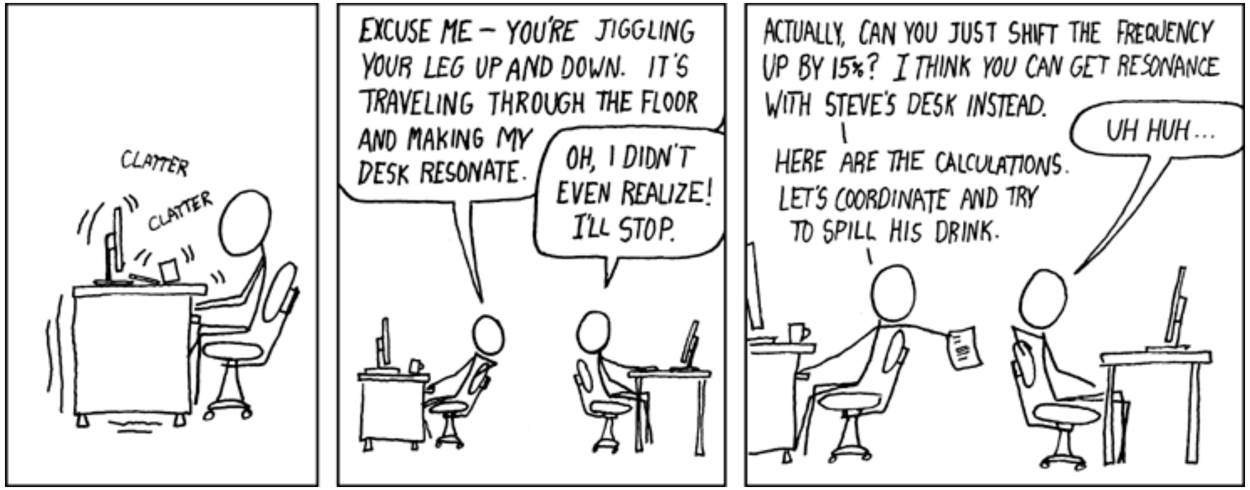
Torsional Oscillator

Episode II: Driven Response

Prof. Jeff Filippini

Physics 401

Spring 2020



XKCD #228

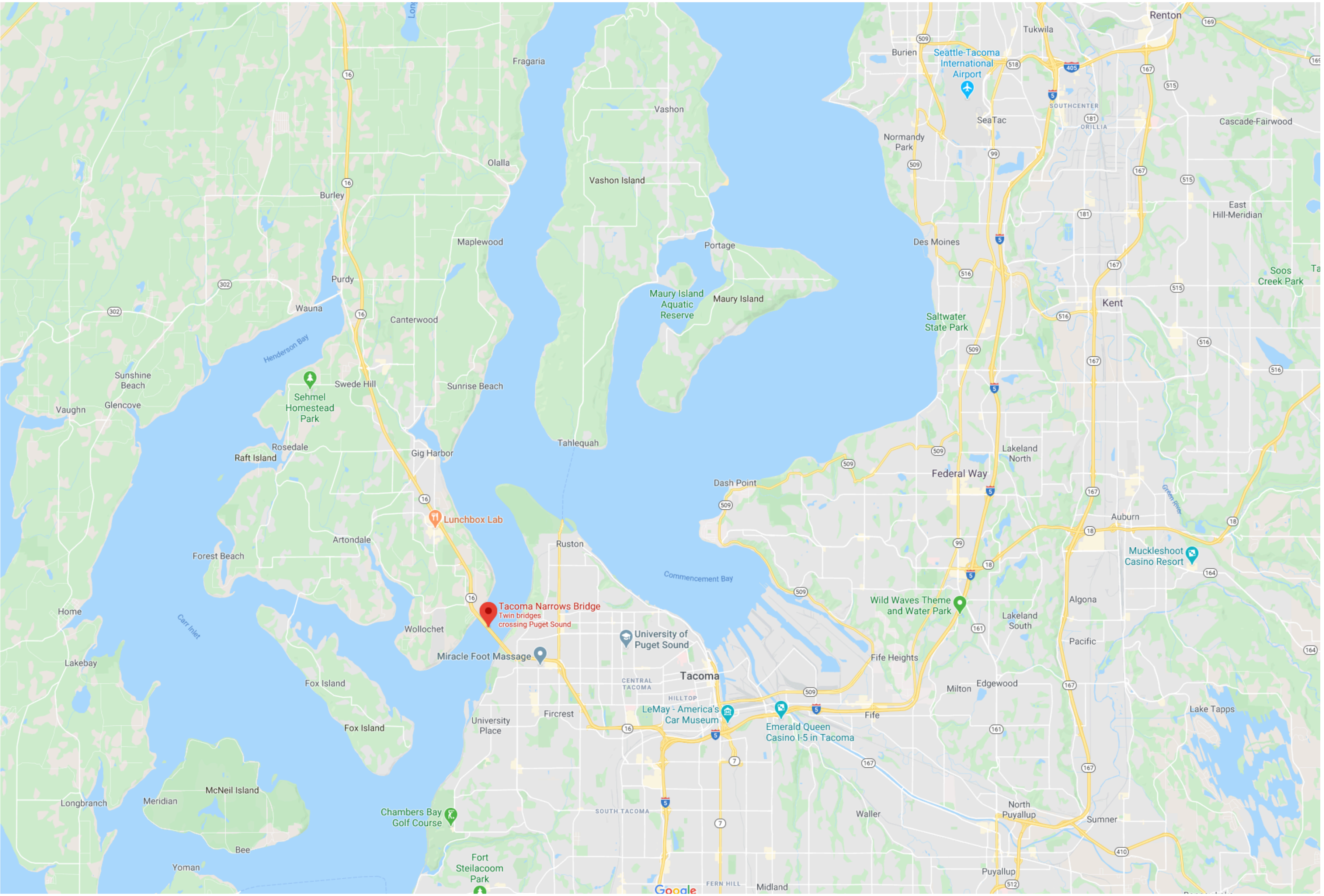
ILLINOIS

The Driven Torsional Oscillator

1. Driven torsional oscillator: Theory
2. Experimental setup and kinematics
3. Resonance
4. Beats
5. Nonlinear effects
6. Comments

Some Historical Examples

Tacoma Narrows (WA) Bridge



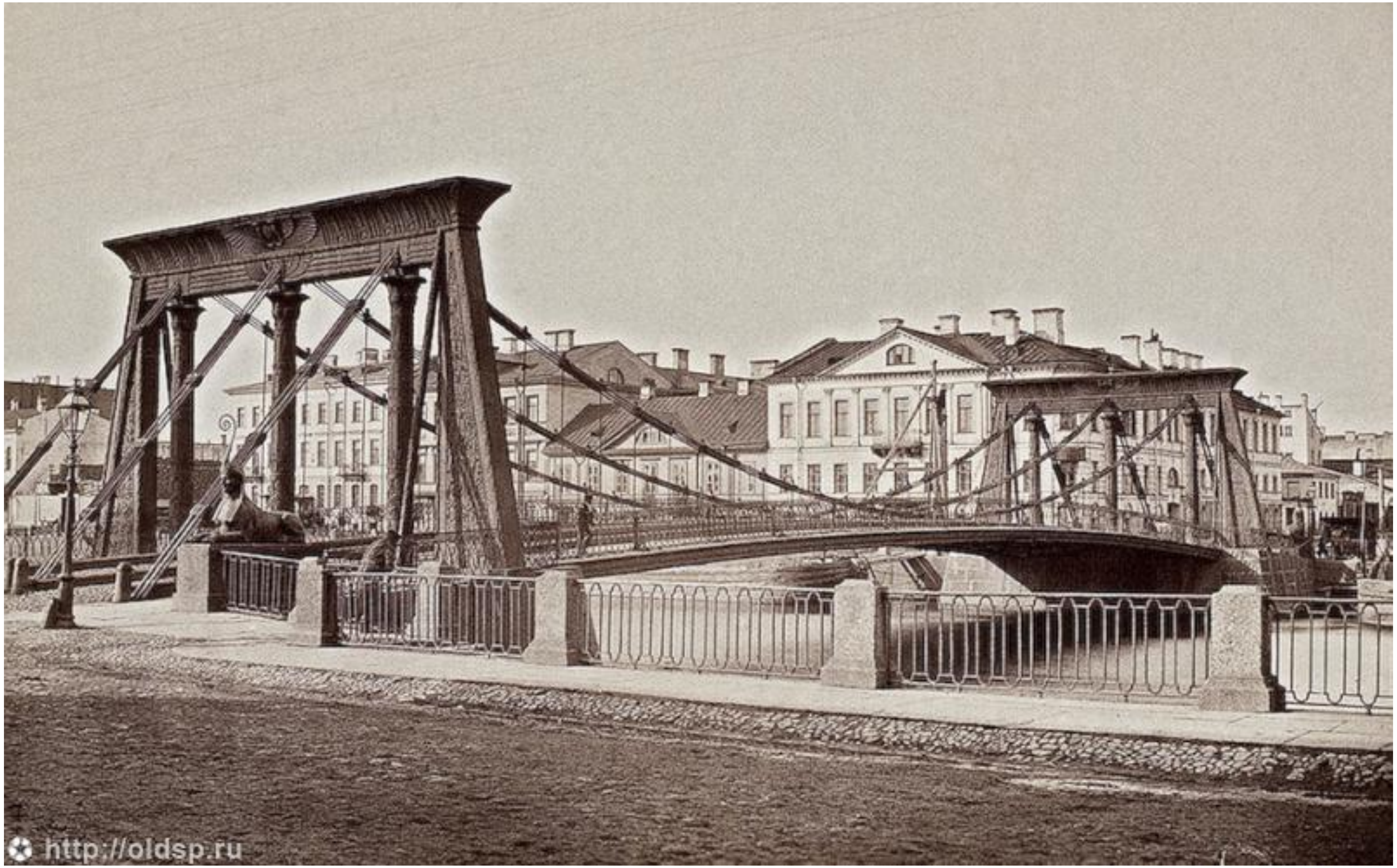
Tacoma Narrows (WA) Bridge - 1940

Tacoma Narrows (WA) Bridge - 1940

Tacoma Narrows (WA) Bridge

Note:1940 failure is *not* best understood as elementary forced resonance (as often described!), but instead a process called aerodynamic flutter. See Billah & Scanlan (1990).

Egyptian Bridge, St. Petersburg (1905)



Egyptian Bridge, St. Petersburg (1905)

Millennium Footbridge, London (2000)

Video #1

Video #2

Millennium Footbridge, London (2000)

Mitigations (2002)

Viscous dampers

Tuned mass inertial dampers

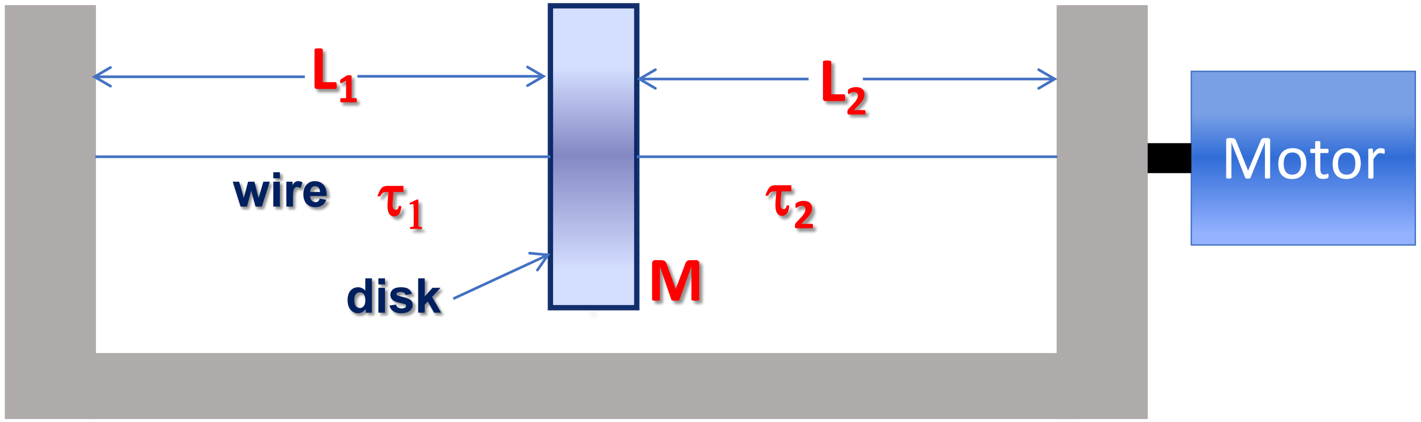
Flutter in Aviation

Milestones in Flight History
Dryden Flight Research Center

PA-30 Twin Commanche
Tail Flutter Test
April 5, 1966

Introducing the Driven Torsional Oscillator

Goals: For a damped, driven torsion oscillator, analyze the response to a sinusoidal drive, the transient response, and the steady state solution



Angular displacement:

$$\theta_0\cos(\omega t)$$

Torque:

$$K\lambda\theta_0\cos(\omega t)$$

$$\lambda = \frac{L_1}{L_1 + L_2}$$

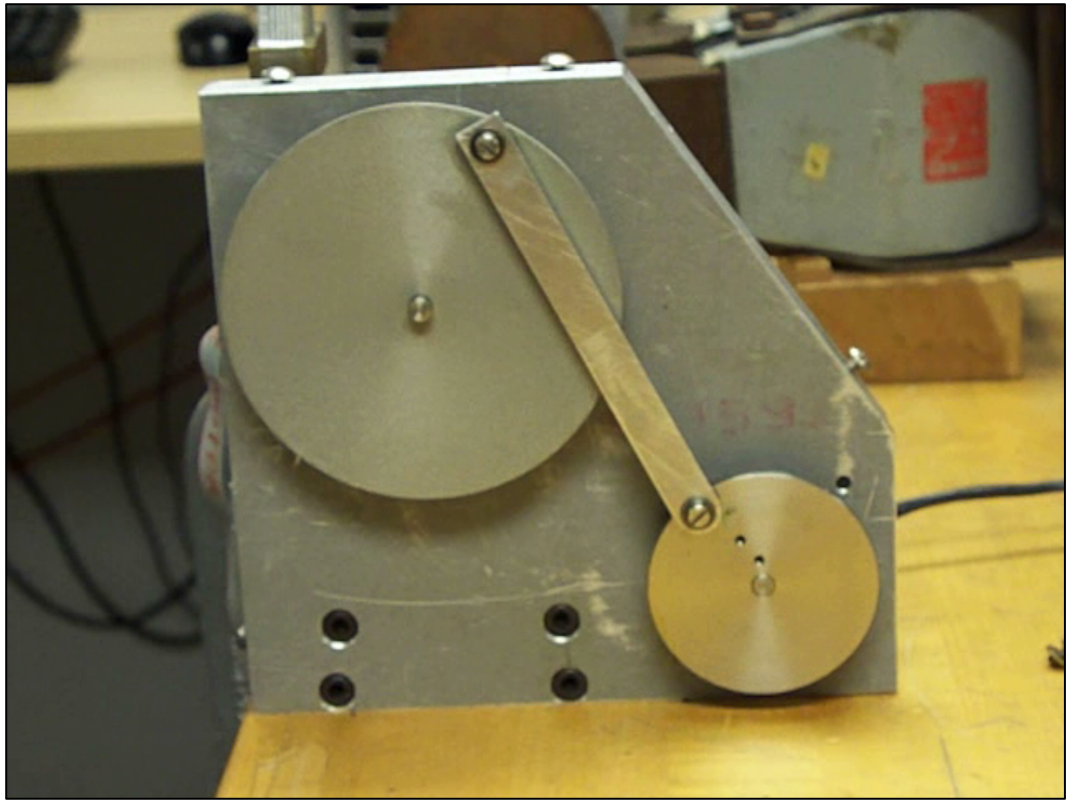
$$I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)$$

Viscous damping

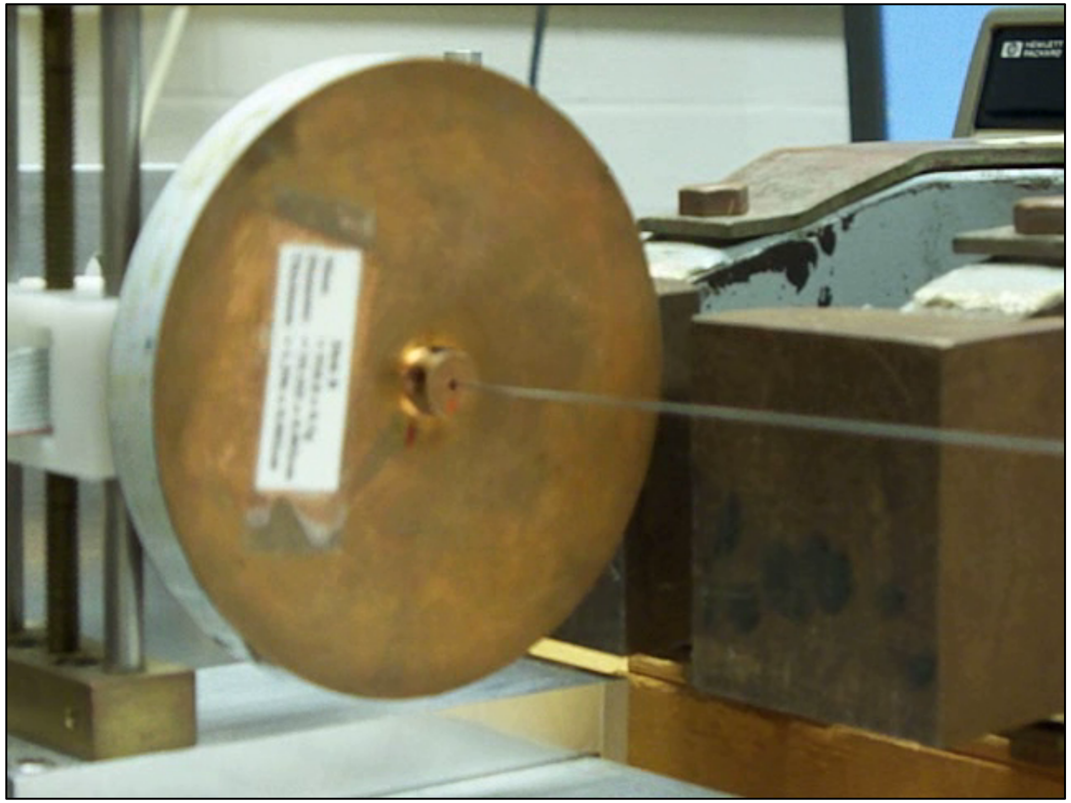
Torque by motor

θ : angular deflection of the disk
I : moment of inertia [kg-m^2]
R : damping constant [N-m-s]
K : torsional spring constant [N-m]

Experimental Setup



Motor



Pendulum

Anatomy of a Solution

$$I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)$$

Solutions are the sum of two components:

Homogeneous 1. **Transient** solution (*last week!*) Temporary, to match initial conditions.

Particular 2. **Steady-state** solution Persistent, due to driving torque τ_m.

Last week

$I\ddot{\theta}+R\dot{\theta}+K\theta=0$	**The homogeneous equation of motion**
$\theta(t)=Ae^{-at}\cos(\omega_1 t-\phi)$	**Transient solution**
$a=R/2I$	**Attenuation constant**
$\omega_o=\sqrt{K/I}$	**Natural (angular) frequency**
$\omega_1=\sqrt{\omega_o^2-a^2}$	**Damped (angular) frequency**

Steady-State Solution

1. **Transient** Solution

Initially the system responds at its characteristic frequency ω_1

$$\theta_t(t) = |A| e^{-at}\cos(\omega_1 t + \phi) \quad \rightarrow \quad \omega_1 = \sqrt{\omega_0^2 - a^2}$$

Once this response dies away, the system responds only at the driving frequency ω

So the steady-state solution must have the same time dependence as the drive

2. **Steady-State** Solution

$$\theta_{ss}(t) = \mathrm{Re}\left(\theta(\omega) e^{i\omega t}\right) \quad \Rightarrow \quad I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)$$

Substituting $\theta_{ss}(t)$ in equation of motion we will find the equations for $\theta(\omega)$

$$\theta(\omega) = \frac{\lambda\omega_0^2\theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 a^2}} e^{-i\beta(\omega)}$$

and

$$\beta(\omega) = \tan^{-1}\left(\frac{2\omega a}{\omega_0^2 - \omega^2}\right)$$

Steady-State Solution

$$\boldsymbol{I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda\theta_0\cos(\omega t)}$$

$$\theta_s(t) = B(\omega)\cos\left(\omega t - \beta(\omega)\right)$$

Steady state solution

$$B(\omega) = \frac{\lambda\theta_o\omega_o^2}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 + \omega^2\gamma^2}}$$

Amplitude function

$$\tan\beta(\omega) = \frac{\omega\gamma}{\omega_o^2 - \omega^2}$$

Phase function

$$\gamma = \frac{R}{I} = 2\frac{R}{2I} = 2a$$

Damping constant

Putting It All Together

So the time-domain form for the steady-state solution is:

$$\theta_{ss}(t)=\frac{\lambda\omega_0^2\theta_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\omega^2a^2}}\cos(\omega t-\beta(\omega))$$

Phase

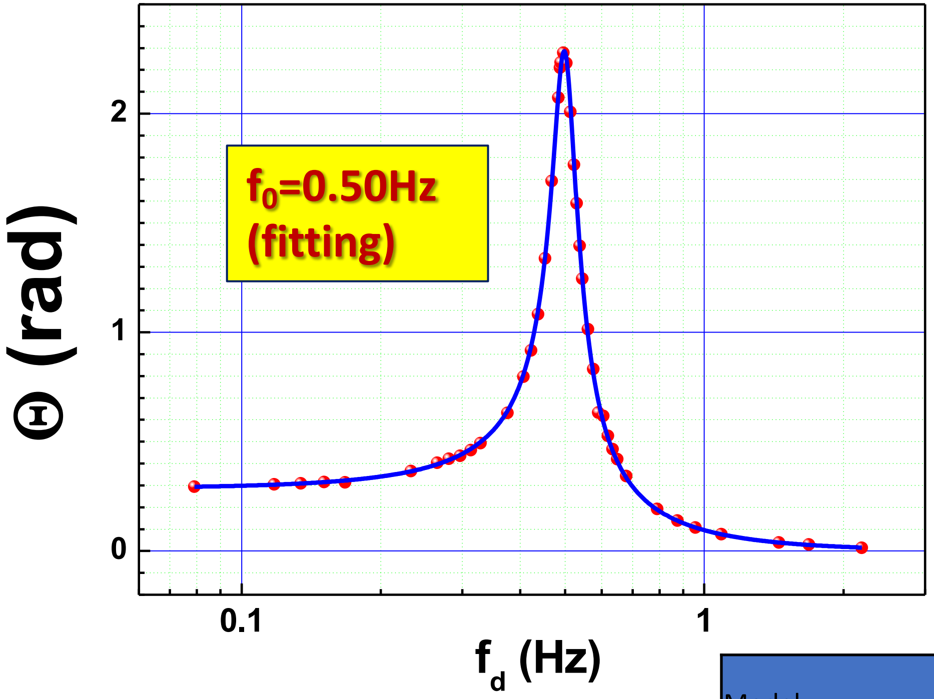
Amplitude ***B(ω)***

With homogeneous and particular solutions now in hand, the general solution to the equation of motion is a sum of these components:

$$\theta(t)=\theta_t(t)+\theta_{ss}(t)=Ae^{-at}\cos(\omega_1 t-\phi)+B\cos(\omega t-\beta(\omega))$$

Coefficients *A* and ϕ are determined by initial conditions

Resonance: Amplitude

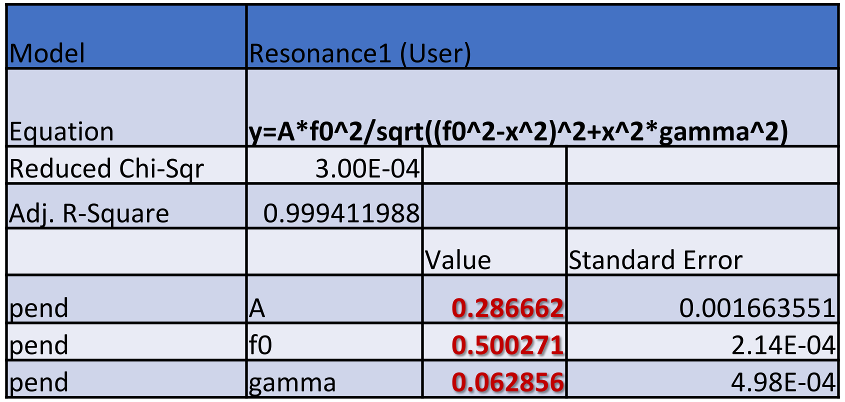


Fitting function:

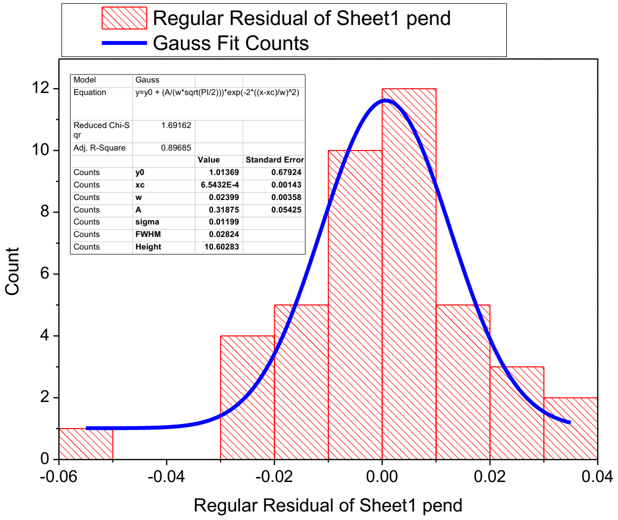
$$\theta(f) = \frac{A \bullet f_0^2}{\sqrt{\left(f_0^2 - f^2\right)^2 + \gamma^2 f^2}}$$

$\omega=2\pi f;\ \gamma=2a$

To create a new fitting function go "*Tools*"→"*Fitting Function Builder*" or press *F8*

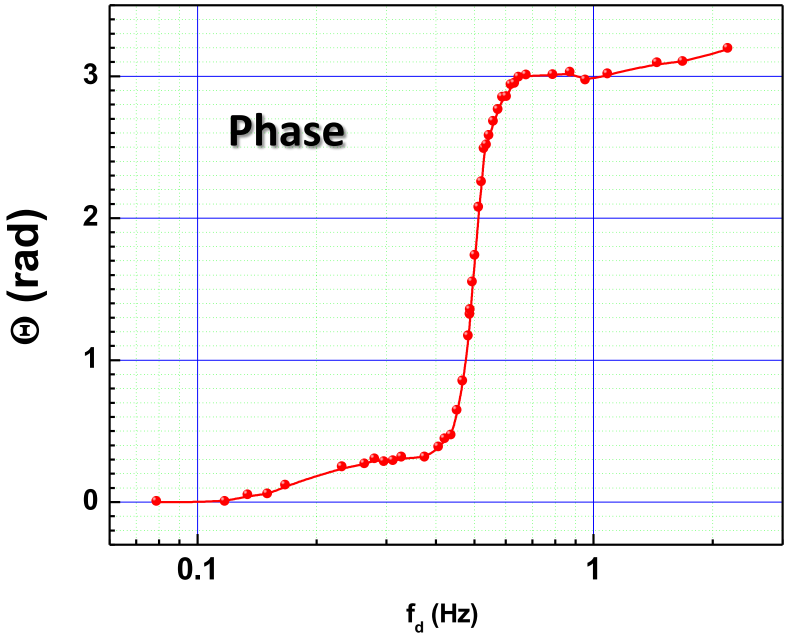


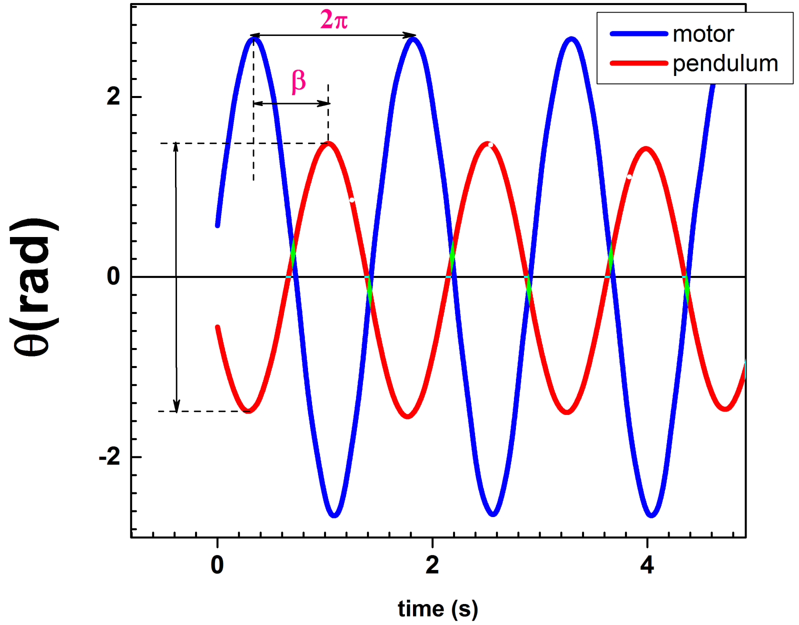
Model	Resonance1 (User)		
Equation	**y=A*f0^2/sqrt((f0^2-x^2)^2+x^2*gamma^2)**		
Reduced Chi-Sqr	3.00E-04		
Adj. R-Square	0.999411988		
		Value	Standard Error
pend	A	**0.286662**	0.001663551
pend	f0	**0.500271**	2.14E-04
pend	gamma	**0.062856**	4.98E-04



Model	Gauss		
Equation	y=y0 + (A/(w*sqrt(PI/2)))*exp(-2*((x-xc)/w)^2)		
Reduced Chi-Sqr	1.69162		
Adj. R-Square	0.89685		
		Value	**Standard Error**
Counts	**y0**	**1.01369**	**0.67924**
Counts	**xc**	**6.5432E-4**	**0.00143**
Counts	**w**	**0.02399**	**0.00358**
Counts	**A**	**0.31875**	**0.05425**
Counts	**sigma**	**0.01199**	
Counts	**FWHM**	**0.02824**	
Counts	**Height**	**10.60283**	

Resonance: Phase

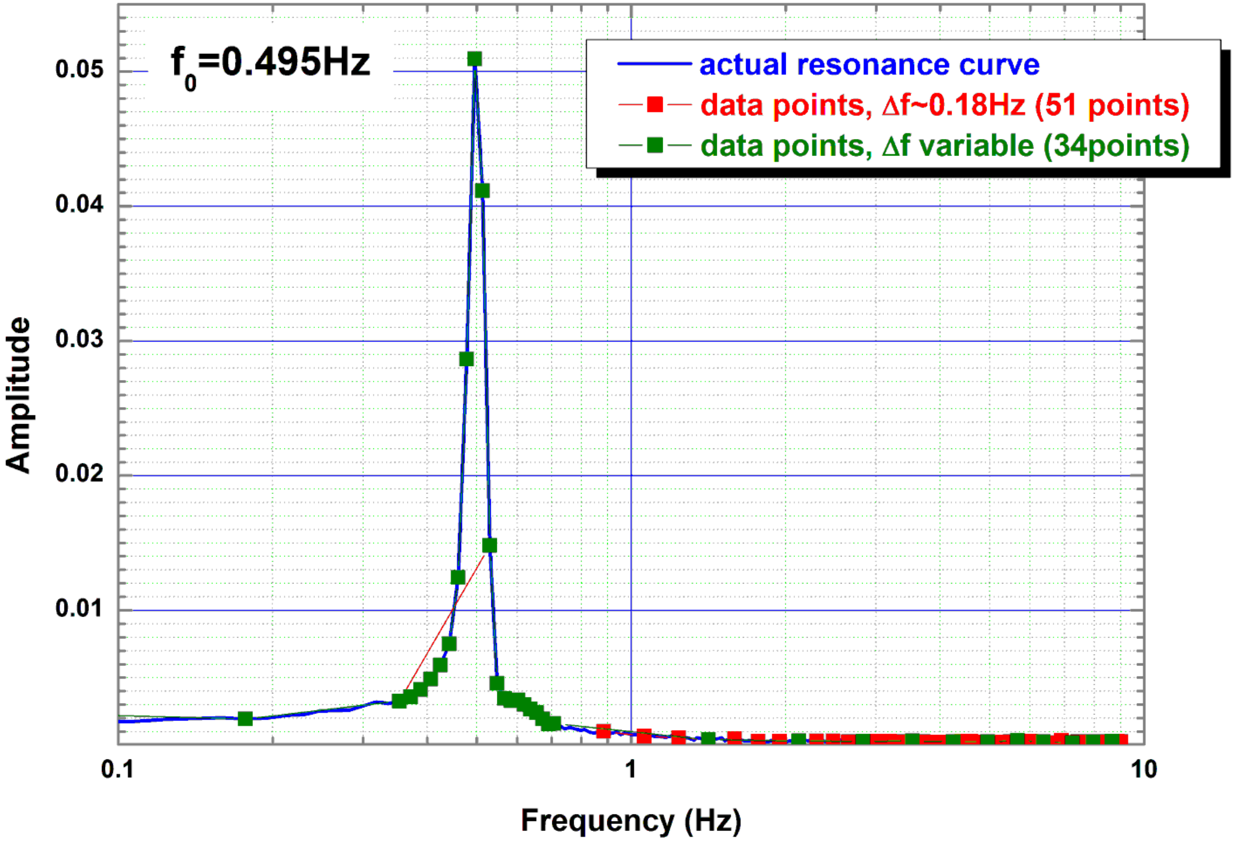




By scanning the driving frequency ***f_d***, we can measure the amplitude and phase shift of the oscillating pendulum as a function of frequency (*i.e.* the **transfer function**).

Both parameters (amplitude and phase) can be extracted by the DAQ program, or by Origin

Resonance: Taking Data

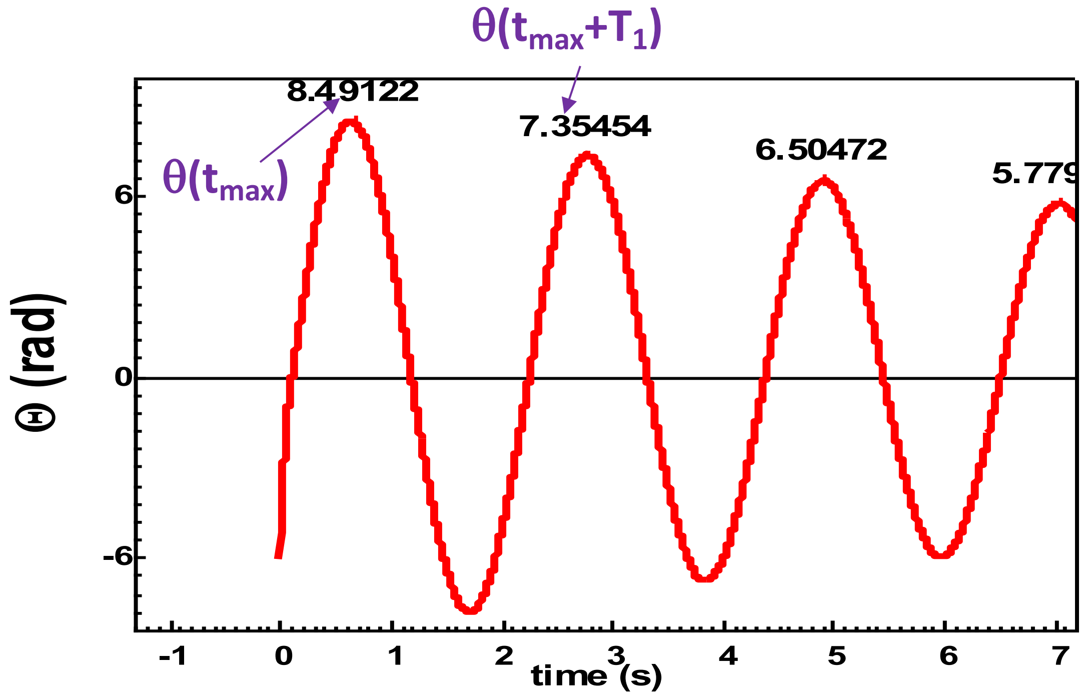


Take care in your choice of step size in frequency in order to capture the resonance’s shape

Quality Factor & Log Decrement

We have discussed two ways of characterizing the rate at which oscillations are damped out:

- **Logarithmic decrement, *δ***: Log of the amplitude ratio between consecutive oscillations
- **Quality factor, *Q***: Ratio of stored energy to energy lost per *radian* of oscillation (*cycle/2π*)



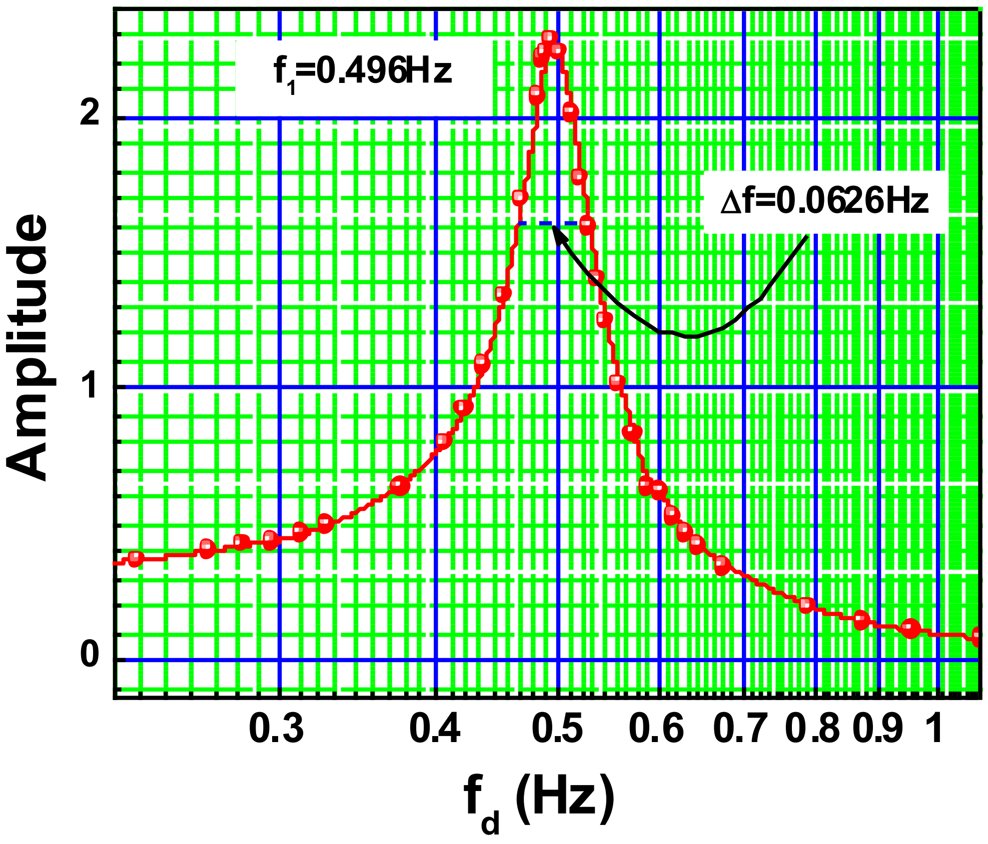
$$\delta \equiv \ln\left(\frac{\theta(t_{max})}{\theta(t_{max}+T_1)}\right) = \ln\left(\frac{e^{-at_{max}}}{e^{-a(t_{max}+T_1)}}\right) = aT_1$$

$$\delta \equiv \ln\left(\frac{8.49}{7.35}\right) \approx \mathbf{0.144}$$

$$Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi}{a}\frac{\omega_1}{2\pi} = \frac{\pi}{a}\frac{1}{T_1} = \frac{\pi}{\delta}$$

$$Q \approx \mathbf{21.8}$$

Quality Factor & Log Decrement



In addition to the time-domain formulation above, there is a (*nearly*) equivalent formulation in the frequency domain.

We can compute $\boldsymbol{Q = \omega_1/\Delta\omega}$ (or $f_1/\Delta f$), where $\Delta\omega$ is the bandwidth of the resonance curve.

$\Delta\omega$ is the width of the resonance curve when it falls to half of its peak **power** level (*not amplitude!*), *i.e.* the full-width at half-maximum (**FWHM**) of power.

Here $\boldsymbol{Q \approx 7.9}$.

Resonance: Angular Displacement Amplitude

Solve for the **amplitude** ⟶

$$\left|\theta_{ss}(t)\right| = \frac{\lambda\omega_0^2\theta_0}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\omega^2 a^2}}$$

… and on **resonance ($\omega = \omega_0$)**, we have:

$$\left|\theta_{ss}(t)\right| = \frac{\lambda\omega_0\theta_0}{2a} = \lambda\theta_0 \bullet Q$$

If we combine…

- high driving amplitude θ_0
- high quality Q
  (equivalently, low damping factor a)

… in a mechanical system, then it will accumulate a lot of oscillation energy, which could result in its destruction!

⟶

Beats: Theory (Symmetric)

Suppose that we measure the sum of two harmonic signals of frequencies **ω_1** and **ω_2**

$$y_1 = A\sin(\omega_1 t + \varphi_1); \quad y_2 = B\sin(\omega_2 t + \varphi_2)$$

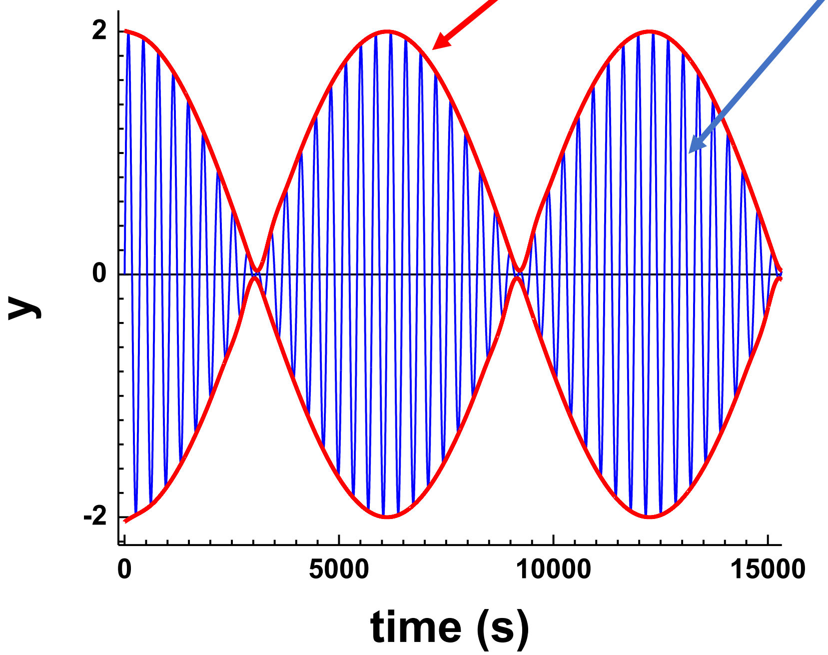
Consider first the case that **A=B** (*equal amplitudes*):

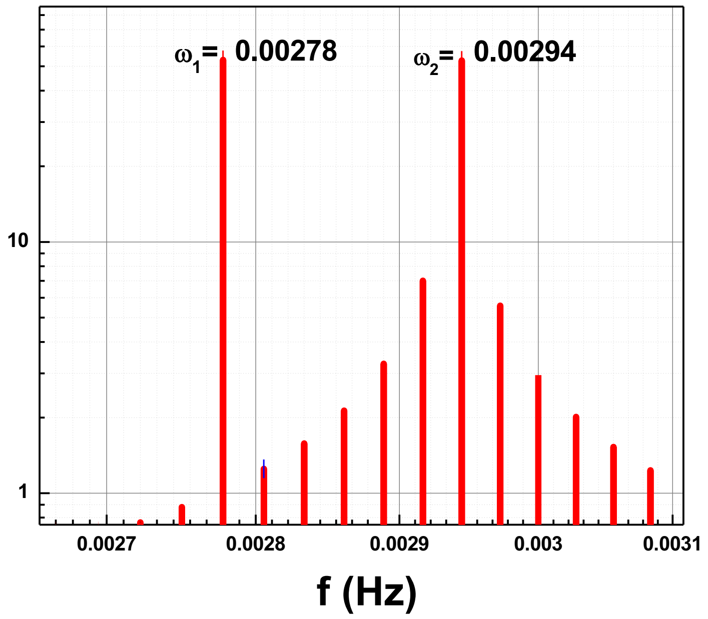
$$y = y_1 + y_2 = 2A\sin\left(\frac{\omega_1+\omega_2}{2}t + \beta_1\right)\cos\left(\frac{\omega_1-\omega_2}{2}t + \beta_2\right)$$

$$\beta_{1,2} \equiv \frac{\varphi_1 \pm \varphi_2}{2}$$

If $\omega_1 \approx \omega_2$ (close frequencies), then take $\boldsymbol{\omega} \equiv \frac{\omega_1+\omega_2}{2} \approx \omega_{1,2}$ and $\boldsymbol{\Omega} \equiv \frac{\omega_1-\omega_2}{2}$

$$\boldsymbol{y = 2A\cos(\Omega t + \beta_2)\ \sin(\omega t + \beta_1)}$$





Beats: Theory (General)

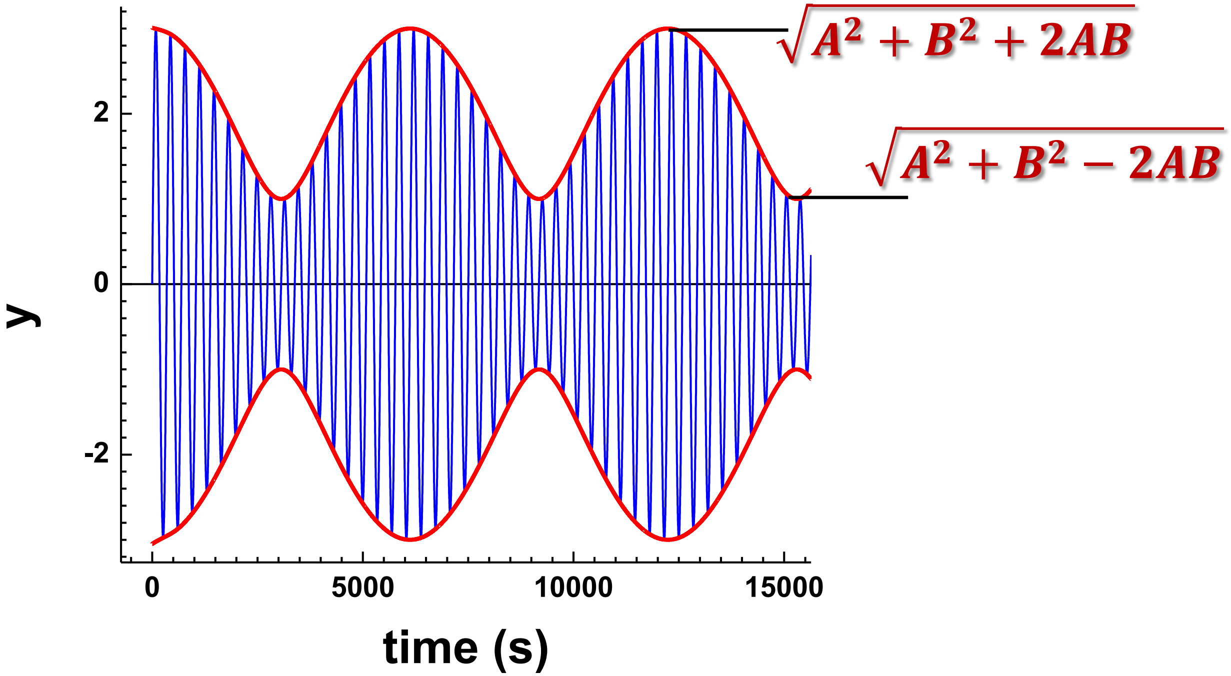
Now consider the more general case where $\boldsymbol{A \neq B}$, and ignore relative phases for simplicity

$$y_1 = A\sin(\omega_1 t);\quad y_2 = B\sin\big((\omega_1+\Omega)t\big)$$

Then we have: $y = y_1 + y_2 = \boldsymbol{C(t)}\sin\big((\omega_1 + \boldsymbol{\beta})t\big)$, where:

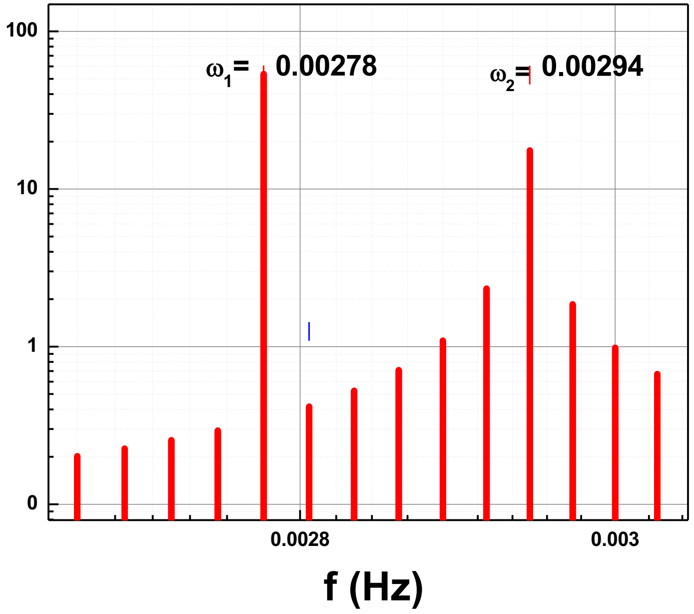
$$C(t) = \sqrt{A^2 + B^2 + 2AB\cos(\Omega t)}$$

$$\beta(t) = \tan^{-1}\left(\frac{B\sin(\Omega t)}{A + B\cos(\Omega t)}\right) + \begin{cases} 0, & A + B\cos(\Omega t) \geq 0 \\ \pi, & A + B\cos(\Omega t) < 0 \end{cases}$$



$\boldsymbol{\sqrt{A^2 + B^2 + 2AB}}$

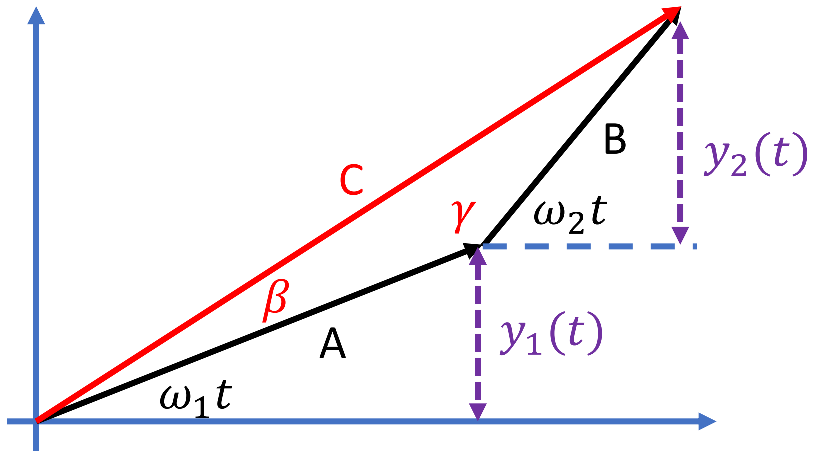
$\boldsymbol{\sqrt{A^2 + B^2 - 2AB}}$



Aside: Deriving the General Beat Formula

Consider the phasor construction below. The two beating sinusoidal signals appear as the heights (*y*-values) of the phasors of lengths *A* and *B*. We seek an expression for the height of the phasor of length *C*.

$$\boldsymbol{y(t) = y_1(t) + y_2(t) = C(t)\sin(\omega_1 t + \beta(t))}$$



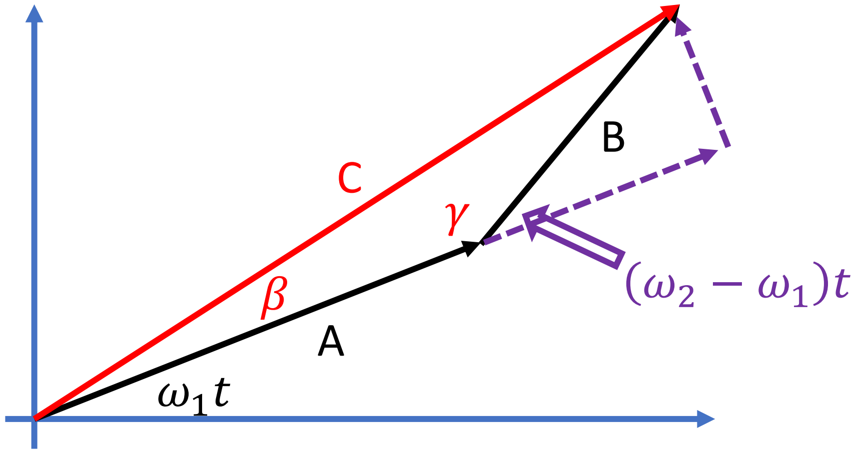
The **amplitude *C(t)*** may be found using the Law of Cosines:

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

$$\gamma = 2\pi - \omega_2 t - (\pi - \omega_1 t) = \pi - (\omega_1 - \omega_2)t$$

$$C^2 = A^2 + B^2 + 2AB\cos\big((\omega_2 - \omega_1)t\big)$$

This is the **envelope**, modulated at the beat frequency

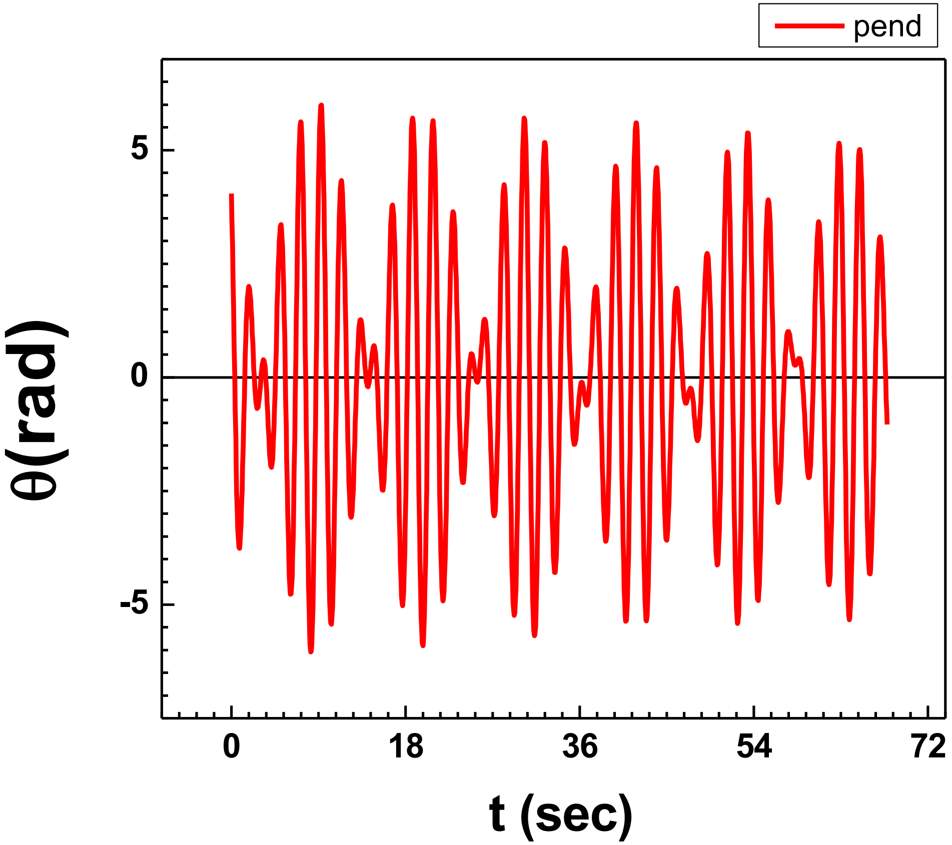


The **phase angle β(t)** may be found by extending a right triangle with hypotenuse *C*. Then we can observe that:

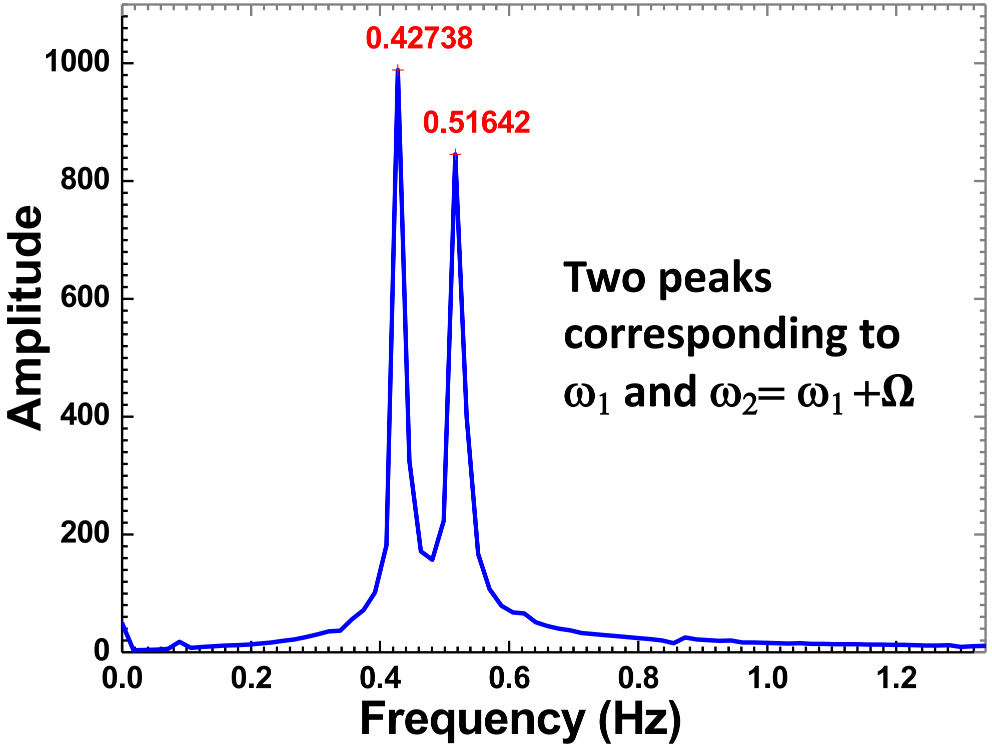
$$\tan\beta = \frac{B\sin\big((\omega_2 - \omega_1)t\big)}{A + B\cos\big((\omega_2 - \omega_1)t\big)}$$

This is a shift in the oscillation phase relative to $y_1(t)$.

Beats: Experiment



Time domain trace



Two peaks corresponding to ω_1 and $\omega_2 = \omega_1 + \Omega$

Beating spectrum

Beats in our Driven Torsional Oscillator

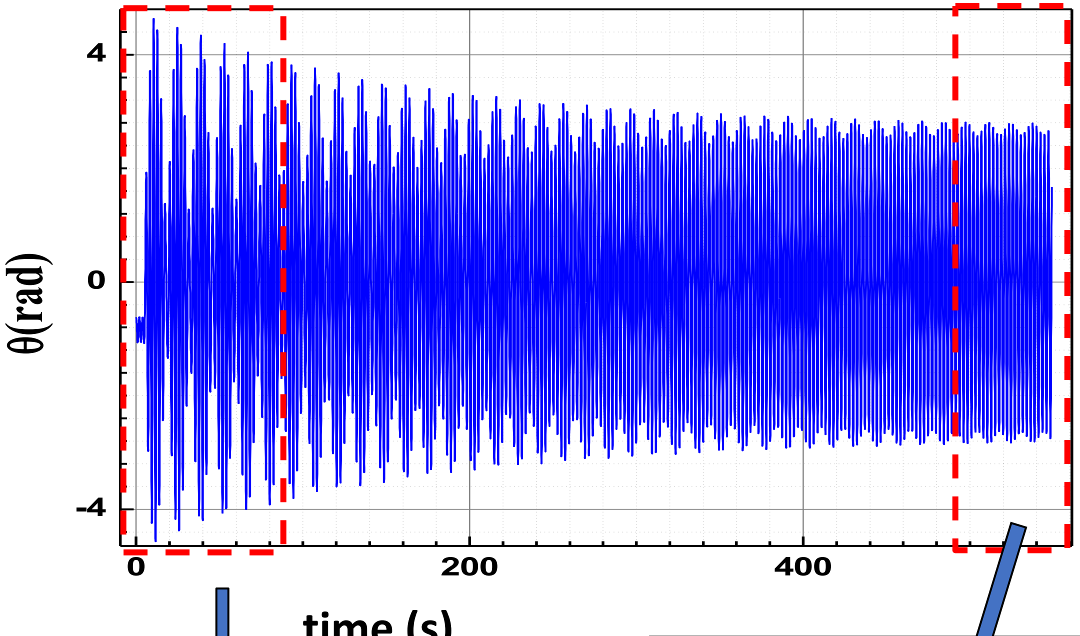
$$\theta(t)=\theta_t(t)+\theta_{ss}(t)=Ae^{-at}\cos(\omega_1 t-\phi)+B\cos(\omega t-\beta(\omega))$$

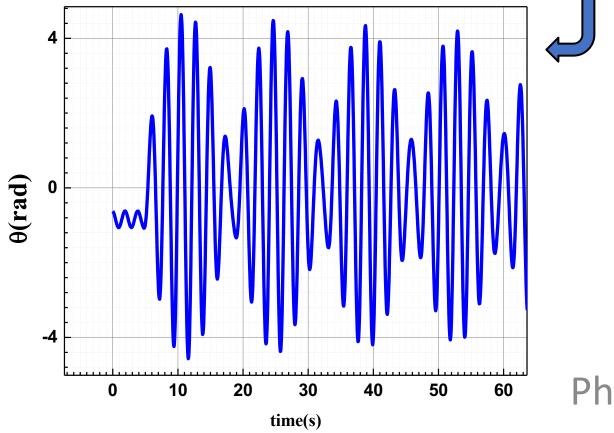
$\theta_t(t)\rightarrow 0$

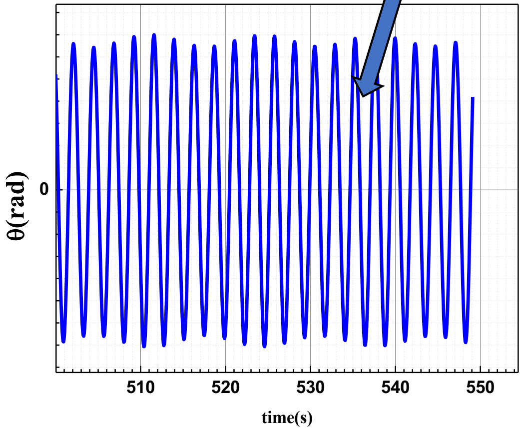
When we change the drive, we introduce a new, second frequency

The beats we see decay over time (*i.e.* they're part of the transient solution). How fast depends upon damping.

When you work on resonance data, **wait** until you see the steady-state oscillations!

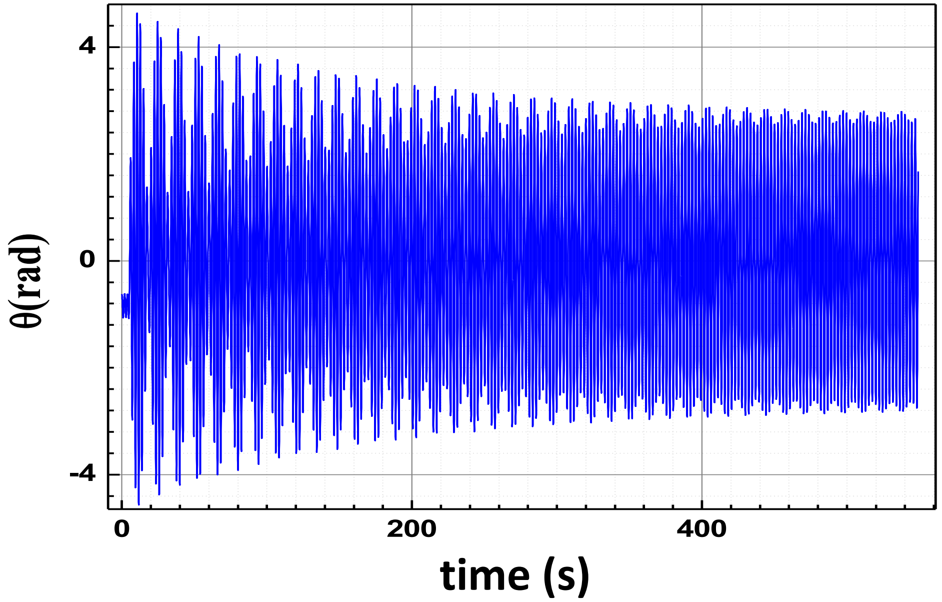


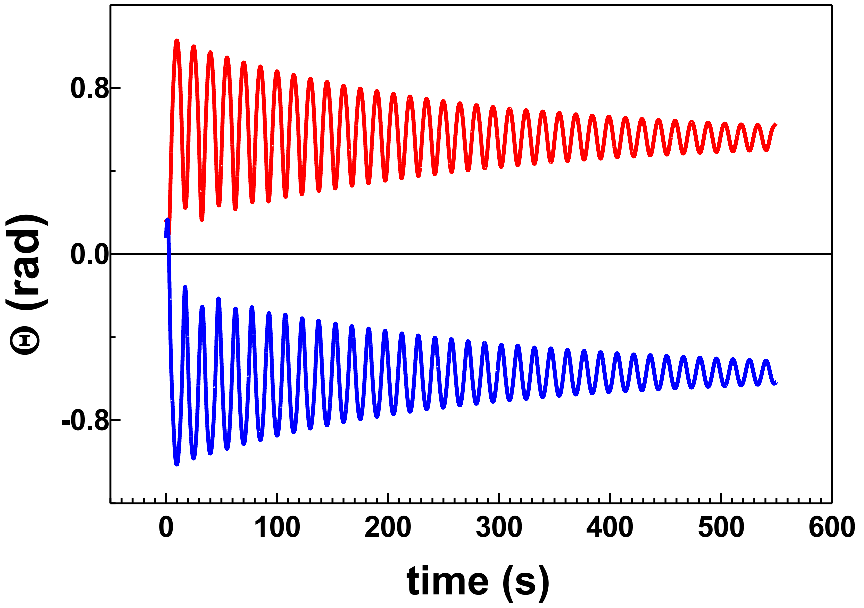




Beat Envelope

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at}\cos(\omega_1 t - \phi) + B\cos(\omega t - \beta(\omega))$$



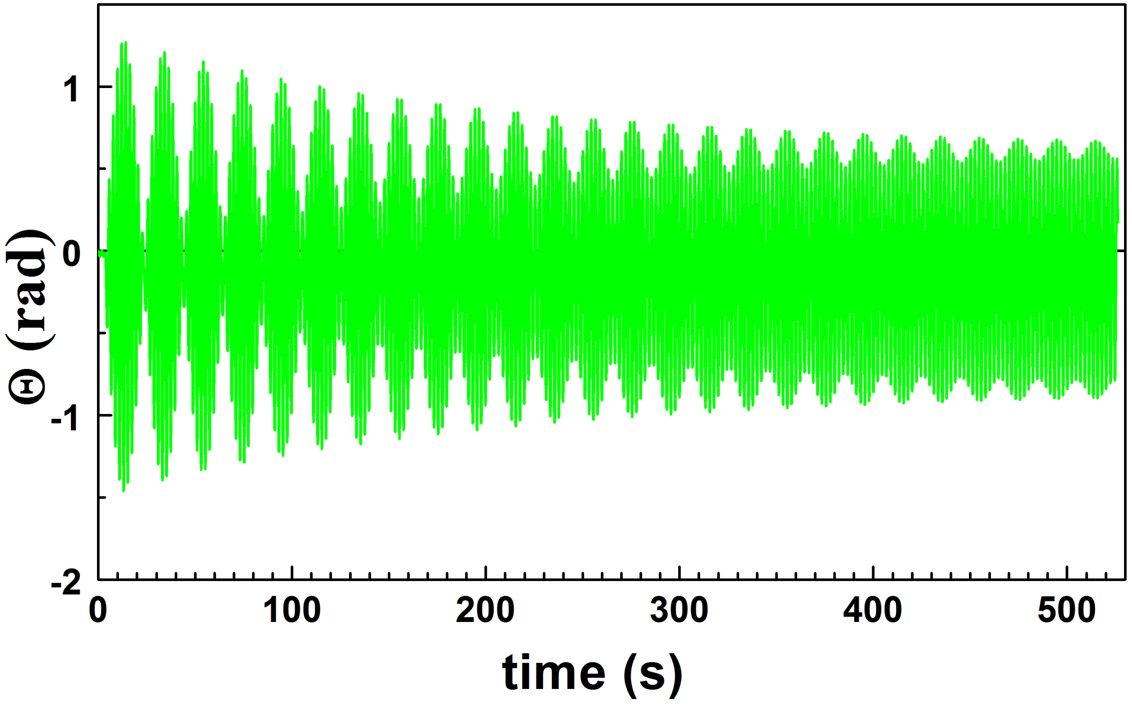


$\theta_t(t) \rightarrow 0$ These decaying beats can be seen clearly in an “envelope” plot

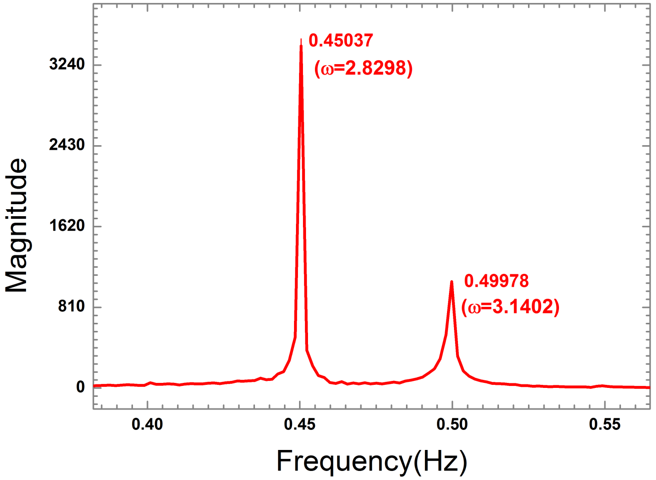
Origin 8.6: *Analysis → Signal Processing → Envelope*

Beats: Fitting

$$\theta(t)=\theta_t(t)+\theta_{ss}(t)=Ae^{-at}\cos(\omega_1 t-\phi)+B\cos(\omega t-\beta(\omega))+C$$



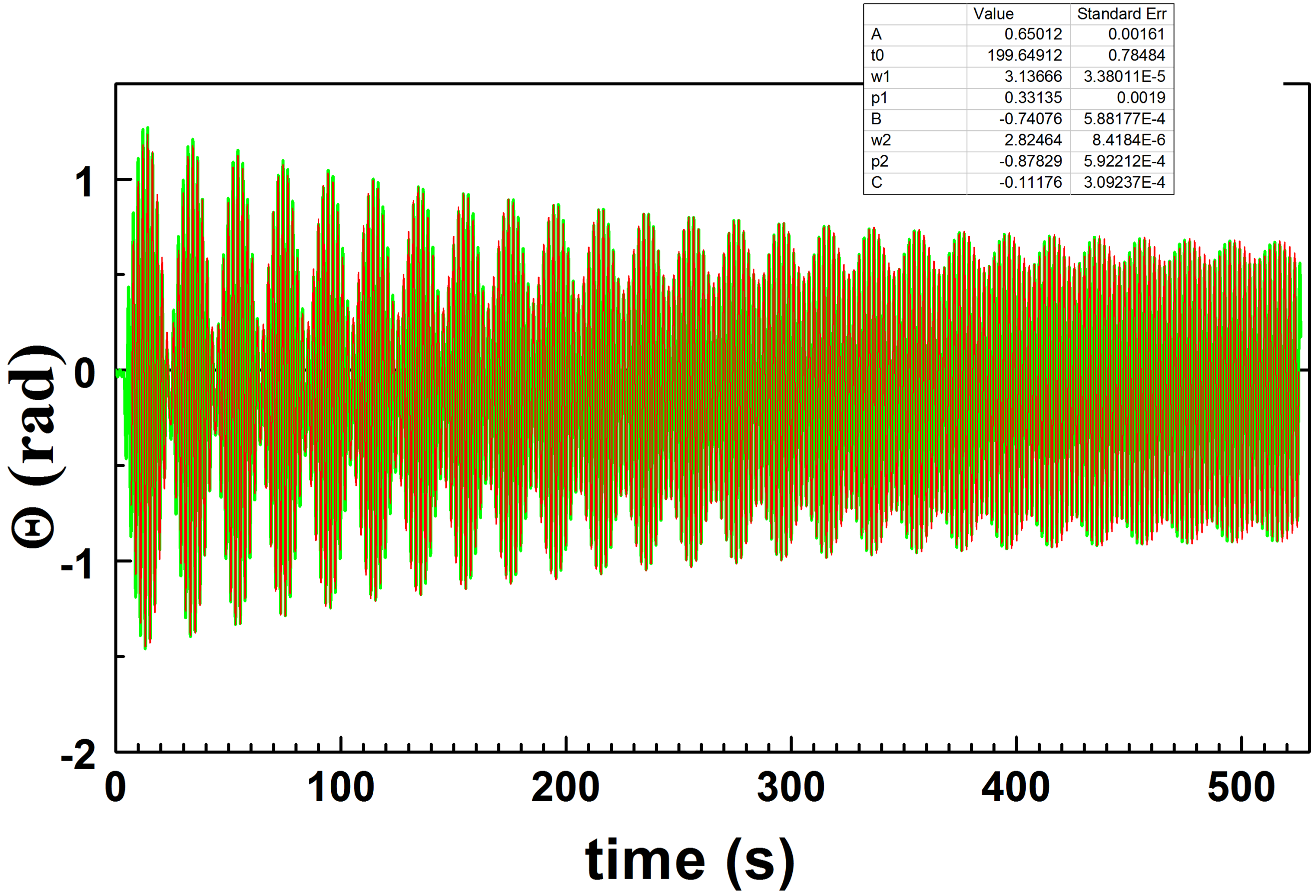
First, we apply an FFT to find ω_1 and ω



Result: ω_1=3.1402 rad^{-1} and ω=2.8298 rad^{-1}

Beats: Fitting

$$\theta(t)=\theta_t(t)+\theta_{ss}(t)=Ae^{-\frac{t}{t_0}}\cos(\omega_1 t-\phi)+B\cos(\omega t-\beta(\omega))+C$$



	Value	Standard Err
A	0.65012	0.00161
t0	199.64912	0.78484
w1	3.13666	3.38011E-5
p1	0.33135	0.0019
B	-0.74076	5.88177E-4
w2	2.82464	8.4184E-6
p2	-0.87829	5.92212E-4
C	-0.11176	3.09237E-4

8 fitting parameters

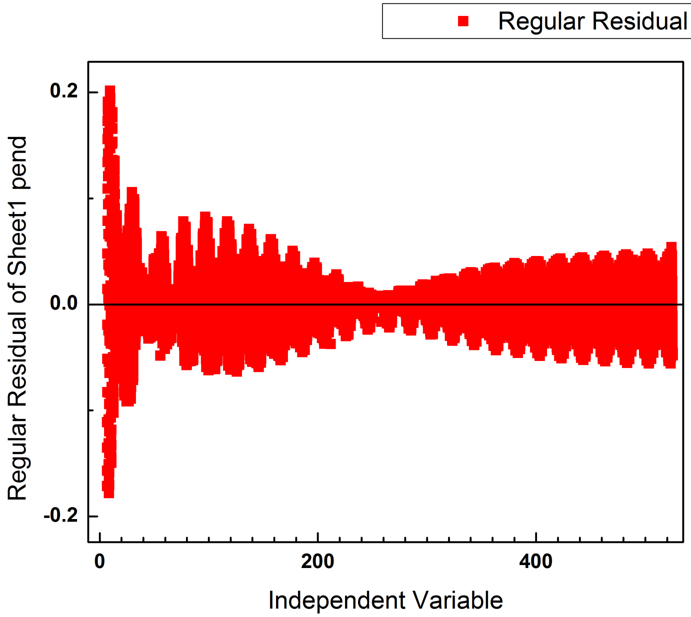
From fit

A	**0.65012**
t_0	**199.64912**
ω_1	**3.13666**
ϕ	**0.33135**
B	**-0.74076**
ω	**2.82464**
β	**-0.87829**
C	**-0.11176**

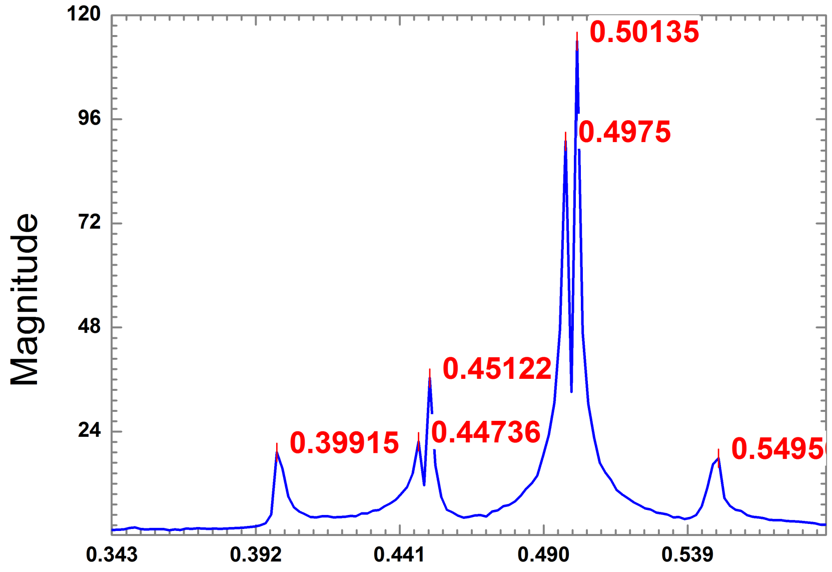
Result from FFT:

ω_1=3.1402 rad^{-1} and ω=2.8298 rad^{-1}

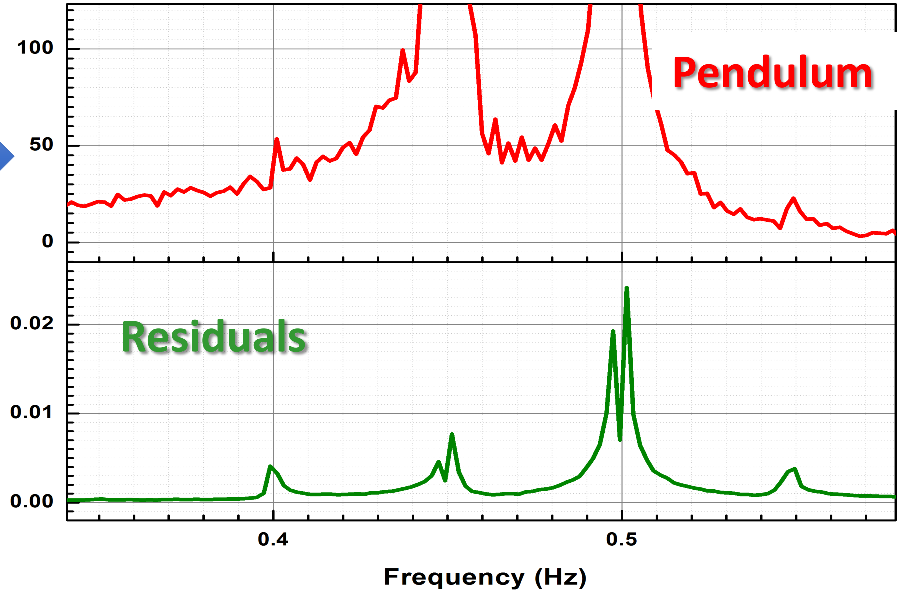
Beats: Fitting - Residuals



FFT



Compare residuals to original pendulum spectrum

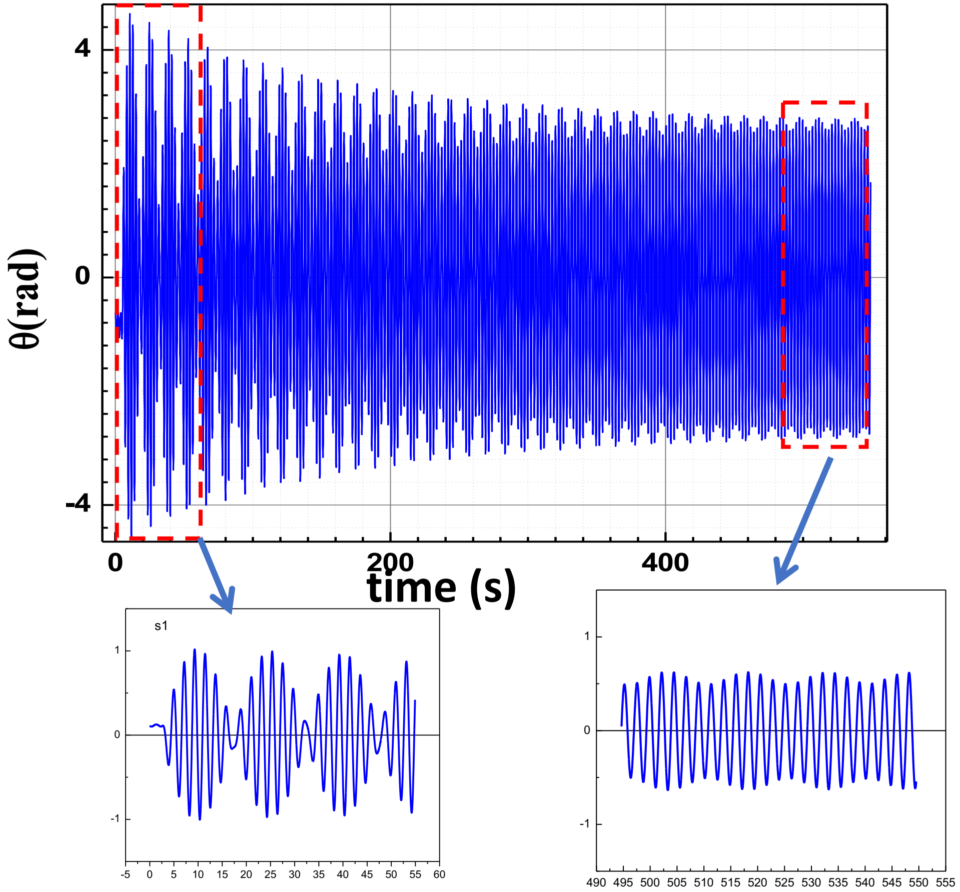


Possible origins for “extra” peaks?

1. Nonlinear behavior of pendulum
2. Motor driving force not perfectly single-frequency
3. Fitting function is not ideal

Beats: Another View

$$\theta(t)=\theta_t(t)+\theta_{ss}(t)=Ae^{-at}\cos(\omega_1 t-\phi)+B\cos(\omega t-\beta(\omega))$$

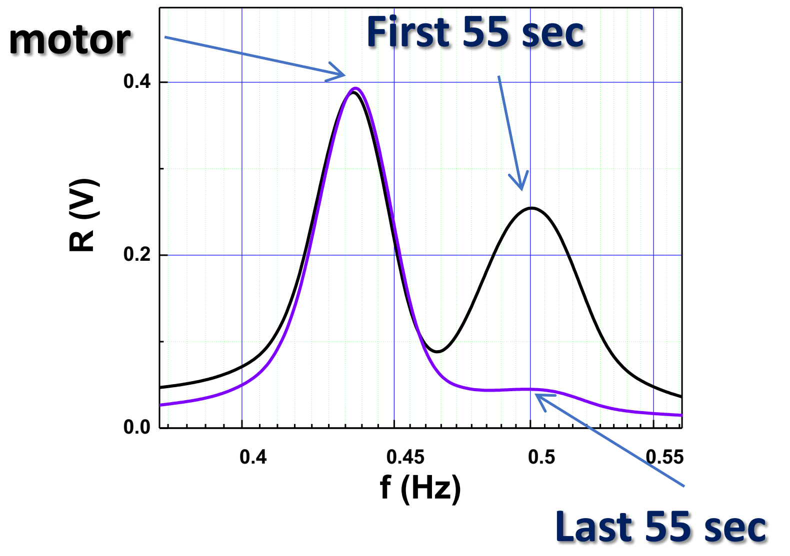


First 55 sec

Last 55 sec

$\theta_t(t) \rightarrow 0$

We also can analyze the decrease of the amplitude of the ω_1 component by analyzing the spectrum as a function of time



Origin 9.0: *Analysis → Signal Processing → FFT*

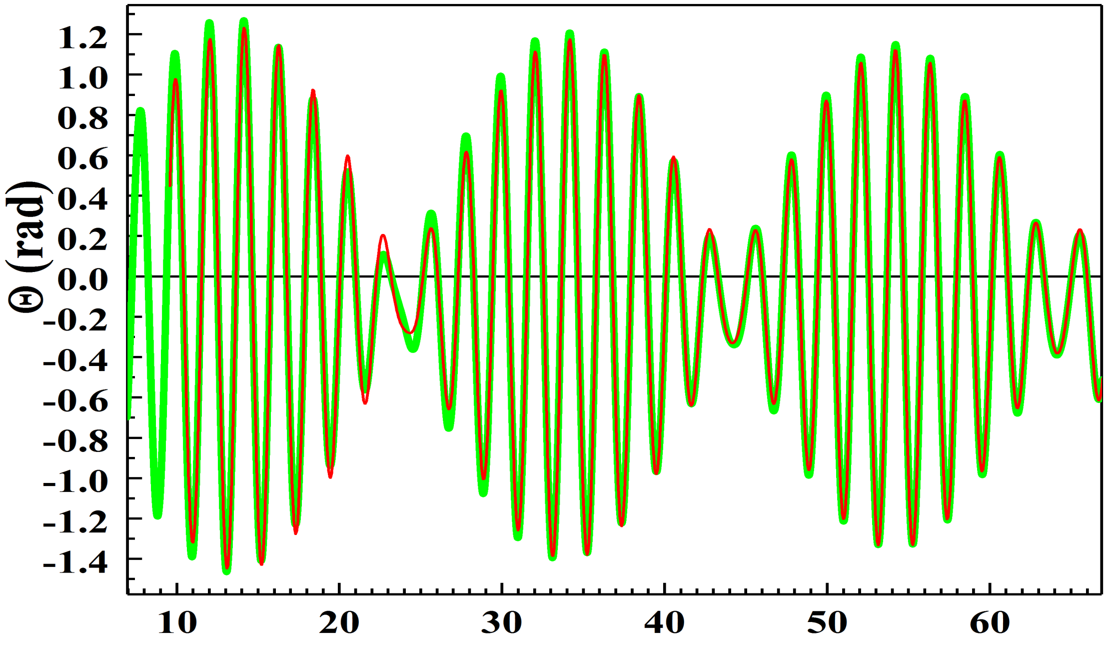
Beats: Fitting

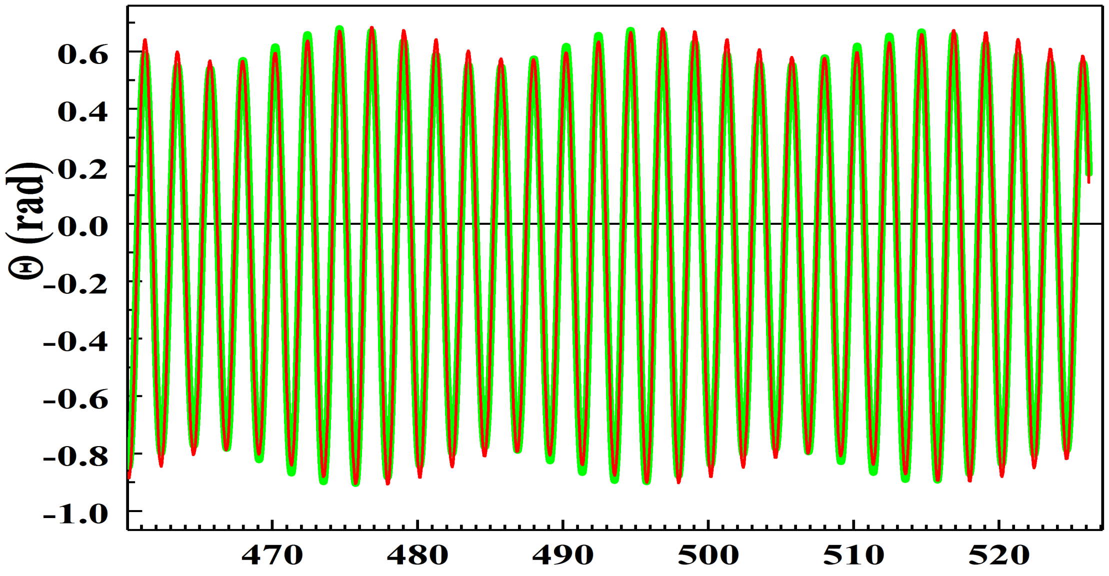
From fitting

ω_1	3.13666
f1	0.4992 Hz
ω	2.82464
f2	0.4496 Hz

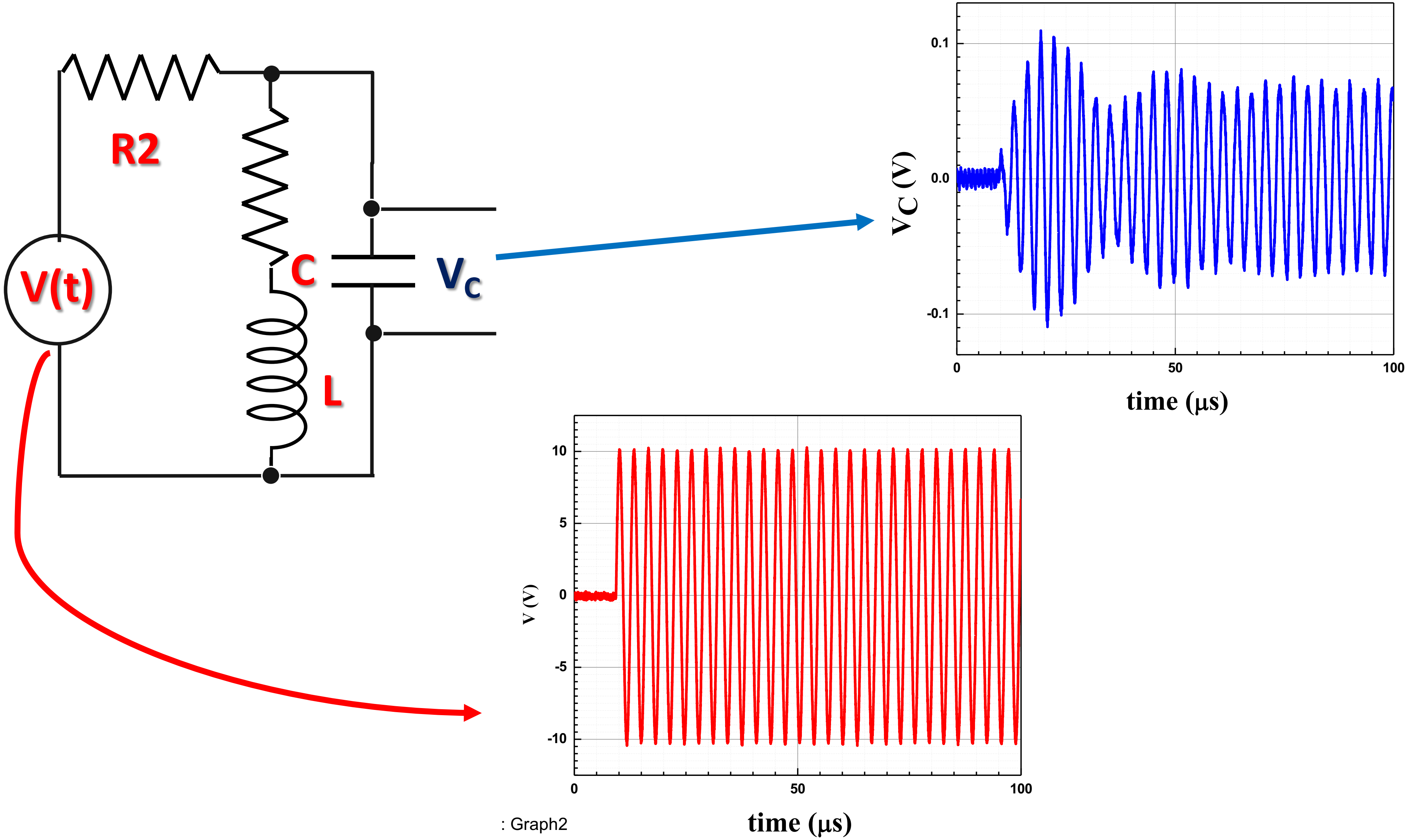
From FFT

f1	0.499 Hz
f2	0.451 Hz

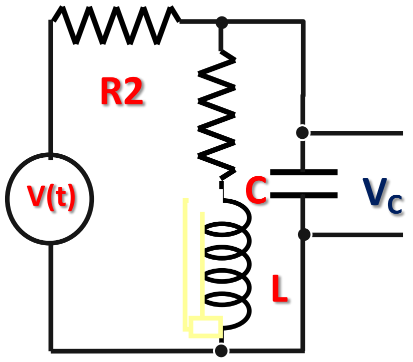


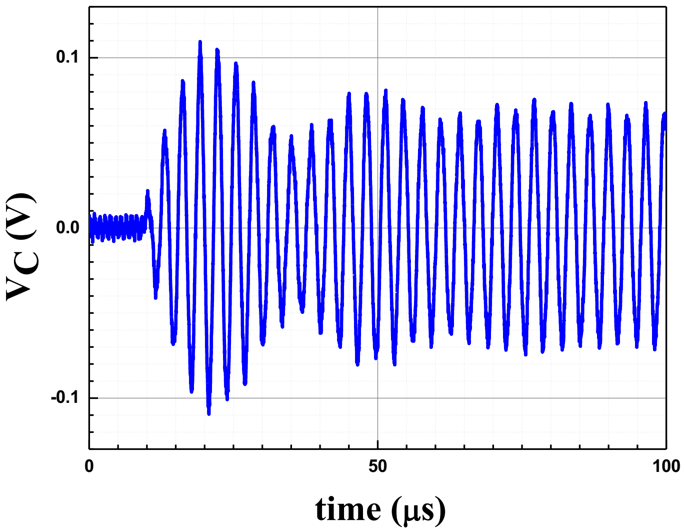


Beats: RLC Circuit

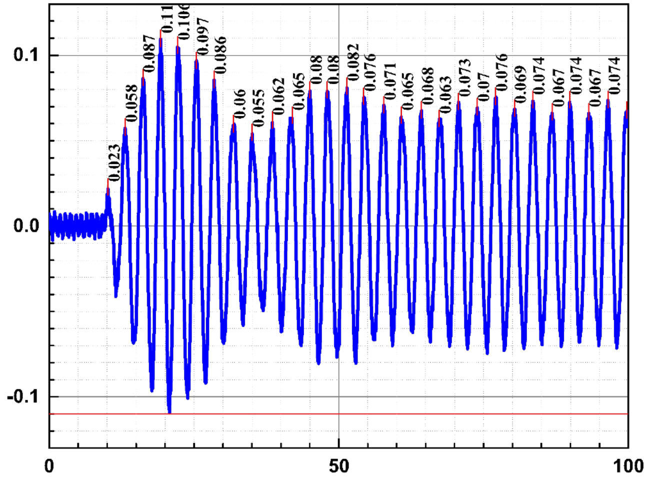


Beats: RLC Circuit

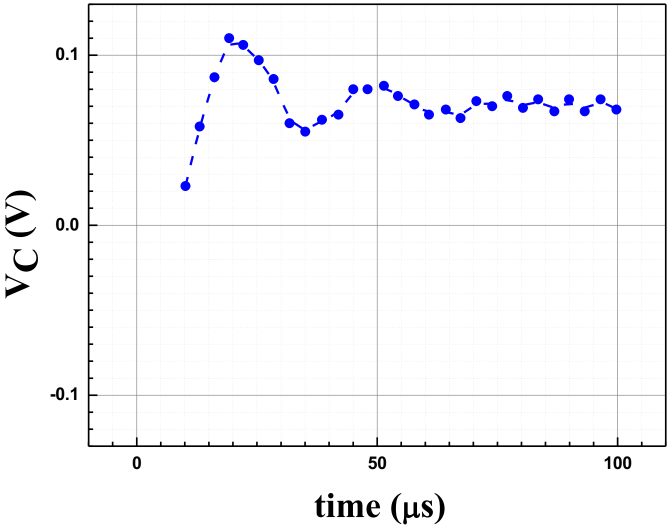




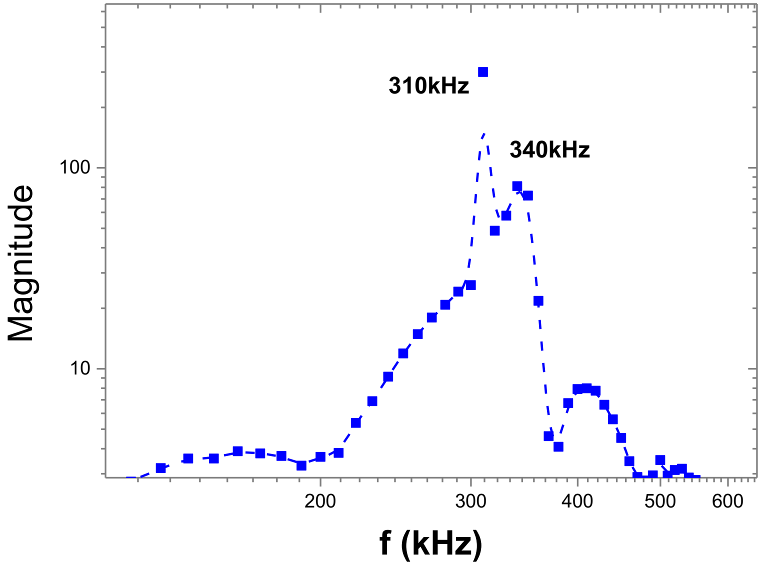
Find peaks



Envelope

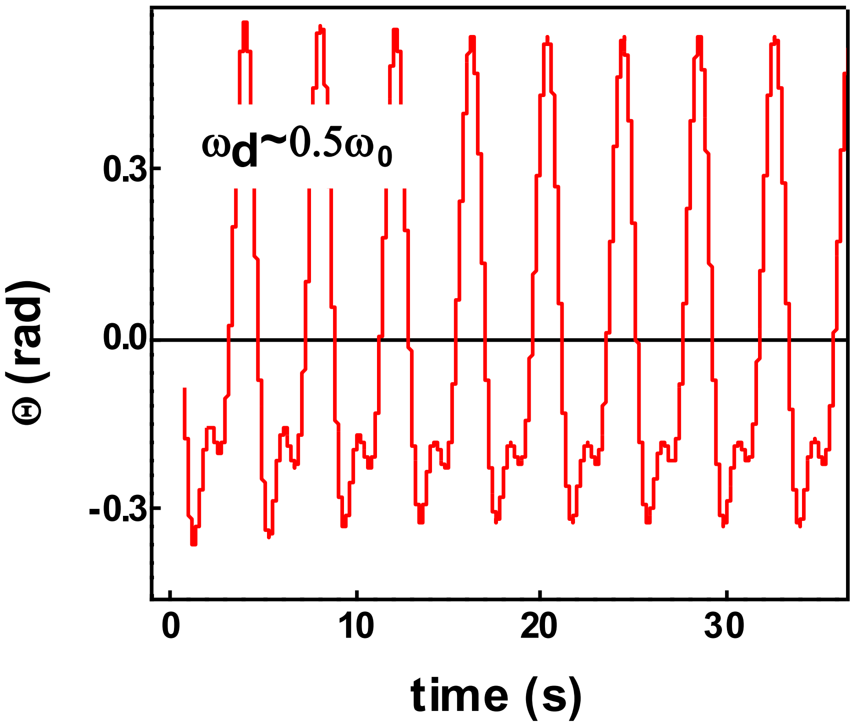


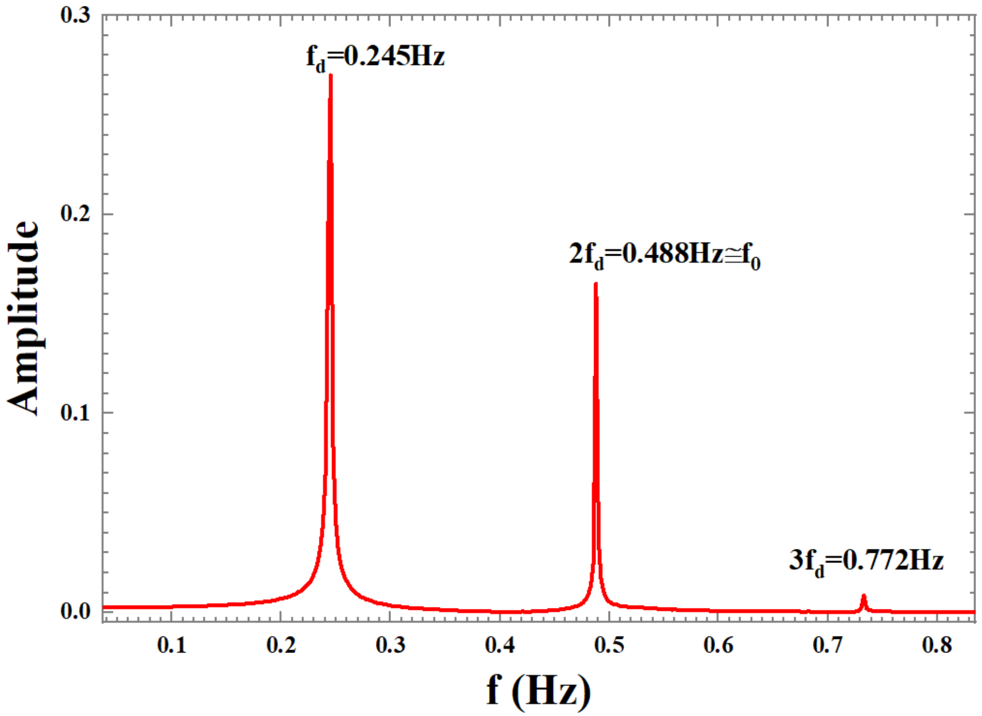
FFT



Harmonics: Experiment

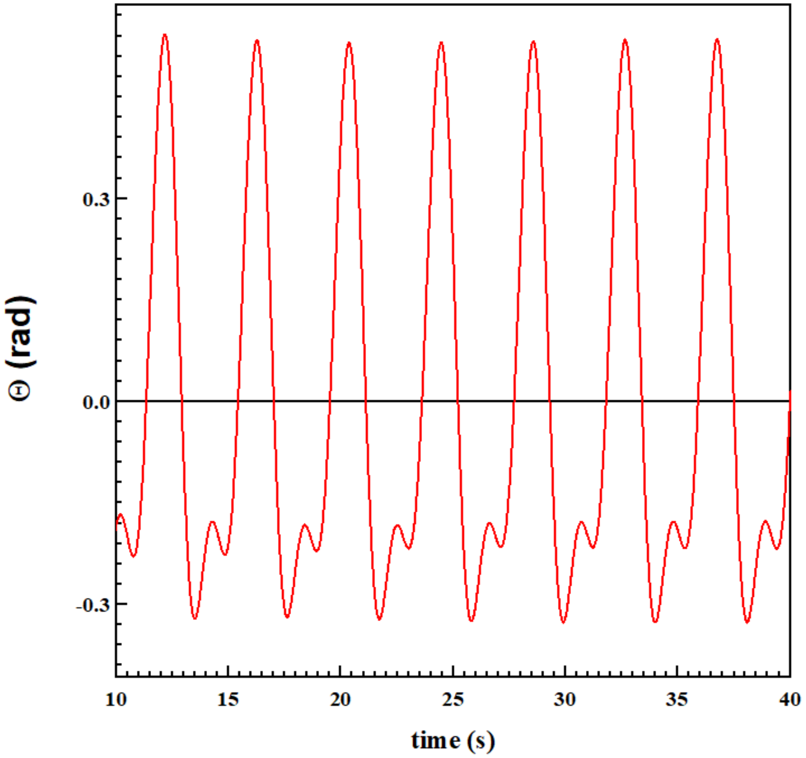
If we drive the oscillator at $f_d = f_0/2$ or $f_d = f_0/3$ (a **sub-harmonic** of the resonant frequency), we observe more complex motion of the pendulum





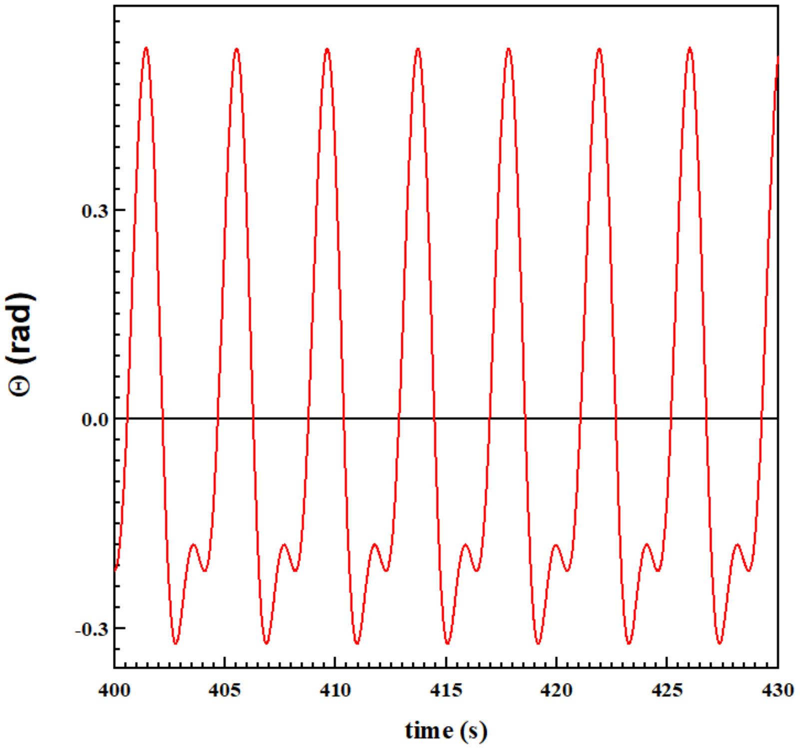
Harmonics: Experiment

This is a steady-state response – it does *not* disappear over time!
Couplings between different frequencies suggest non-linearity somewhere in the system…



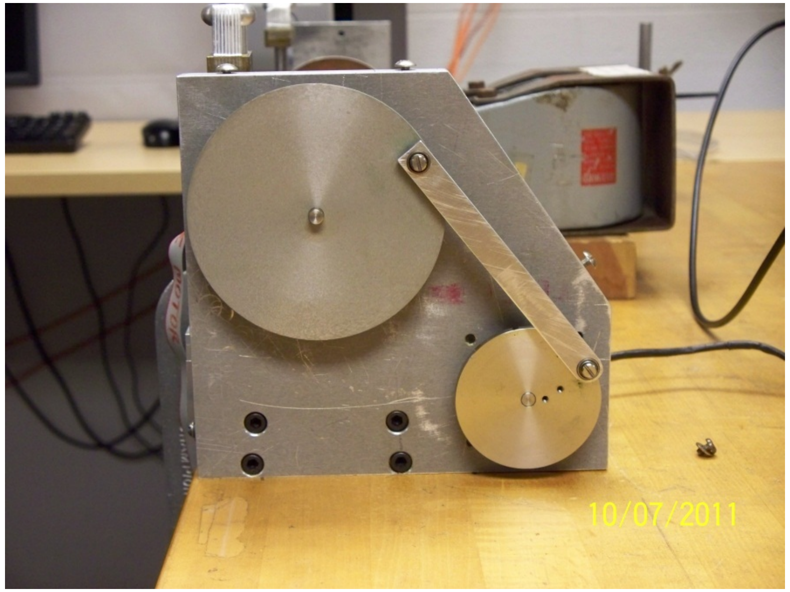
Beginning of time record

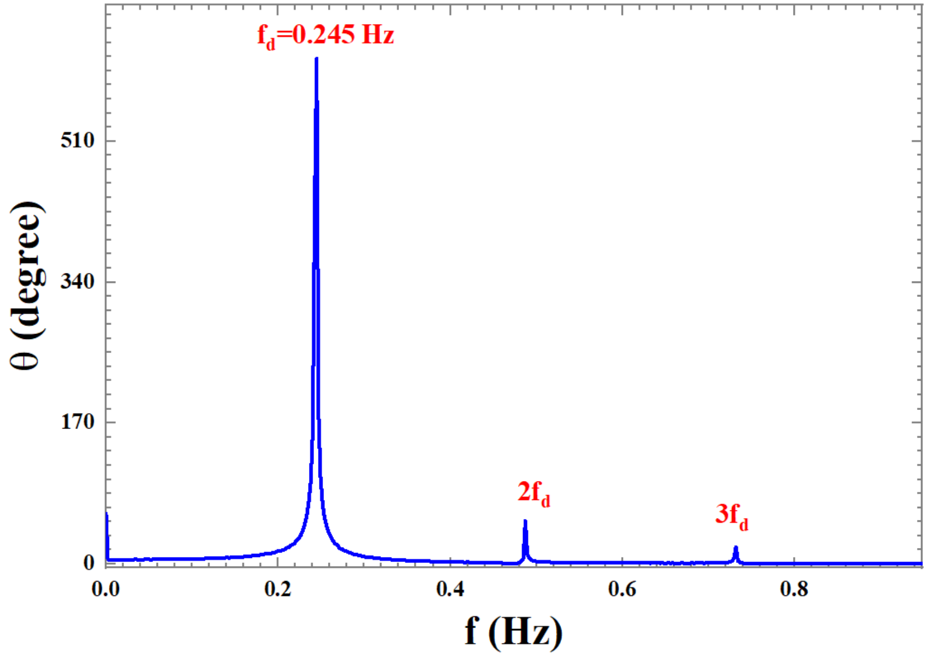
ω_d~$0.5\omega_0$

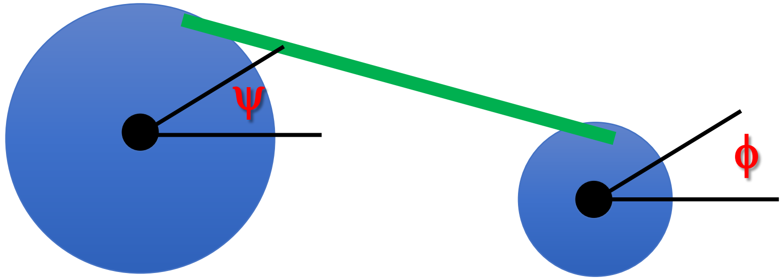


End of time record

Origin of Harmonics







Drive includes tiny components at **harmonics** (*multiples*) of the nominal drive frequency. If close to resonance, these can excite the resonator and be amplified.

A detailed analysis by P. Debevec (UIUC Physics) has shown that even if $\phi = \phi_0 \sin(\omega_d t)$ exactly, our drive torque will still contain several harmonics of ω_d.