Midterm Exam

1) [15 Pts.] Two harmonic waves traveling on a one-dimensional vibrating string are described by wave functions $f_1(x,t)$ and $f_2(x,t)$. At some snapshot in time the waves have the form 
$f_1 = (2mm) \cos 1.1x/cm$ and $f_2 = (2mm) \cos 0.9x/cm$. Make a sketch—with labels on the axes—of the string at this instant in time, properly accounting for interference between the two waves.

2) [20 pts.] An unpolarized ray of light is incident on an interface between two materials with refractive indices $n_1$ and $n_2$, as shown. The ray is incident at the Brewster angle, $\theta_W$, and the transmitted ray makes a 30$^\circ$ angle with the surface normal.
   a) Determine the Brewster angle, $\theta_W$.
   b) Determine the ratio of the two indices of refraction, $n_2/n_1$.

3) [30 pts.] A material is characterized by a complex, frequency-dependent polarizability $\chi_e(\omega) = \chi_0 e^{i\gamma(\omega)} - 1$. An electromagnetic plane with frequency $\omega$ and amplitude $E_0$, propagating along the $\hat{Z}$ direction, enters the material at the point $z = 0$.
   a) What is the phase velocity of this wave?
   b) As the wave propagates, energy is absorbed by the material. How far must the wave propagate for its amplitude to fall to $E_0/2$?
   c) Now two waves with frequencies $\omega_1$ and $\omega_2$ enter the medium. At $t = 0$, the two waves are initially in-phase. How does their phase difference evolve as a function of time and distance into the material, $z$?

4) [35 pts.] A beam expander is a device comprising a diverging lens with $f_1 < 0$ followed by a converging lens with $f_2 > 0$. Its function is to take a parallel beam with width $w_1$ and widen it into another parallel beam with width $w_2$.
   a) Determine the distance, $d$, in terms of the focal lengths of the two lenses.
   b) Compute the system matrix, $M$, for this device (assume it resides in air).
   c) Show that a ray entering the system parallel to the optic axis will also emerge parallel to the optic axis.
   d) Determine the ratio $w_2/w_1$ in terms of $f_1$, $f_2$, and $d$. 
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4. \( f = (2\text{ mm}) \cos (1.1 \text{ x/cm}) \)
   \( f = (2\text{ mm}) \cos (0.9 \text{ x/cm}) \)

\[ f = f_1 + f_2 = (2\text{ mm}) \left[ \cos (1.1 \text{ x/cm}) + \cos (0.9 \text{ x/cm}) \right] \]

Define \( k_0 = \frac{k_1 + k_2}{2} = \frac{1.1/\text{cm} + 0.9/\text{cm}}{2} = 1/\text{cm} \)

\( \Delta k = \frac{k_1 - k_2}{2} = \frac{0.2/\text{cm}}{2} = 0.1/\text{cm} \)

Use complex notation:

\[ \tilde{f} = (2\text{ mm}) \left[ e^{ik_0 x} + e^{i(k_0 - \Delta k) x} \right] \]

\[ = (2\text{ mm}) e^{ik_0 x} \left[ e^{i\Delta k x} + e^{-i\Delta k x} \right] = 4 e^{ik_0 x} \cos \Delta k x \]

\[ f = \text{Re} \left[ \tilde{f} \right] = (4\text{ mm}) \cos \Delta k x \cos k_0 x \]

(2)

a) The Brewster angle occurs when this angle is 90°.

So, \( \Theta_w = 180° - 90° - 30° = 60° \)

\( \Theta_w = 60° \)

b) Snell's law \( n_1 \sin \theta_1 = n_2 \sin \theta_2 \)

\( n_1 \sin 60° = n_2 \sin 30° \)

\[ n_1 \frac{\sqrt{3}}{2} = n_2 \frac{1}{2} \]

\[ \frac{n_2}{n_1} = \sqrt{3} \]
The polarizability has the funny form
\[ \chi_{\alpha}(\omega) = \chi_0 e^{-i\omega/\omega_0} - 1 \]

For the rest of the problem we need to know the index of refraction
\[ n = \sqrt{1 + \chi_{\alpha} e^{-i\omega/\omega_0}} - 1 = \sqrt{\chi_0 e^{-i\omega/\omega_0}} \]

The real and imaginary parts are
\[ N(\omega) = N_{\text{Re}}(\omega) + i N_{\text{Im}}(\omega) \]
\[ N_{\text{Re}}(\omega) = \sqrt{\chi_0 \cos \left( \frac{\omega_0}{\omega} \right)} \]
\[ N_{\text{Im}}(\omega) = \sqrt{\chi_0 \sin \left( \frac{\omega_0}{\omega} \right)} \]

a) The phase velocity \( v = \frac{c}{N(\omega)} = \frac{c}{\sqrt{\chi_0 \cos \left( \frac{\omega_0}{\omega} \right)}} \)

Also acceptable: \( v = \frac{c}{\sqrt{\chi_0}} e^{-i\phi(\omega)/2} \)

b) Inside the material,
\[ E(2; t) = E_0 e^{-i\left( \frac{N_{\omega_2}(\omega)\omega_2}{c} \right) - kx_2/c} \]

The amplitude of the wave falls to \( \frac{E_0}{2} \) when
\[ e^{-kx_2/c} = \frac{1}{2} \]
\[ \frac{kx_2}{c} = \log 2 \]
\[ x_2 = \frac{c \log 2}{k(\omega)} \]

\[ z = \frac{c \log 2}{\omega \sqrt{\chi_0 \sin (\phi)e^{2}}/2} \]

C) Now we have two waves:
\[ E_1(t) = E_0 e^{-i\left( \frac{N_{\omega_1}(\omega)\omega_1}{c} \right) - kx_1/c} \]
\[ E_2(t) = E_0 e^{-i\left( \frac{N_{\omega_2}(\omega)\omega_2}{c} \right) - kx_2/c} \]

The phase difference is
\[ \phi = \frac{N_{\omega_1}(\omega)\omega_1 - \omega_1 t - N_{\omega_2}(\omega)\omega_2 - \omega_2 t}{c} \]
For a beam expander to function, the virtual image created by the first lens must reside at the focal point of the second:

\[ d = \frac{f_2}{2} - |f_1| \]

Hence, \( |f_1| + d = f_2 \)

\[ d = \frac{f_2}{2} - |f_1| \]

\[ d = f_2 + f_1 \]

b) \( M = \begin{pmatrix} 1 & -\frac{f_2}{f_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

\[ = \begin{pmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{pmatrix} \]

C) Ray parallel to optic axis described by

\[ \mathbf{v}_{\text{enter}} = \begin{pmatrix} \theta \\ 0 \end{pmatrix} = \begin{pmatrix} \theta \\ y_{\text{enter}} \end{pmatrix} \]

\[ \mathbf{v}_{\text{exit}} = M \mathbf{v}_{\text{enter}} \]
\[
\begin{pmatrix}
\frac{-f_1}{f_2} & 0 \\
\frac{f_1 + f_2}{f_2} & \frac{f_2}{f_2}
\end{pmatrix}
\begin{pmatrix}
\theta_{\text{exit}} \\
\theta_{\text{exit}}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\theta_{\text{exit}} \\
\frac{-f_2}{f_1} \theta_{\text{exit}}
\end{pmatrix}
= \begin{pmatrix}
\theta_{\text{exit}} \\
\theta_{\text{exit}}
\end{pmatrix}
\]

Still parallel since \( \theta_{\text{exit}} = 0 \).

1) By inspection,

\[
\frac{\omega_2}{\omega_1} = \frac{\theta_{\text{exit}}}{\theta_{\text{exit}}} \text{ from above}
\]

\[
= \frac{-f_2}{f_1}
\]

Note that this quantity is positive since \( f_1 < 0 \).