Midterm Exam

1) [15 Pts.] Determine the (a) amplitude, (b) period, (c) wavelength, (d) phase velocity and (e) direction of propagation of the following pressure wave:

\[ P(x,t) = (5 \text{ Pa}) \sin \left[ (4 \text{ m}^{-1}) \cdot x + (8 \text{ s}^{-1}) \cdot t \right] \]

2) [20 Pts.] A spherical surface with radius \( R \) divides regions of water (refractive index, \( n \)) and air as shown. A fish swims a distance \( L = 20 R \) from the interface (assume paraxial rays).

a) Determine the location of the image of the fish. What is the physical meaning of the sign?

b) Calculate the transverse magnification.

3) [30 Pts.] A ray of light containing two different wavelengths, \( \lambda_1 \) and \( \lambda_2 \), enters a medium at angle \( \theta_0 \). The medium is dispersive, i.e., its index of refraction is wavelength-dependent: \( n(\lambda) = 1 / \left( n_0 + \alpha \lambda \right) \), where \( n_0 \) and \( \alpha \) are constants. The ray splits and strikes a screen residing a distance \( L \) from the surface, as shown.

a) What is the distance between the two rays, \( \Delta y \), when they strike the screen? You may assume that the angles in this problem are small, e.g., \( \sin \theta_0 \approx \theta_0 \).

4) [35 Pts.] Two lenses are placed concentrically along the z axis, separated by a distance, \( L \), as shown in the figure. The left lens is convex with focal length \( f_1 = 1 / R_1 \), while the right is concave with \( f_2 = -1 / R_2 \), where the quantities \( R_1 \) and \( R_2 \) are both greater than zero.

a) Determine the system matrix, \( M \), for this arrangement of lenses.

b) A ray traveling from the left strikes the first lens at angle \( \theta_1 \) and height \( y_1 \) above the optical axis. At what angle, \( \theta_2 \), and height, \( y_2 \), does this ray emerge from the second lens?

c) What condition must be met for this ray to intersect the optic axis after emerging from the second lens?
1) The wave in question is
\[ P(x,t) = (5 \cos kx) \sin [(4m^{-1})x + (8s^{-1})t] \]
By inspection, \( k = 4\text{m}^{-1} \) and \( \omega = 8\text{s}^{-1} \)
   a) \( A = 5\text{Pa} \)
   b) \( T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4} \text{seconds} \)
   c) \( \lambda = \frac{2\pi}{k} = \frac{2\pi}{4} = \frac{\pi}{2} \text{meters} \)
   d) \( \nu = \frac{\omega}{k} = \frac{8\text{s}^{-1}}{4\text{m}^{-1}} = 2\text{m/s} \)
   e) The wave is propagating in the -\( x \) direction.

2) This is not a complete lens, but a half-lens of the sort discussed in lecture #2. In the paraxial limit, a half-lens is described by
\[ \frac{n_1 + n_2}{s_0} = \frac{n_2 - n_1}{s_i} \]
   a) In this case
\[ \frac{n}{s_0} + \frac{1}{s_i} = \frac{1 - n}{20R} \]
\[ \frac{1}{s_i} = \frac{1}{R} - \frac{n}{20R} \]
\[ = \frac{1}{R} - \frac{20n}{20R} = \frac{1}{R} \left( 1 - \frac{20n}{20} \right) \]
\[ s_i = R \frac{1}{1 - \frac{20n}{20}} = R \frac{20}{20 - 20n} \]
For water, \( n > 1 \), which means that \( s_i > 0 \). This means the image is virtual.
   b) \[ M_1 = \frac{n_1 s_i}{n_2 s_0} = R \frac{20n}{20n - 20} \]
\[ = \frac{n}{20n - 20} \]
3) In this problem, two rays strike the surface at the same angle but are refracted by different amounts.

\[ \Delta y = L \Delta \theta = L \alpha \theta_0 (\lambda_1 - \lambda_2) \]

Each ray satisfies Snell's law:

\[ \sin \theta_0 = n(\lambda_1) \sin \theta_1, \quad \sin \theta_0 = n(\lambda_2) \sin \theta_2 \]

a) The problem says we can assume small angles, so

\[ \theta_0 = n(\lambda_1) \theta_1, \quad \theta_0 = n(\lambda_2) \theta_2 \]

\[ \theta_0 = \frac{1}{n_0 + \alpha \lambda_1} \theta_1, \quad \theta_0 = \frac{1}{n_0 + \alpha \lambda_2} \theta_2 \]

\[ \theta_1 = (n_0 + \alpha \lambda_1) \theta_0, \quad \theta_2 = (n_0 + \alpha \lambda_2) \theta_0 \]

The angular splitting between the two rays is, then,

\[ \Delta \theta = \theta_1 - \theta_2 = \left( n_0 \theta_0 + \alpha \lambda_1 \right) \theta_0 - \left( n_0 \theta_0 + \alpha \lambda_2 \right) \theta_0 \]

\[ = \alpha \theta_0 (\lambda_1 - \lambda_2) \]

The spacing between the two rays on the screen, then, is

\[ M = R_2, \quad T_1, \quad R_1 \]

\[ \text{refract at lens 1} \]

\[ \text{refract at lens 2} \]

\[ \text{propagate from 1 to 2} \]

\[ M = \begin{pmatrix} 1 & 0 & 1 & d \\ \frac{-1}{s_2} & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{s_1} & 1 \end{pmatrix} \]
\[
\begin{pmatrix}
\frac{1}{f_2} & -\frac{d}{f_2} \\
-\frac{1}{f_2} & 1 + \frac{d}{f_2}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{f_1} & 0 \\
-\frac{1}{f_1} & 1
\end{pmatrix}
\end{align}
= \begin{pmatrix}
1 - \frac{d}{f_1} & d \\
-(\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2}) & 1 - \frac{d}{f_2}
\end{pmatrix}
\]

\[
M = \begin{pmatrix}
1 - \frac{d}{f_1} & d \\
-(\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2}) & 1 - \frac{d}{f_2}
\end{pmatrix}
\]

b. \[
\begin{pmatrix}
y_2 \\ \theta_2
\end{pmatrix}
= M \begin{pmatrix}
y_1 \\ \theta_1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 - \frac{d}{f_1} & d \\
-(\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2}) & 1 - \frac{d}{f_2}
\end{pmatrix}
\begin{pmatrix}
y_1 \\ \theta_1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(1 - \frac{d}{f_1}) y_1 + \theta_1 d \\
y_1 (\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2}) + \theta_1 (1 - \frac{d}{f_2})
\end{pmatrix}
\]

C. If \( y_2 > 0 \), ray 2 will intersect the optical axis if \( \theta_2 < 0 \). Likewise, if \( y_2 < 0 \), \( \theta_2 > 0 \). Hence

\[
y_2 > 0, \quad -y_1 (\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2}) + \theta_1 (1 - \frac{d}{f_2}) < 0
\]

or

\[
y_2 < 0, \quad -y_1 (\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_2}) + \theta_1 (1 - \frac{d}{f_2}) > 0
\]