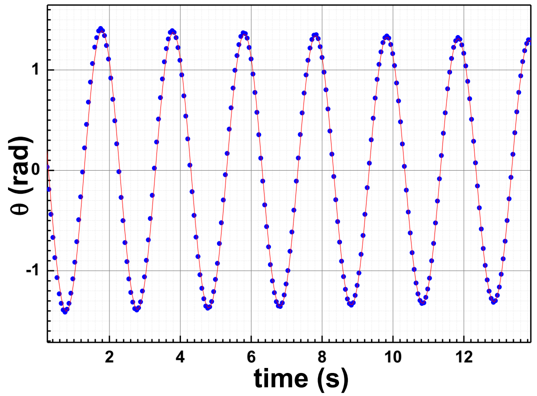
Fitting the "No Damping" Case

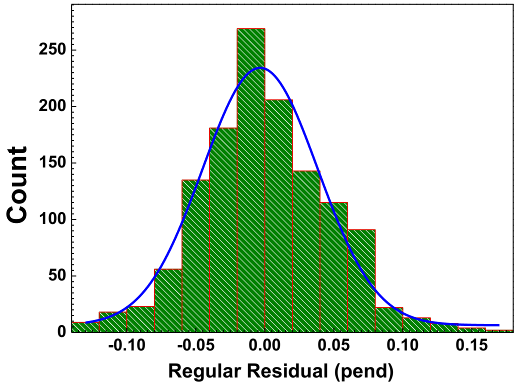
There's always *some* damping, so we should fit with the **SineDamp** function

$$y = y_0 + A\,exp(-x/t_0)\,\sin\left(\pi\frac{(x-x_c)}{w}\right); \qquad \omega_0 = \frac{\pi}{w}$$









$$\omega_0 = 3.126\ \frac{rad}{s}$$

$$f_0 = \frac{\omega_0}{2\pi} \approx 0.4975\ Hz$$

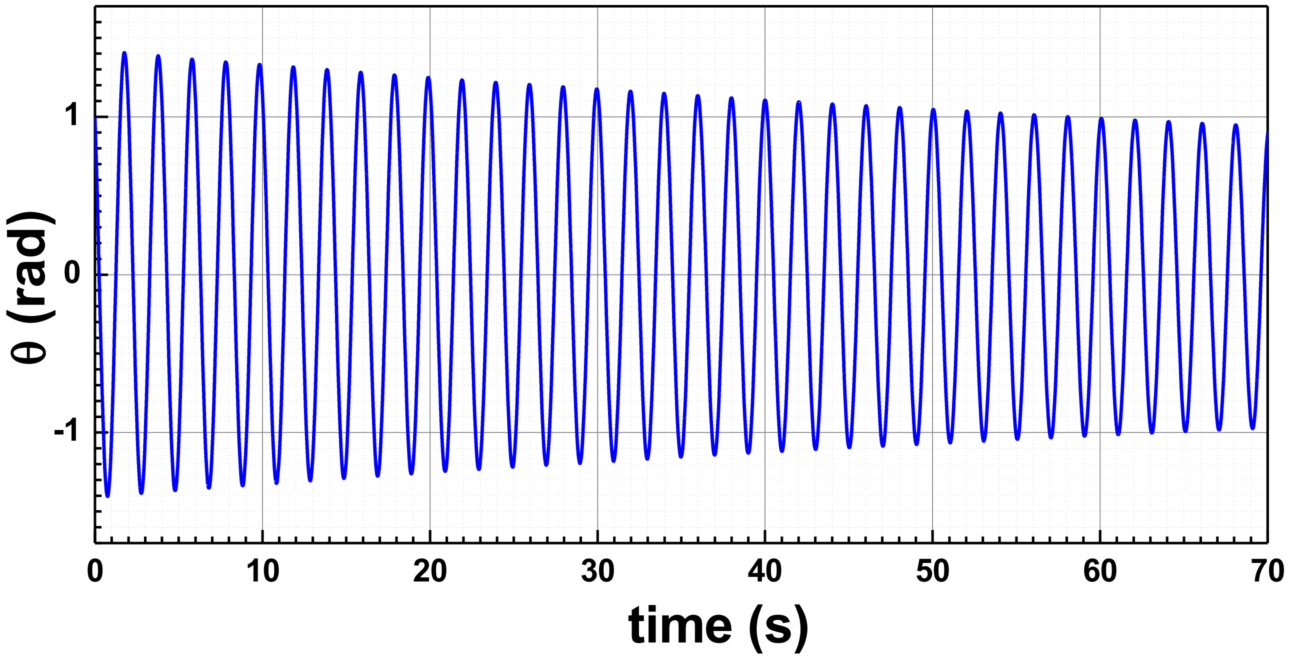
$$K = I\omega_0^2$$

$$K \approx 1.12\times10^{-2}\ \frac{N\cdot m}{rad}$$

SineDamp: $y = y_0 + Aexp\left(\frac{-x}{t_0}\right)sin\left(\pi\frac{(x-x_c)}{w}\right)$

	Value	Standard Error
y0	-0.0024	0.0013
xc	-0.7236	9.3E-4
w	1.00517	2.5E-5
t0	178.02	2.44
A	1.409	0.004

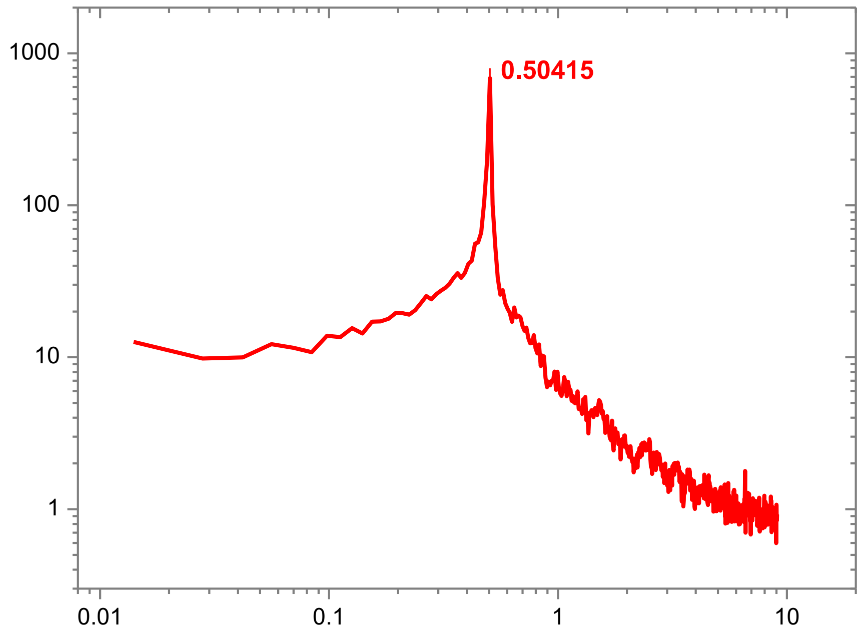
Fitting the "No Damping" Case



From SineDamping fitting:

- $\omega_0 = 2\pi f_0 = 3.123 \frac{rad}{s}$
- $f_0 = 0.497\ Hz$

We can also fit the resonance frequency by FFT-ing the raw data



Three Damping Mechanisms

1. Viscous (Magnetic) Damping

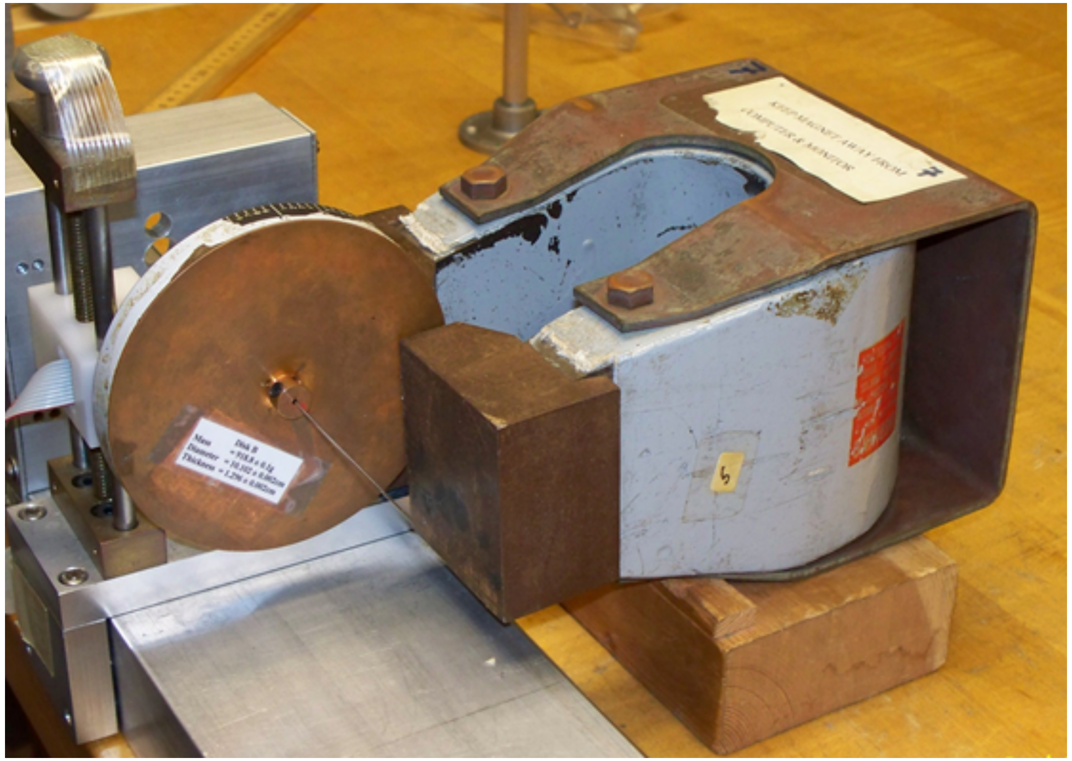
2. Coulomb Damping

3. Turbulent Damping

Three kinds of drag forces, all with technological applications

Three Damping Mechanisms

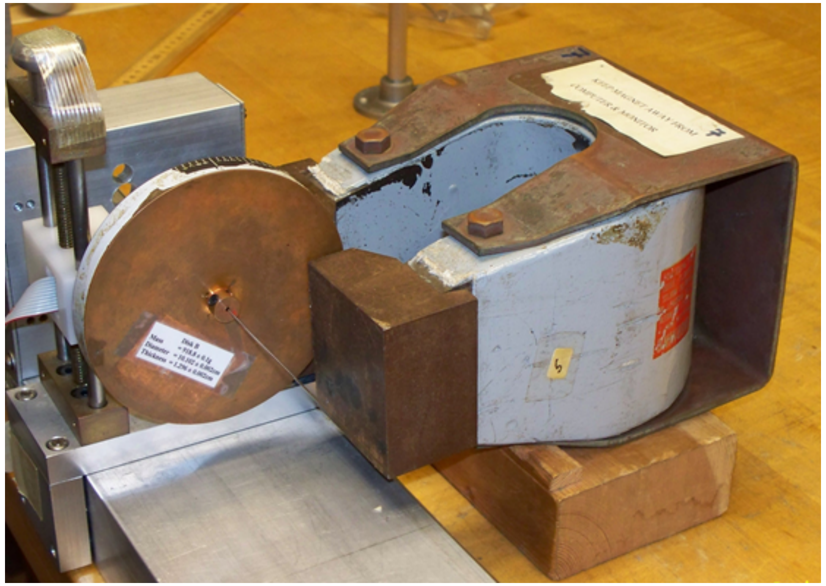
1. **Viscous (Magnetic) Damping**



2. Coulomb Damping

3. Turbulent Damping

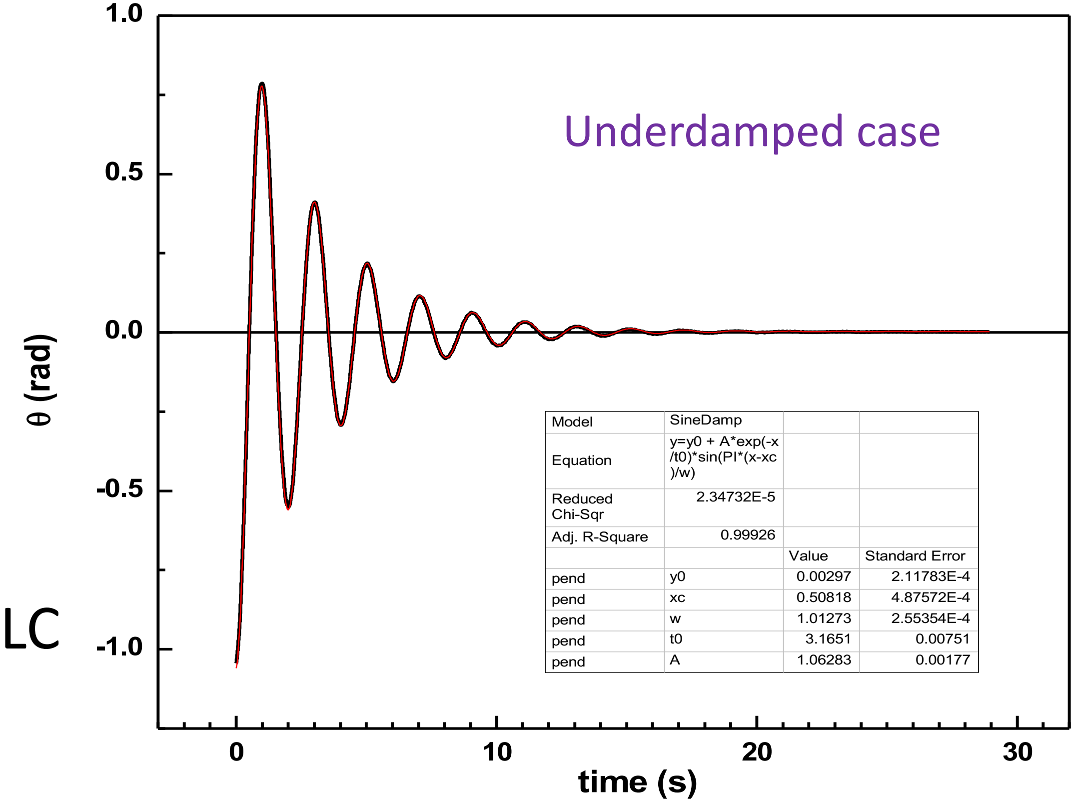
Viscous (Magnetic) Damping



$$I\frac{d^2\theta}{dt^2} + K\theta + \boldsymbol{R\frac{d\theta}{dt}} = 0$$

Solutions are identical to those of an RLC circuit (linear, three damping regimes)

Conductor moving through magnetic field generates **eddy currents**, which dissipate momentum as **heat**



Viscous Damping: Logarithmic Decrement

$$I\frac{d^2\theta}{dt^2} + K\theta + \boldsymbol{R}\frac{\boldsymbol{d\theta}}{\boldsymbol{dt}} = 0$$

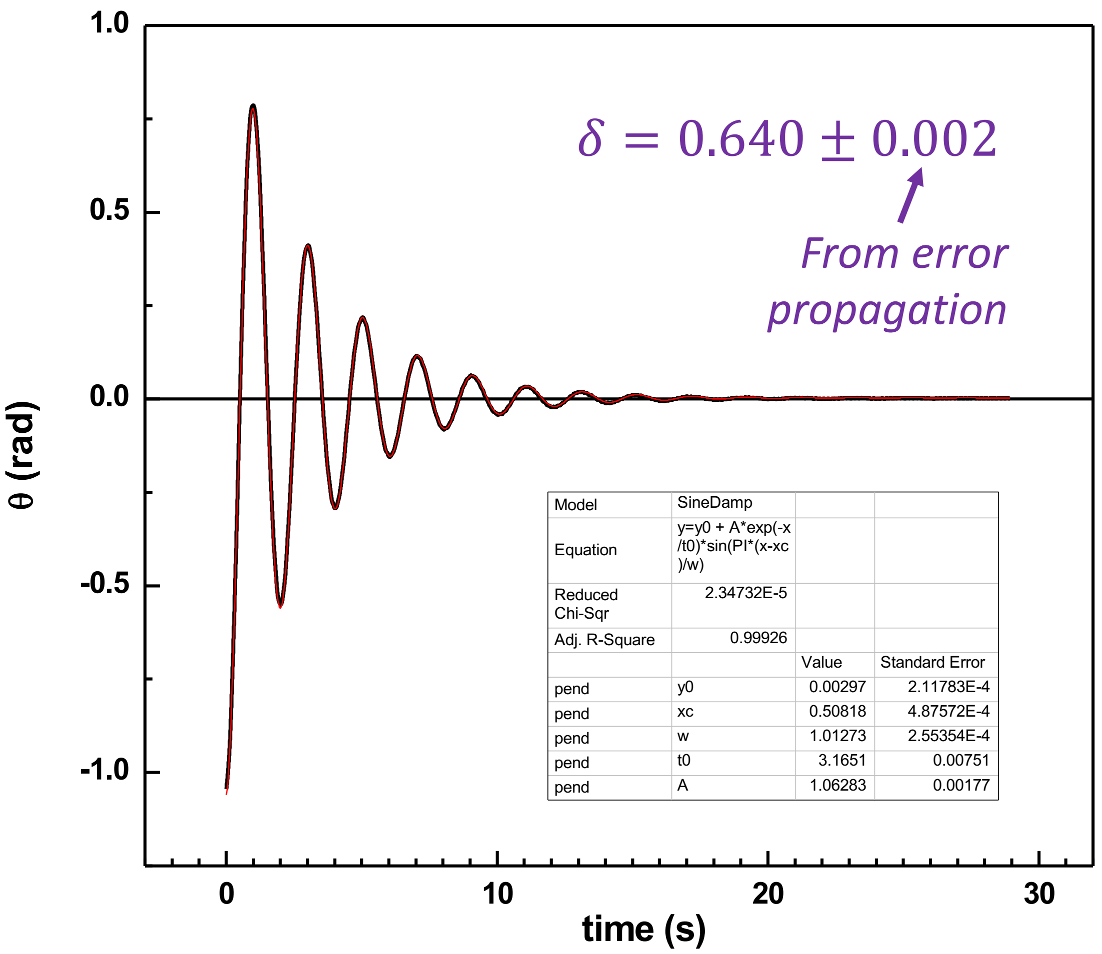
Logarithmic loss per oscillation

$$\delta = \ln\left(\frac{\theta_{n+1}}{\theta_n}\right) = \frac{T}{t_0}$$

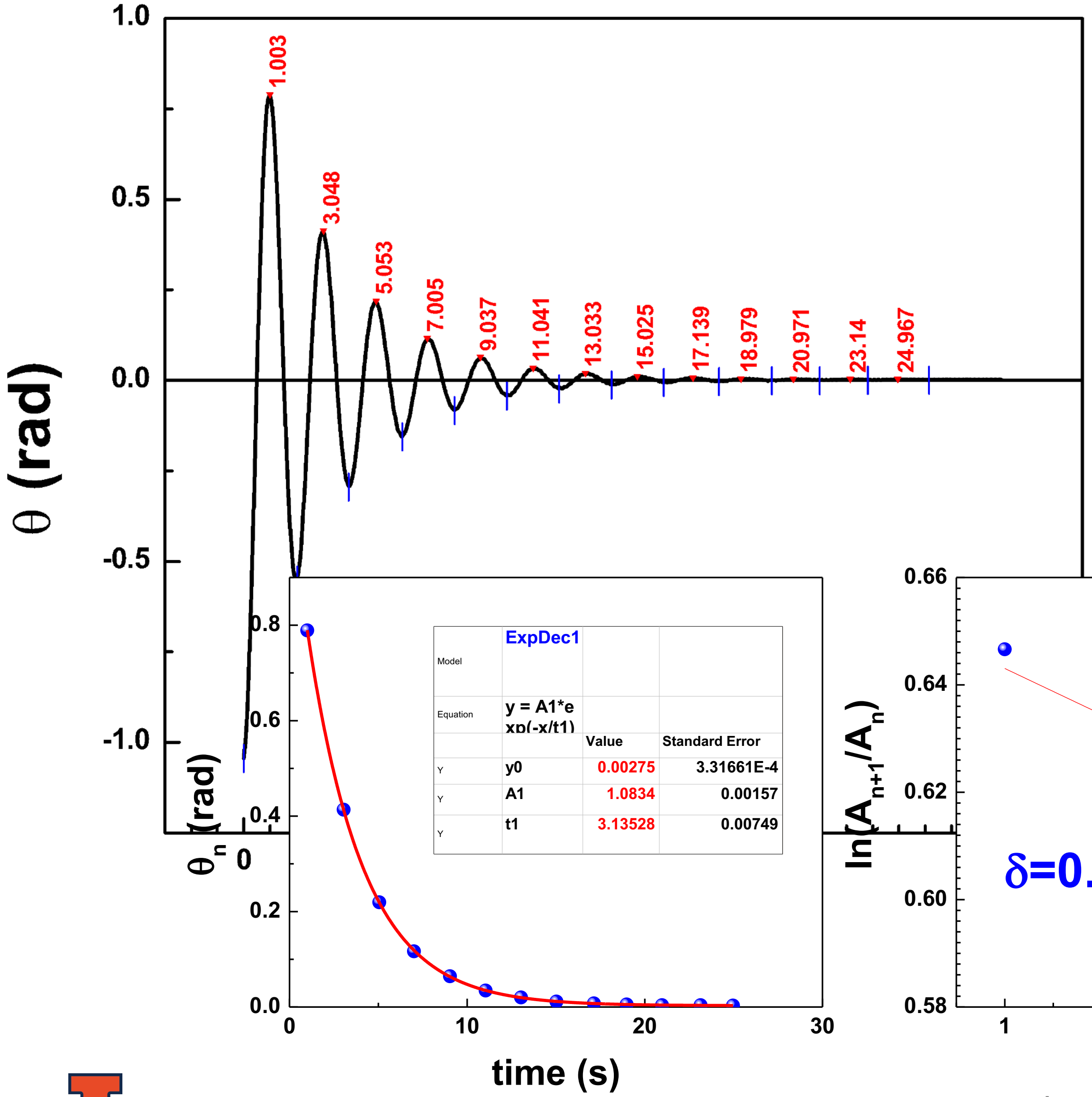
for period ***T***, characteristic exponential decay time $\boldsymbol{t_0}$ (*from writeup,* $a = 1/t_0$)

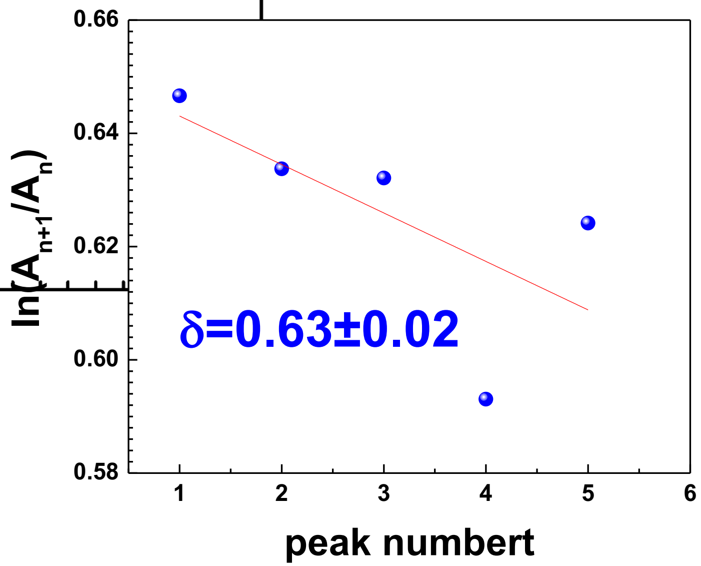
SineDamp fitting function:

$T = 2w$ and $\delta = \frac{2w}{t_0}$

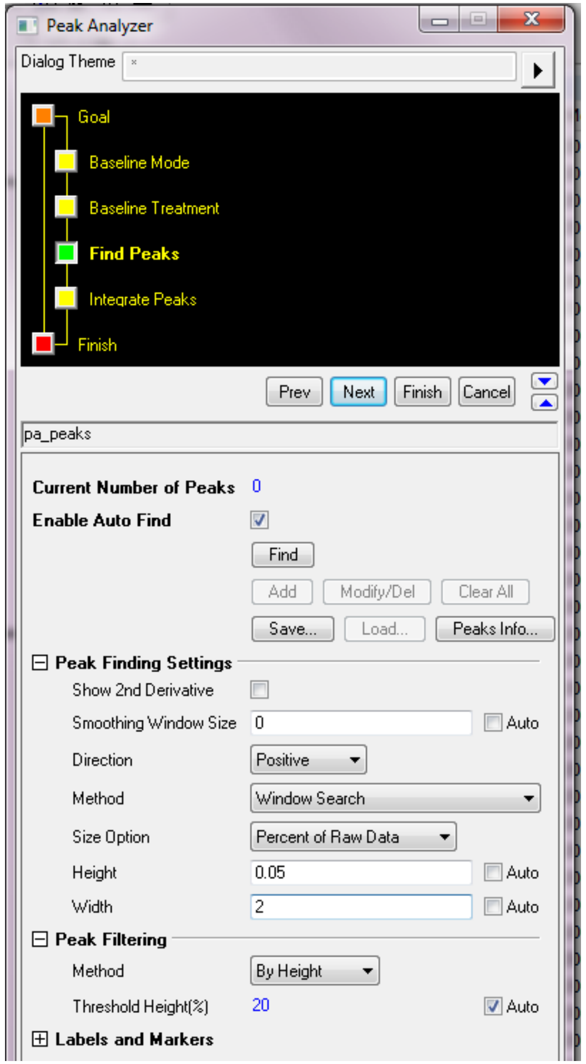


Viscous Damping: Logarithmic Decrement





We can find amplitudes to fit using Peak Analyzer

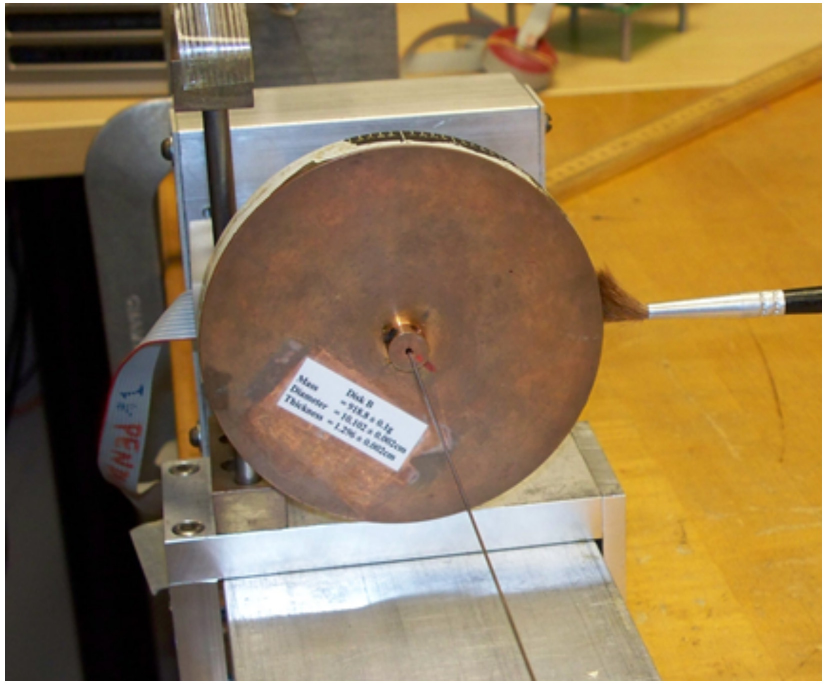


Three Damping Mechanisms

1. Viscous (Magnetic) Damping

2. **Coulomb Damping**

3. Turbulent Damping



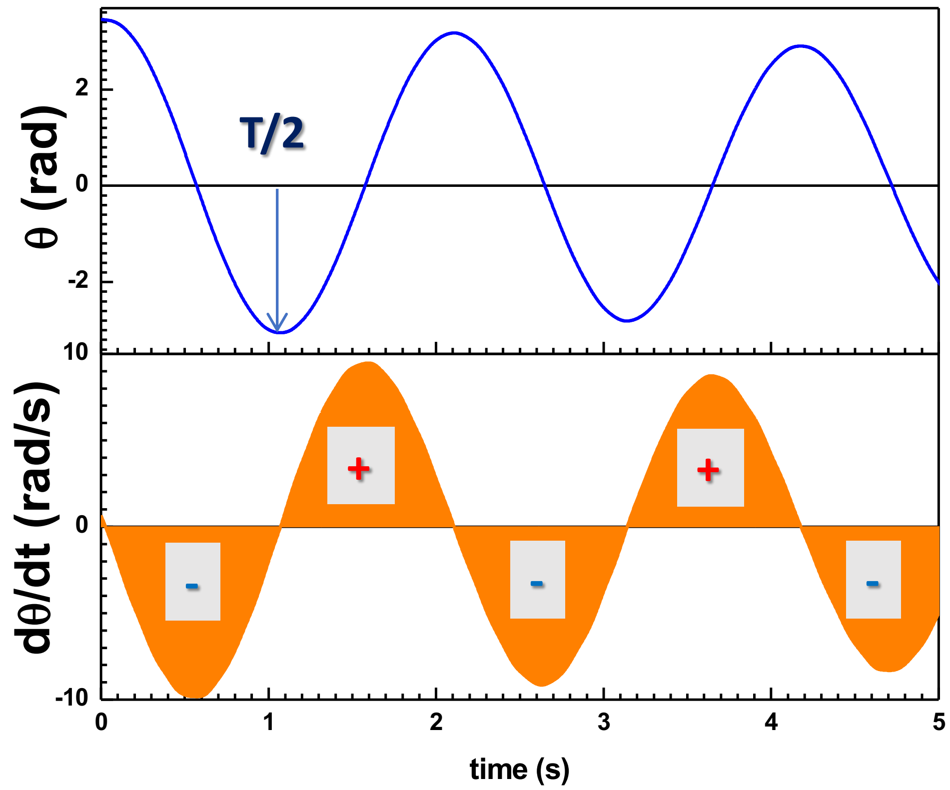
Coulomb Damping: Theory

$$I\frac{d^2\theta}{dt^2} + K\theta + \tau_{Coulomb} = 0$$
$$\tau_{Coulomb} = C \,\mathrm{sgn}\, \dot{\theta}$$

"Coulomb damping" is a historical term for ordinary **friction**.

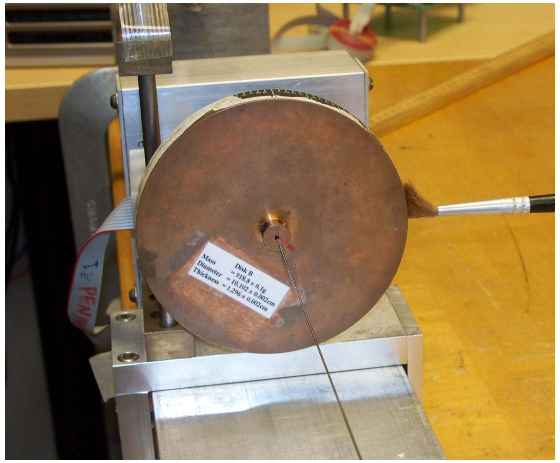
Constant torque opposing motion.

Amplitude decreases *linearly* by 4C/K each oscillation period!
But diff. eqn. is now nonlinear

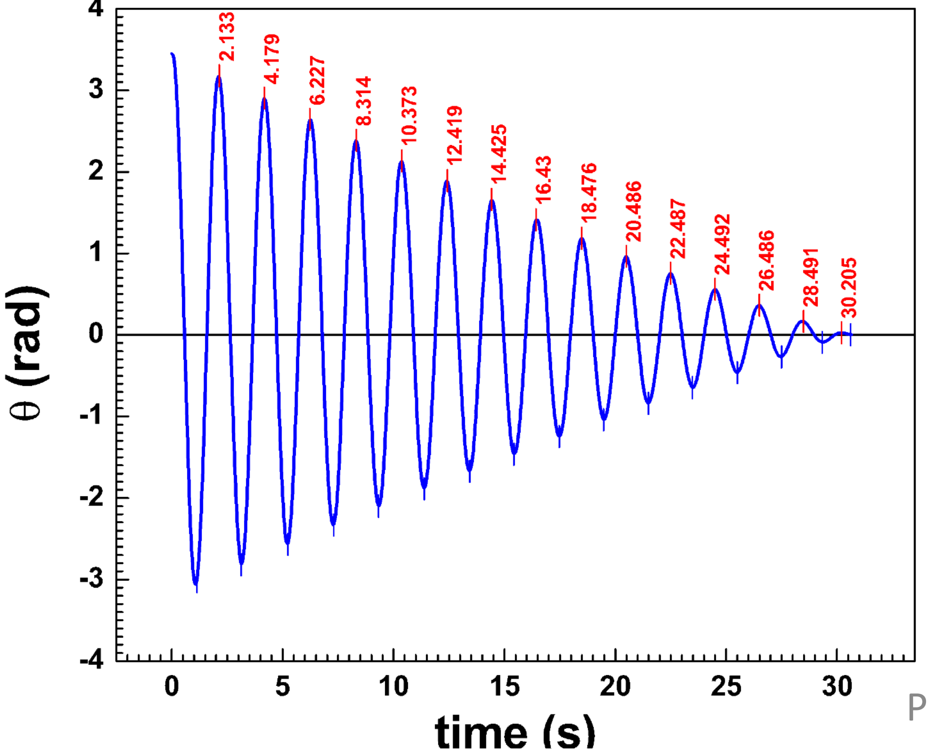


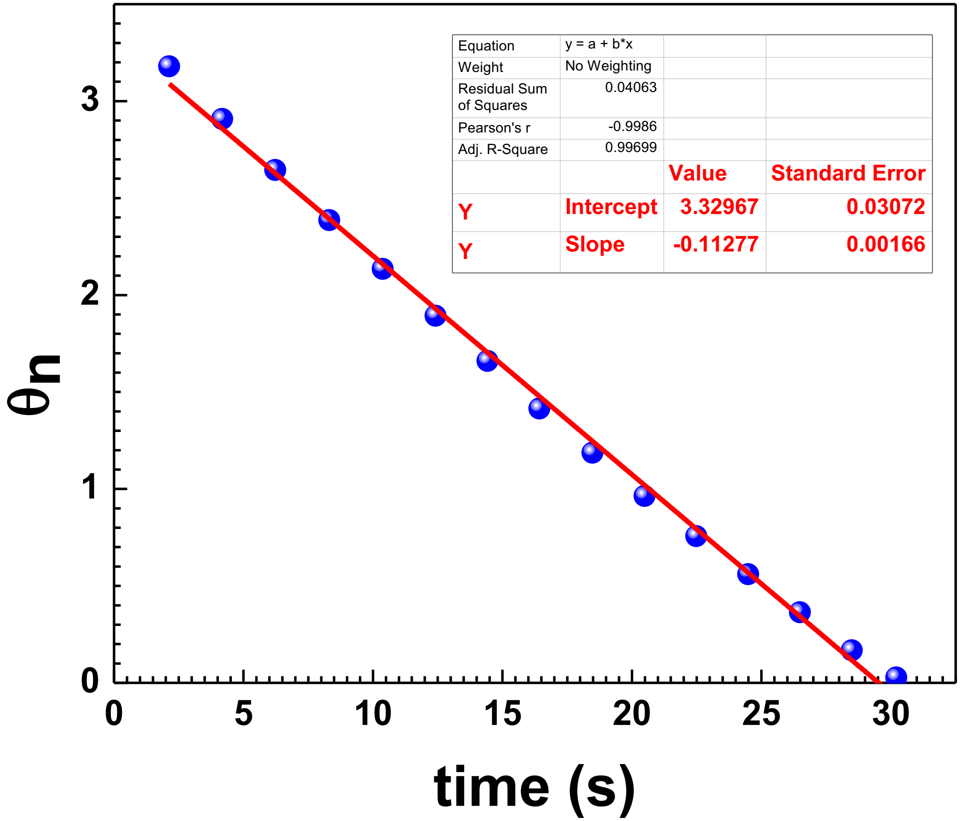
Kinetic friction damps motion
Static friction at turnarounds

Coulomb Damping: Experiment



Amplitude decreases *linearly* by 4C/K each oscillation period!

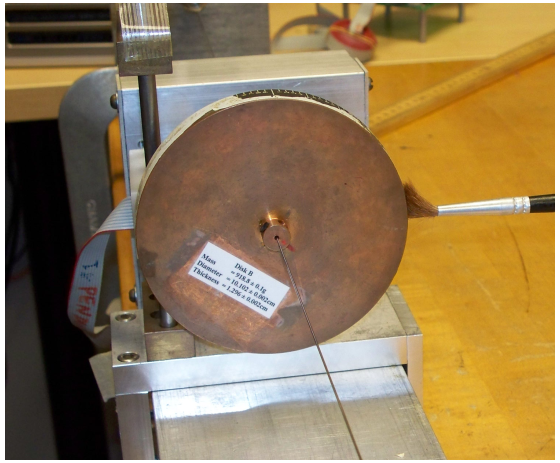


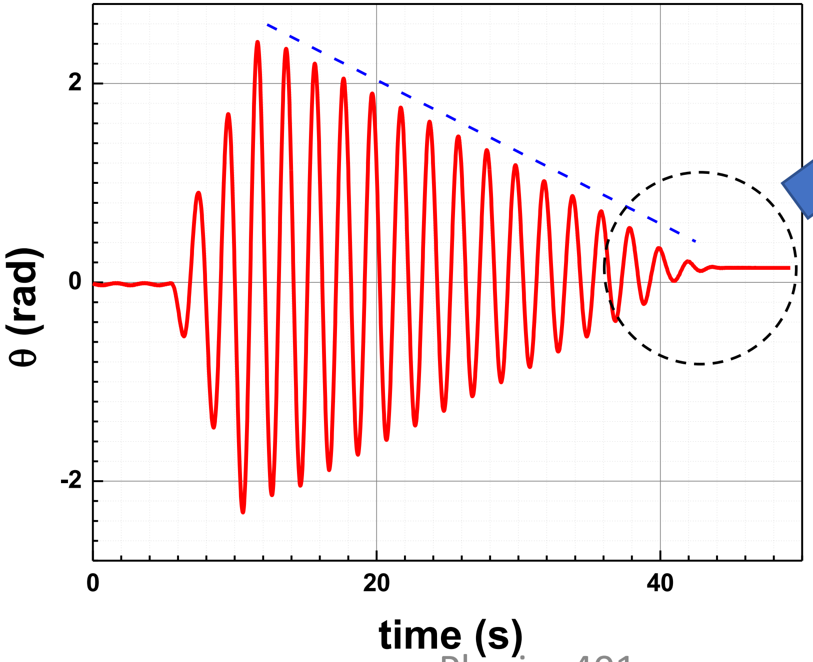


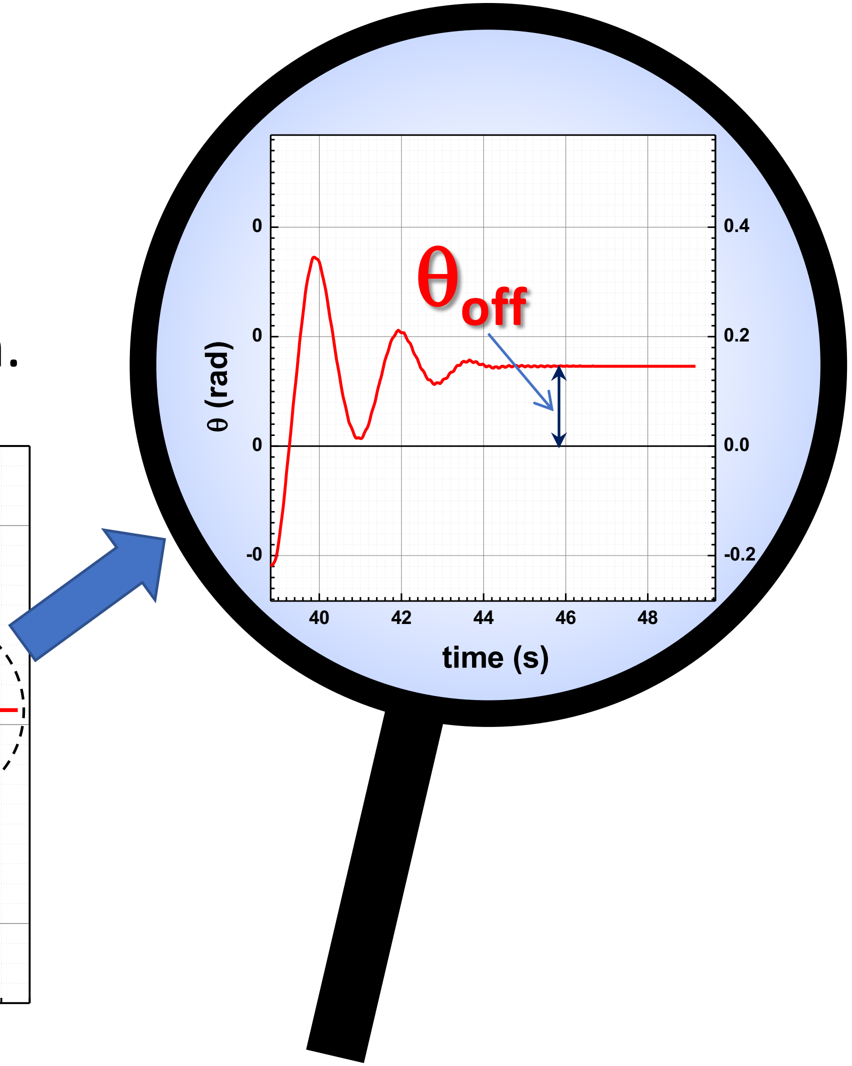
Coulomb Damping: Damping and Stopping

Since the damping torque is constant, eventually it exceeds the spring torque ($K\theta < C$) and the pendulum stops *away from equilibrium* ($\theta \neq 0$).

At a turnaround, static friction may prevent motion.





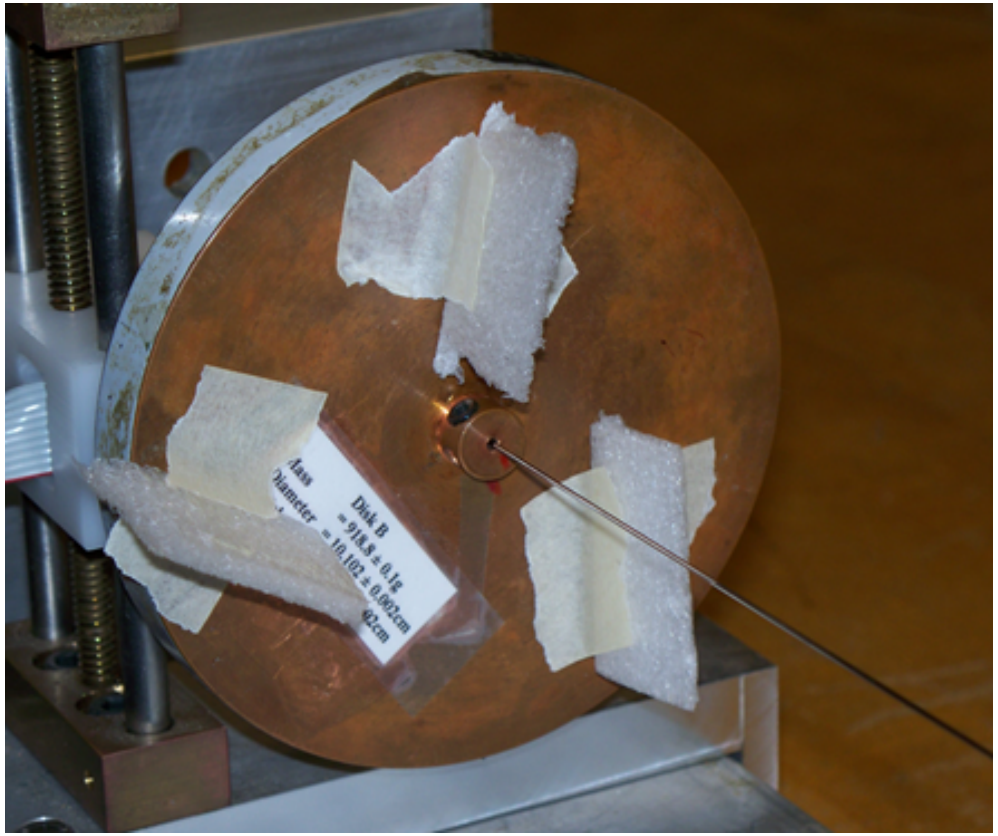


Three Damping Mechanisms

1. Viscous (Magnetic) Damping

2. Coulomb Damping

3. **Turbulent Damping**



Turbulent Damping: Theory

$$I\frac{d^2\theta}{dt^2} + K\theta + \tau_{Turb} = 0$$
$$\tau_{Turb} = C_t\, \mathrm{sgn}(\dot{\theta})\, |\dot{\theta}|^n$$

Limiting cases

***n*=0**: Coulomb damping

***n*=1**: Viscous damping

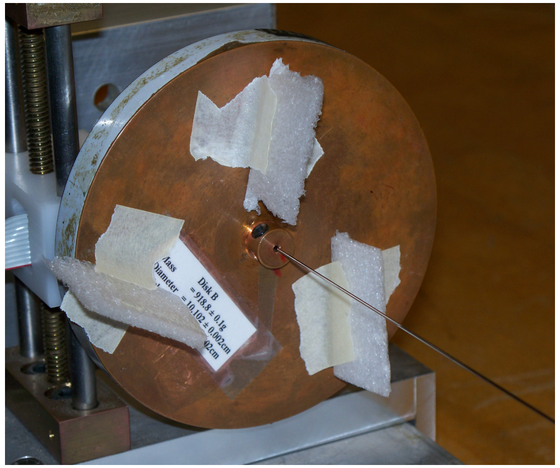
Wikipedia

With a power law exponent $n \neq 1$, the differential equation is nonlinear.

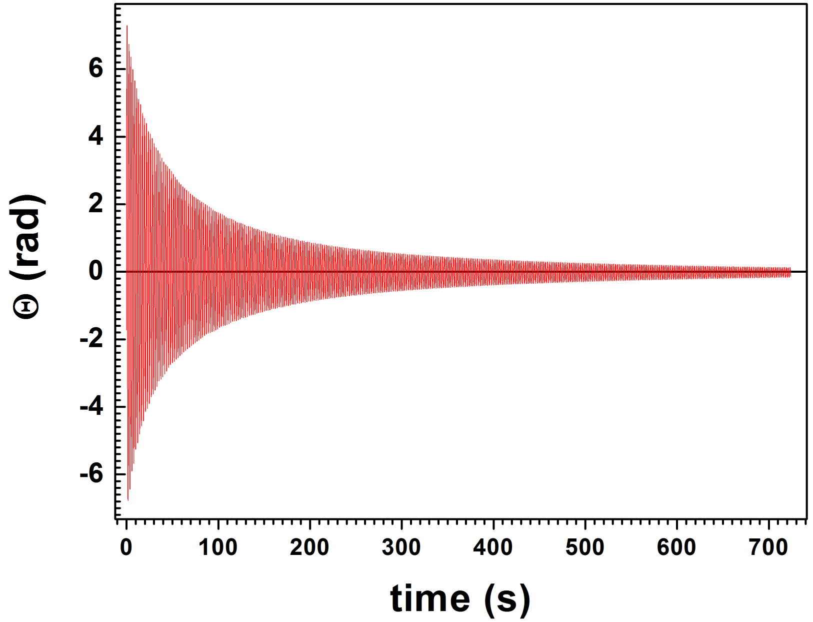
For $n > 1$ the logarithmic decrement decreases as the oscillation amplitude decreases – the oscillation decays "more slowly" as amplitude decreases.

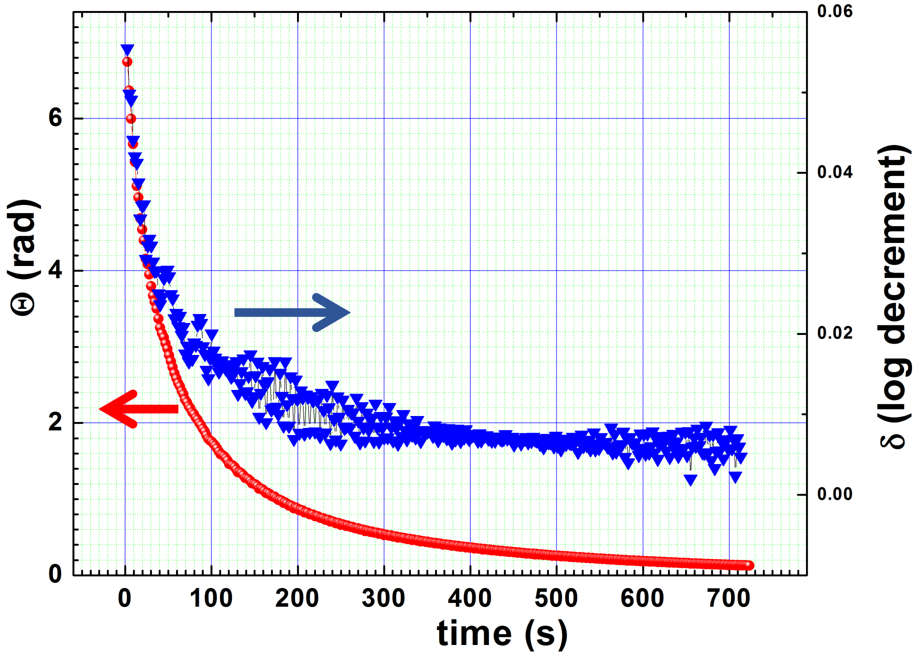
For $n = 2$ one can show the log-decrement is $\delta = \frac{8C}{3I}\theta_0$.

Turbulent Damping: Experiment

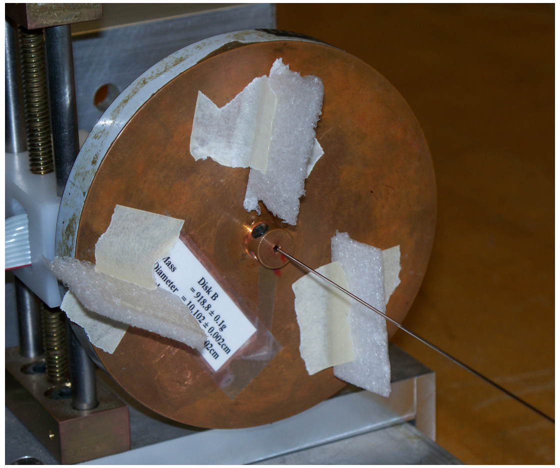


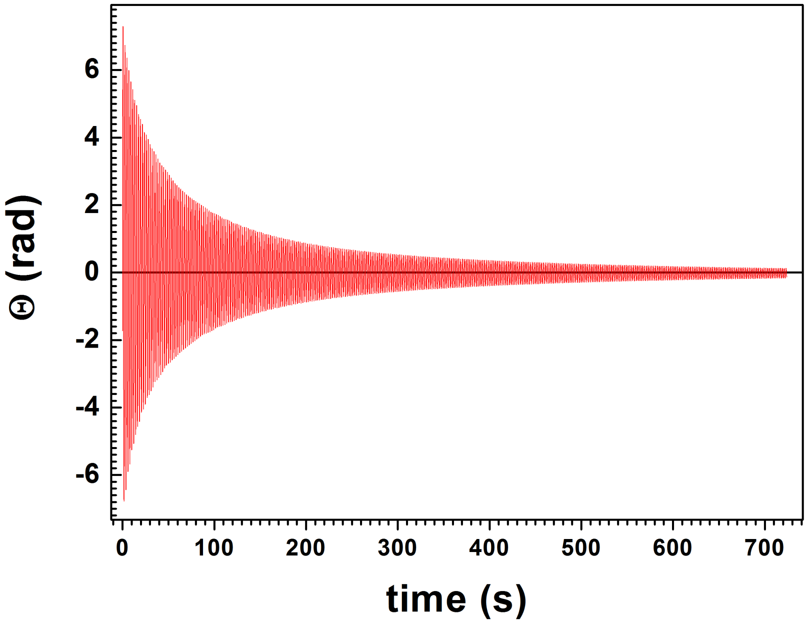
By analyzing the envelope of the damped oscillation over time, we can calculate the changing log-decrement

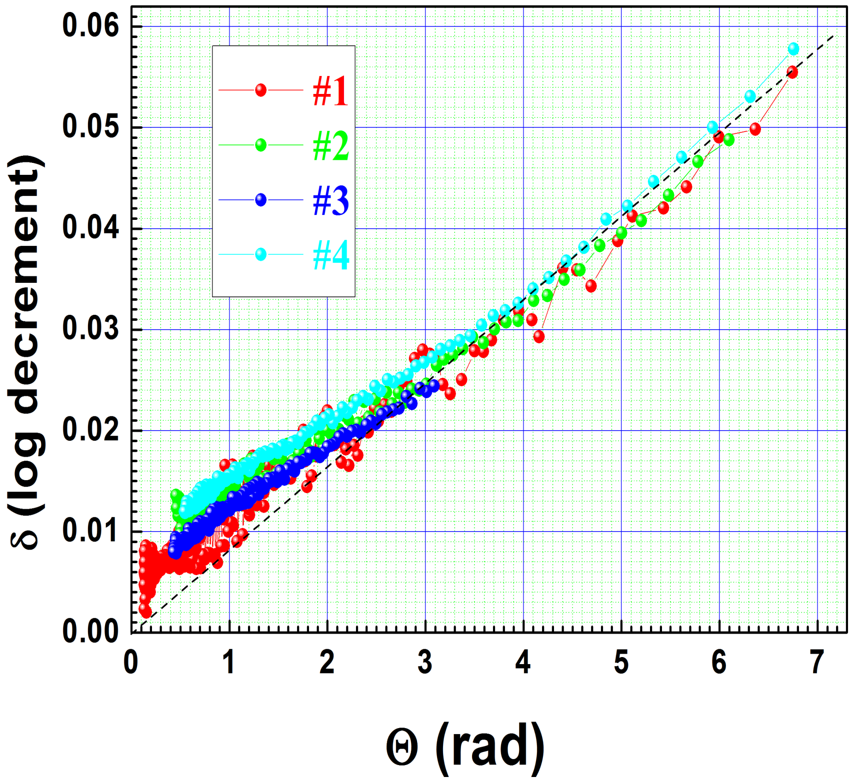




Turbulent Damping: Experiment







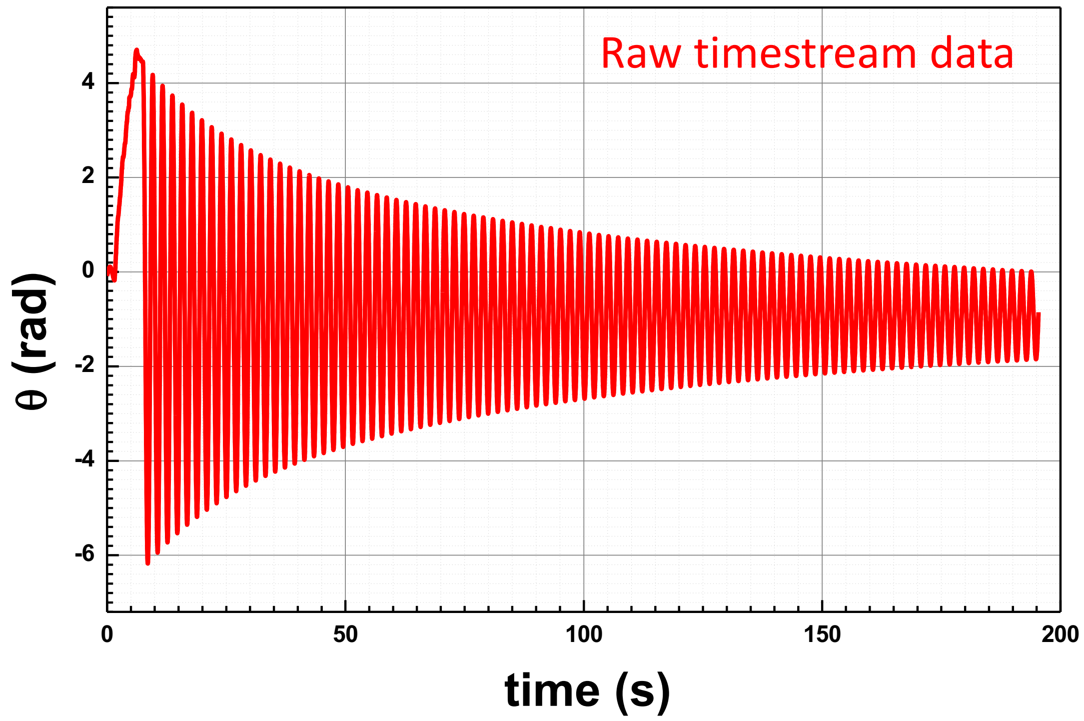
Damping: Analysis Reminders

Each technique gives somewhat different outputs:

1. Nonlinear fitting to SineDamp function yields resonance frequency and decrement coefficient (for amplitude decay)
2. Examining the FFT power spectrum yields resonance frequency

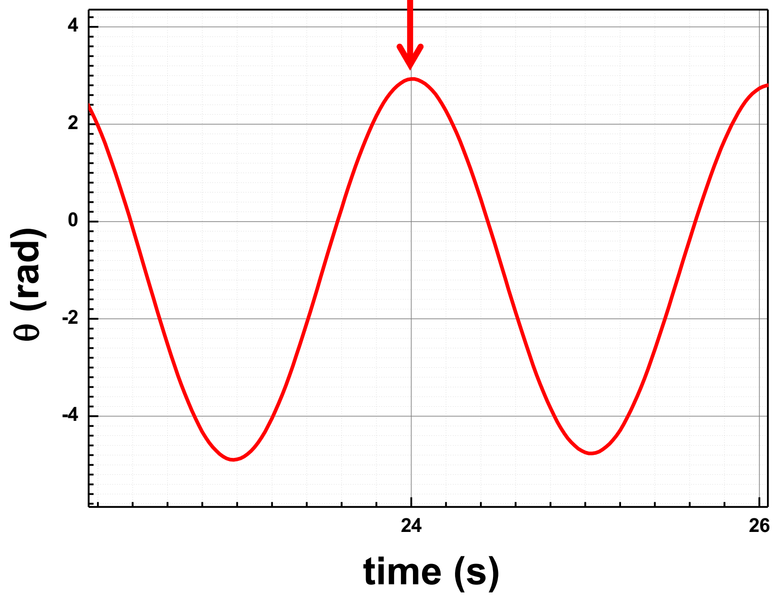
   Quality factor (decrement coefficient) is encoded in peak width
3. Using the Origin Peak Analyzer yields amplitudes and positions of the damped sine wave maxima, so we can plot and fit the envelope
4. You can also directly obtain the envelope of the damped sine wave using Origin (*optional*)

Data Analysis: Finding the Peaks



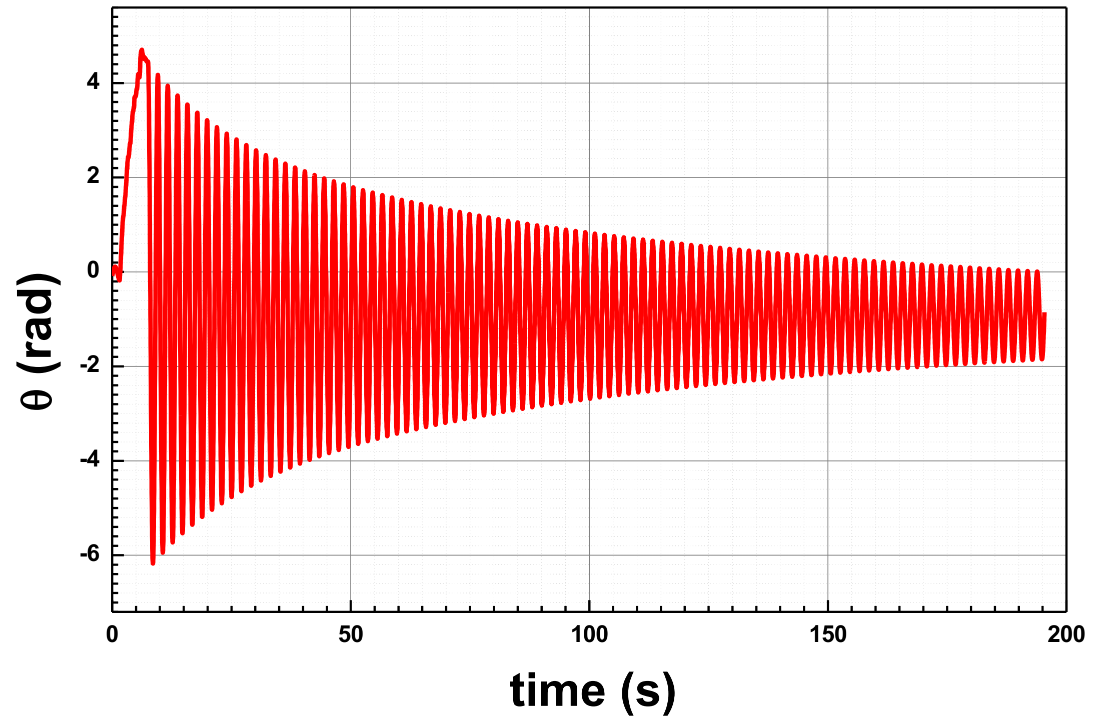
Goal: Find the positions and amplitudes of each oscillation peak

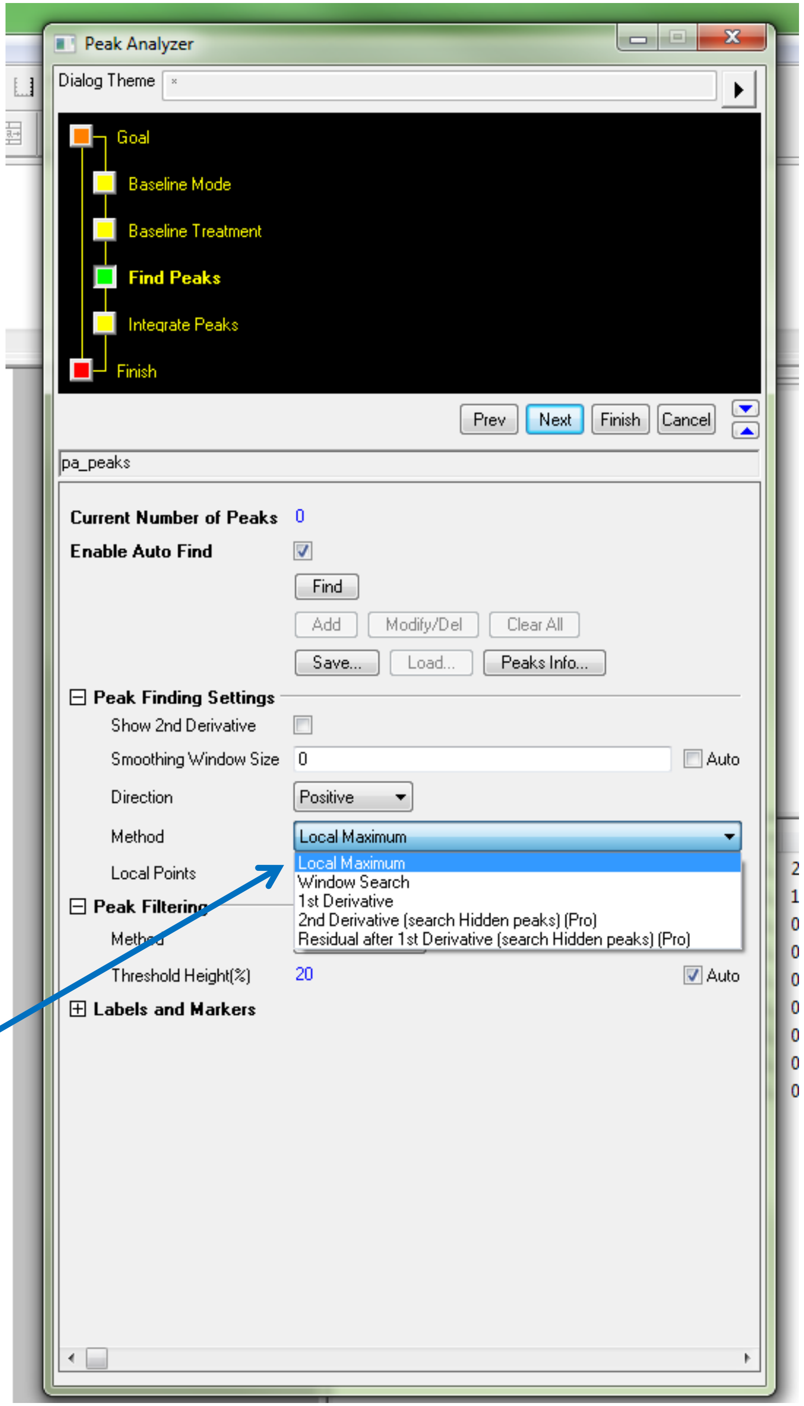
(x_i, y_i)



Technique #1: use the "**FindPeaks**" option

Data Analysis: Finding the Peaks



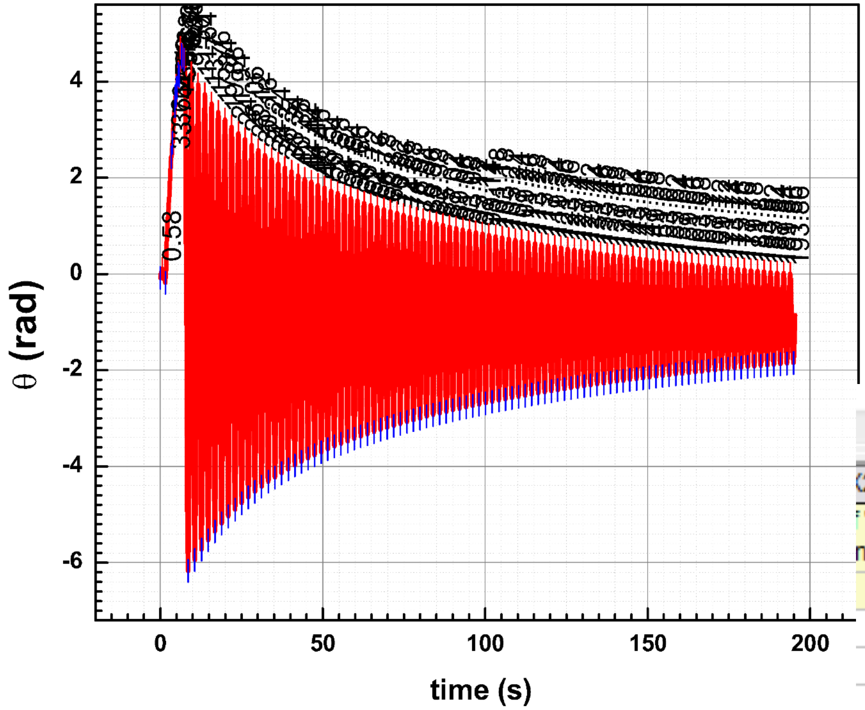


The **Local Maximum** option of Peak Analyzer works well for oscillations at relatively low noise levels

Data Analysis: Finding the Peaks

Files related to this project may be found in:

\\engr-file-03\PHYINST\APL Courses\PHYCS401\Students\6. Torsional oscillator\Turbulent damping.opj



New plot with labels, after peak-finding

Peaks data can be found in a worksheet

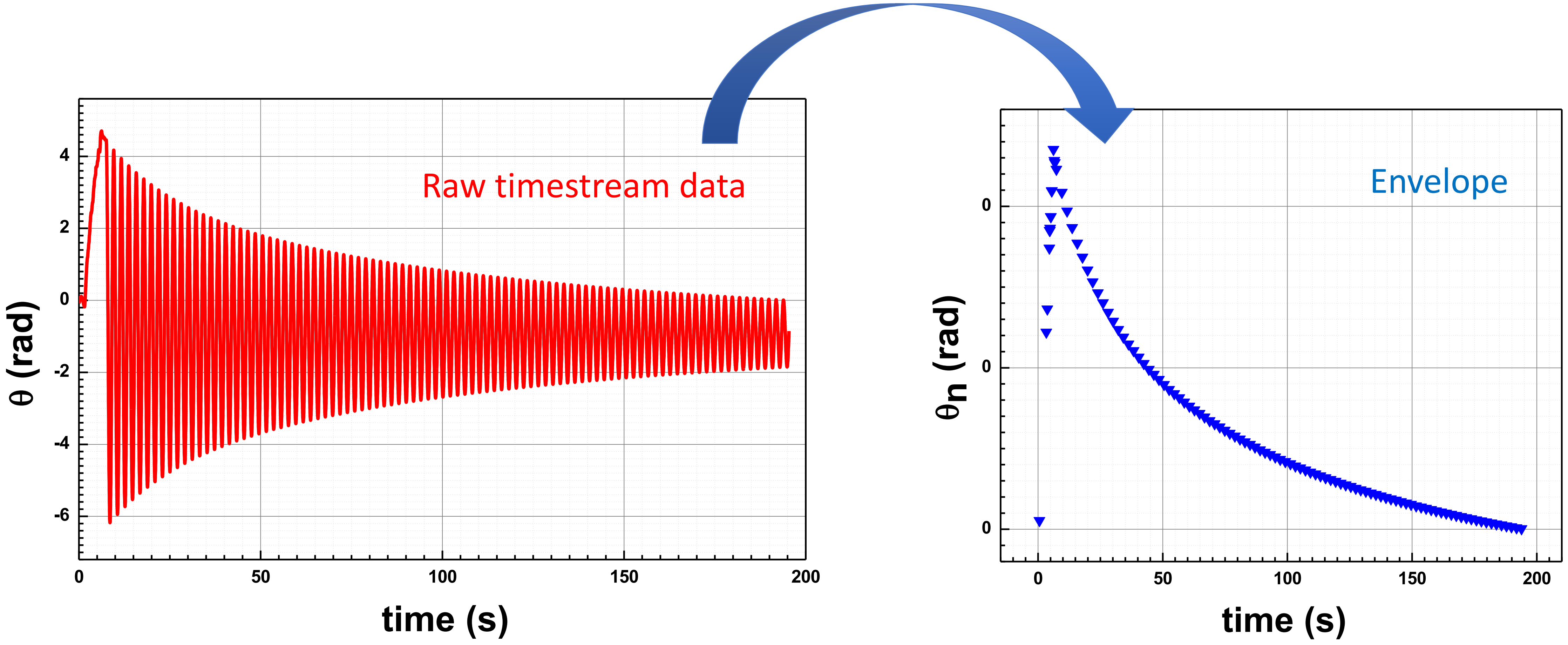
Using these data you can plot the dependence of amplitude on time

(2)	pcy(Y2)	pmx(X3)	pmy(Y3)
"Pend_theta ns)"	Peak Centers of "Pend_theta (radians)"	Base Markers of "Pend_theta (radians)"	Base Markers of "Pend_theta (radians)"
	Y	X	Y
0.58	0.10738	0	-0.0813
3.36	2.44056	1.54	-0.17948
3.76	2.72742	1.54	-0.17948
4.54	3.4829	3.44	2.42676
[illegible]	[illegible]	3.44	2.42676
[illegible]	[illegible]	[illegible]	[illegible]2128
4.88	3.72834	[illegible]	[illegible]2128
5.1	3.87177	4.54	3.4829
5.16	3.87253	4.54	3.4829

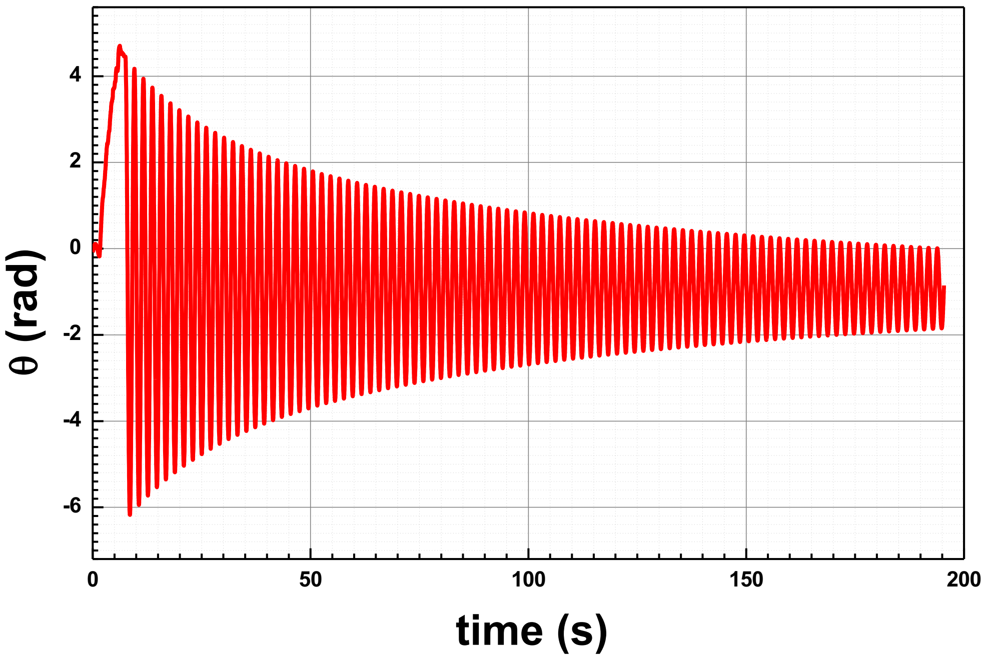
"Positive" peaks

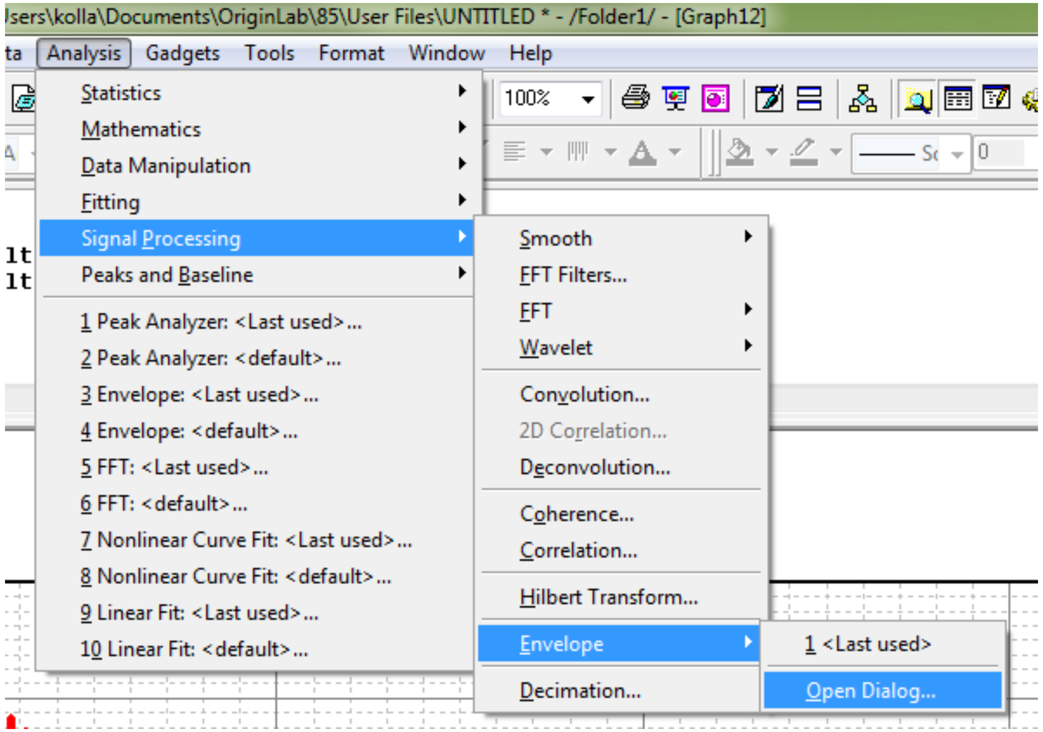
"Negative" peaks

Data Analysis: Finding the Peaks



Data Analysis: Finding the Peaks

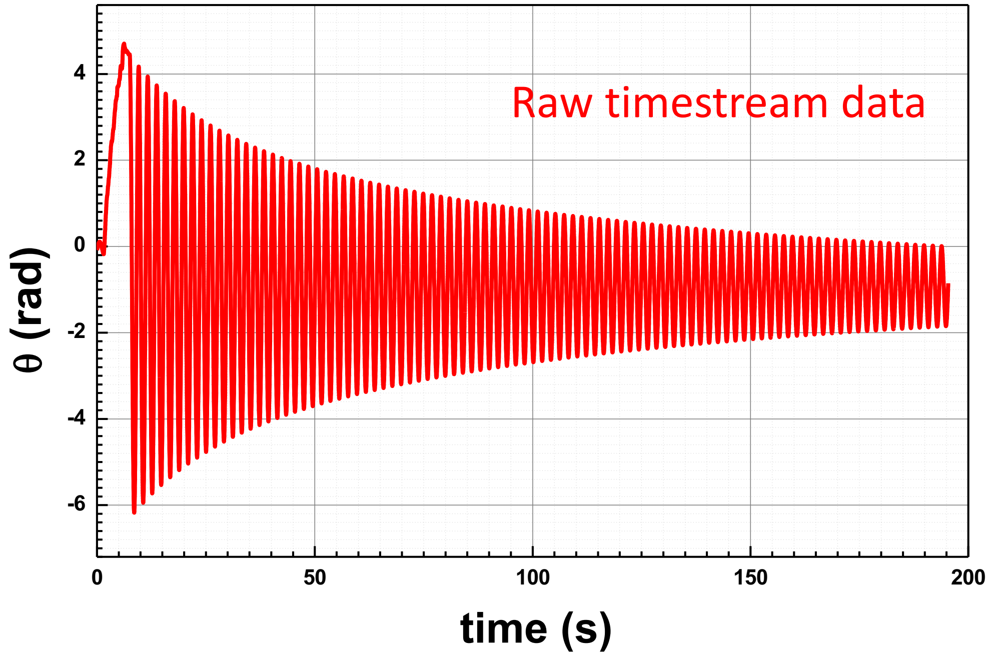


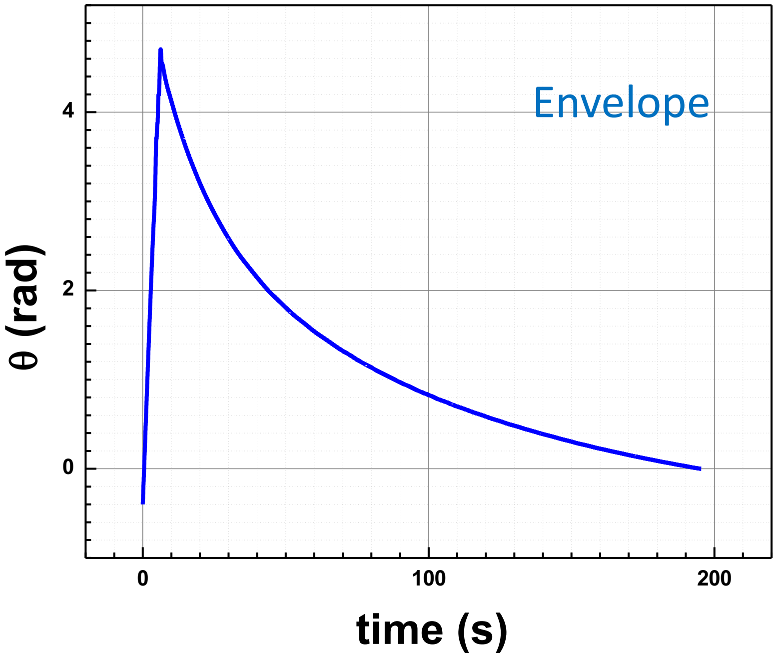


Technique #2: use the "**Envelope**" option

Origin will create a worksheet with envelope data, interpolated to the same *x*-values as the raw data

Data Analysis: Finding the Peaks



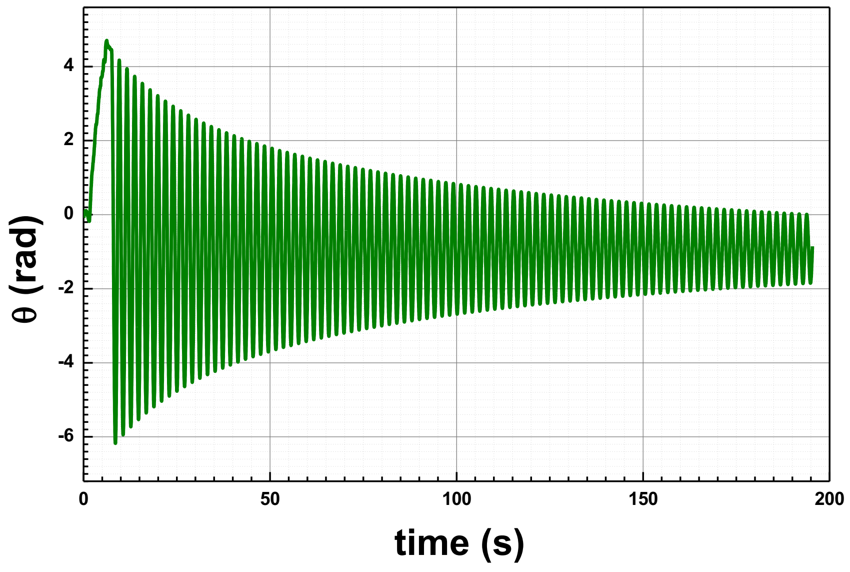


	A(X1)	B(Y1)	C(Y1)	X1(X2)	Y1(Y2)
ts				Upper Envelope of "Pend_theta (ra	Upper Envelope of "Pend_theta (radians)"
ie	Time	Pend_theta (radians)	Motr_theta (radians)	Envelope X 1	Envelope Y 1
1	0	-0.0813	0	0	-0.39883
2	0.02	-0.07286	0	0.02	-0.38144
3	0.04	-0.06443	0	0.04	-0.36403
4	0.06	-0.05522	0	0.06	-0.34662
5	0.08	-0.04679	0	0.08	-0.32921

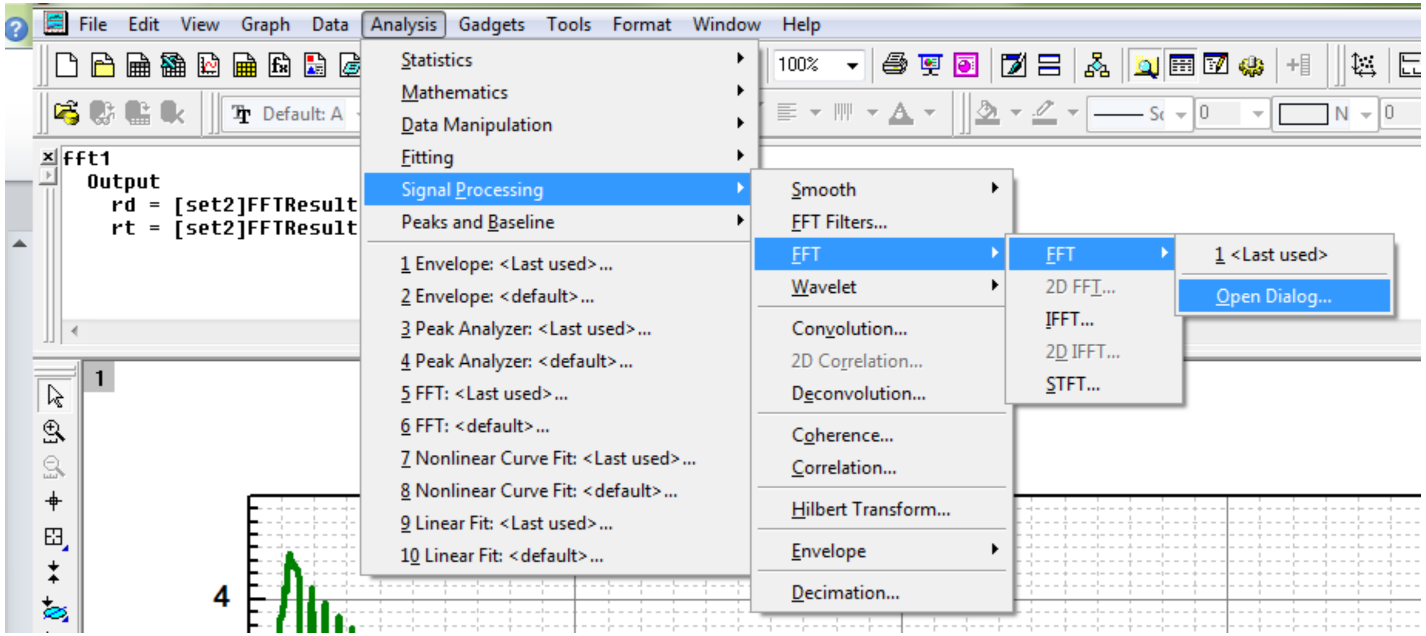
Original data

Envelope data

Data Analysis: Fourier Transform

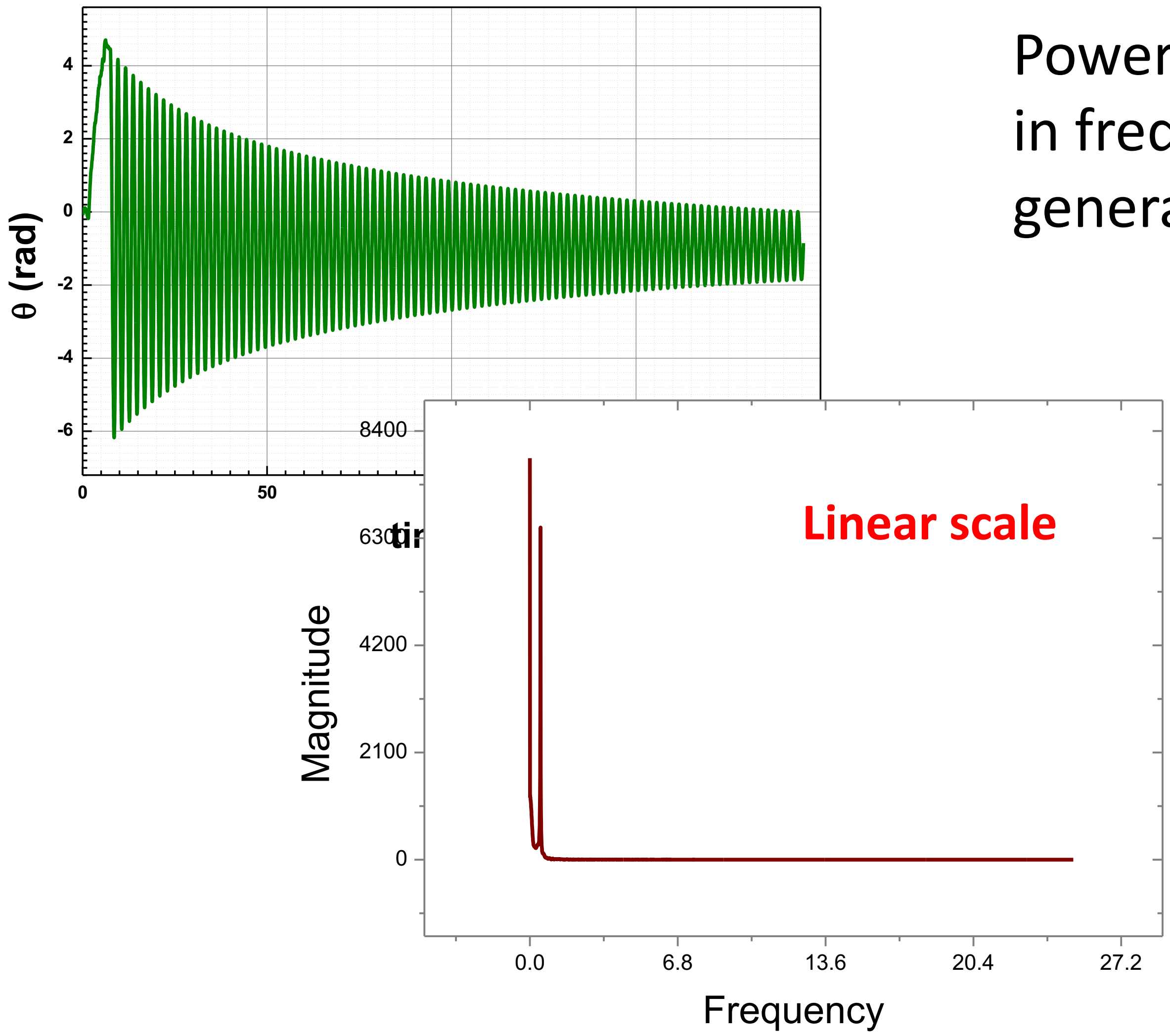


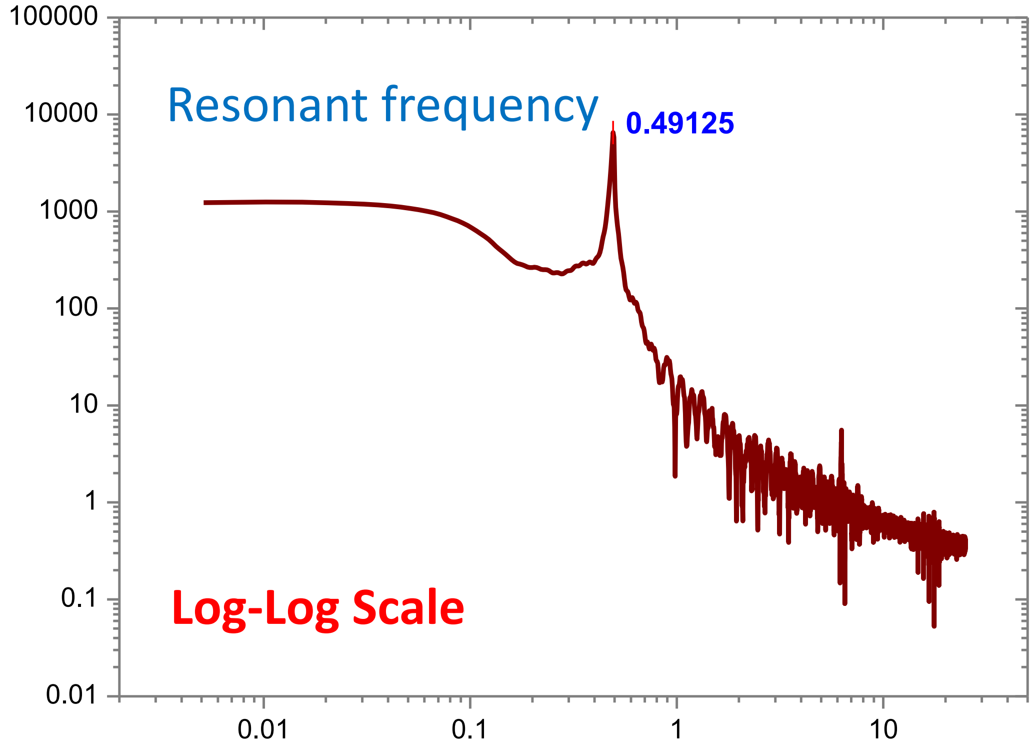
Technique #3: Use the Fourier Transform to compute the data's power spectrum and identify the resonant frequency.



Data Analysis: Fourier Transform

Power spectra have a wide dynamic range in frequency and amplitude, and so generally look better on a log-log scale





Appendices

Appendix: Reminder on Writing Reports

Take a moment to look back at the writing guidelines from Lecture 1

1. **Abstract**: brief; what, why, how results
2. **Introduction**: Motivation? What physics is involved?
   Theory/formulas linking measurements to underlying principles?
3. **Procedure**: Measurement concept/apparatus, operations
4. **Results**: Main findings, data analysis, error analysis
   If you fit a function, justify the function somewhere!
5. **Conclusions**: Main findings, compare to established results

Throughout: Complete sentences, logical narrative, sufficient detail

Appendix: Further Notes on Oil Drop Analysis

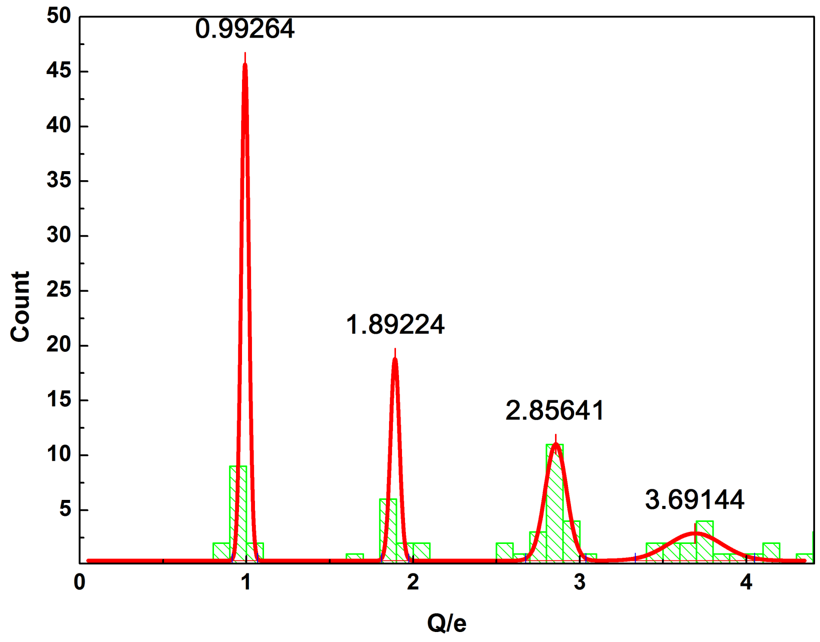
Measured value

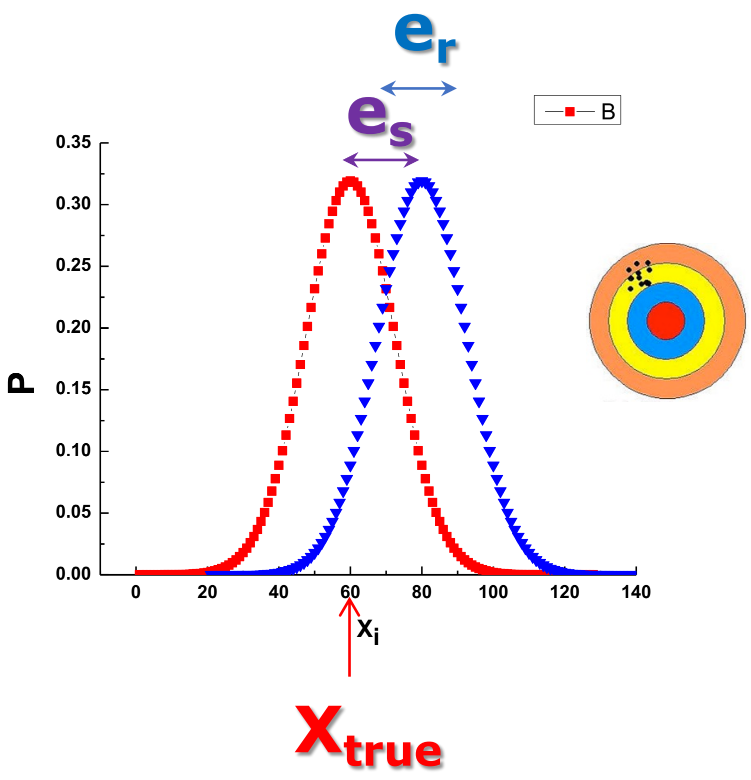
Systematic error

$$x_{meas} = x_{true} + e_s + e_r$$

Correct value

Random error





Appendix: Further Notes on Oil Drop Analysis

$$Q_{meas} = Q_{true} + e_s + e_r$$

$$Q = ne = F\,S\,T = \frac{1}{f_c^{2/3}}\;\frac{9\pi d}{V}\sqrt{\frac{2\eta^3 x^3}{g\rho}}\;\sqrt{\frac{1}{t_g}\left[\frac{1}{t_g}+\frac{1}{t_{rise}}\right]}$$

$$\frac{1}{f_c^{2/3}} \approx 1 - \left(\frac{t_g}{\tau_g}\right)^{\frac{1}{2}}$$

$$\Delta Q = \sqrt{\left(\frac{dQ}{dF}\right)^2(\Delta F)^2 + \left(\frac{dQ}{dS}\right)^2(\Delta S)^2 + \left(\frac{dQ}{dT}\right)^2(\Delta T)^2} \approx \sqrt{\left(\frac{dQ}{dS}\right)^2(\Delta S)^2 + \left(\frac{dQ}{dT}\right)^2(\Delta T)^2}$$

Generally negligible

$$= \sqrt{(FT)^2(\Delta S)^2 + (FS)^2(\Delta T)^2} = Q\sqrt{\left(\frac{\Delta S}{S}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

Appendix: Further Notes on Oil Drop Analysis

$$Q_{meas} = Q_{true} + \boldsymbol{e_s} + e_r$$

$$\boldsymbol{Q = ne = F\,S\,T = \frac{1}{f_c^{2/3}}\ \frac{9\pi d}{V}\sqrt{\frac{2\eta^3 x^3}{g\rho}}\ \sqrt{\frac{1}{t_g}\left[\frac{1}{t_g}+\frac{1}{t_{rise}}\right]}}$$

$$\Delta Q \approx Q\sqrt{\left(\frac{\Delta S}{S}\right)^2+\left(\frac{\Delta T}{T}\right)^2}$$

$$\frac{\Delta S}{S}=\sqrt{\left(\frac{\Delta d}{d}\right)^2+\left(\frac{\Delta V}{V}\right)^2+\left(\frac{3}{2}\frac{\Delta x}{x}\right)^2+\left(\frac{3}{2}\frac{\Delta\eta}{\eta}\right)^2+\left(\frac{1}{2}\frac{\Delta\rho}{\rho}\right)^2+\left(\frac{1}{2}\frac{\Delta g}{g}\right)^2}\approx\sqrt{\left(\frac{\Delta d}{d}\right)^2+\left(\frac{3}{2}\frac{\Delta x}{x}\right)^2}$$

$$\Delta T=\sqrt{\left(\frac{3/2}{t_g^{5/2}}+\frac{1/2}{t_g^{3/2}}\frac{1}{t_{rise}}\right)^2\Delta t_g^{\,2}+\left(\frac{1}{t_g^{1/2}}\frac{1}{t_{rise}^{\,2}}\right)^2\Delta t_{rise}^{\,2}}$$

Appendix: Further Notes on Oil Drop Analysis

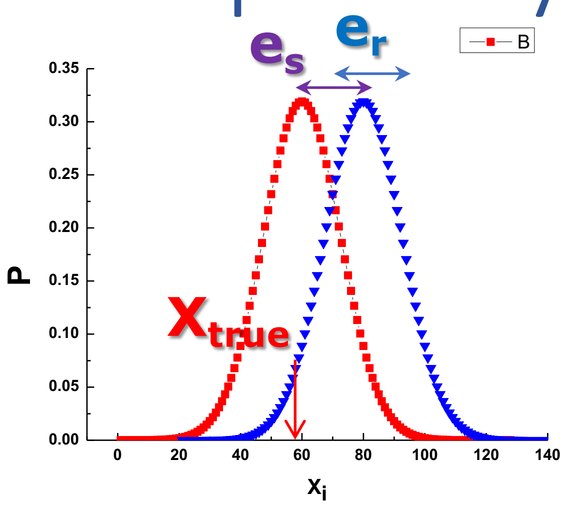
Measured value

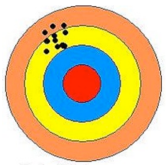
Systematic error

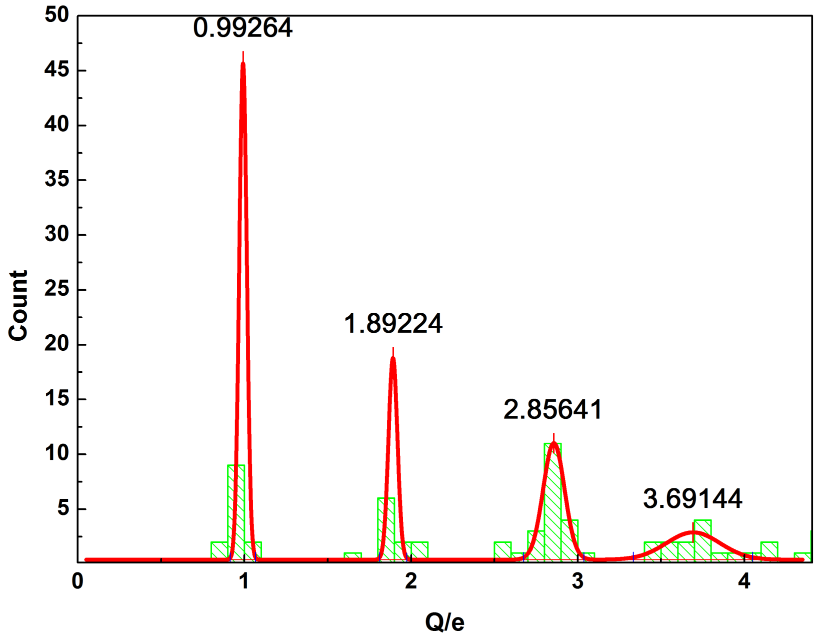
$$x_{meas} = x_{true} + e_s + e_r$$

Correct value

Random error







Set of N measurements $\{x_i\}$:

Mean: $\mu = \frac{1}{N}\sum_{i=0}^{N-1} x_i$

Standard deviation: $\sigma^2 = \frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \mu)^2$

Standard deviation of mean: $\sigma_X = \frac{\sigma}{N}$