Midterm Exam

1) [25 pts.] A ray of light traveling inside a piece of diamond ($n_d = 2.42$) strikes an internal surface at angle $\theta$, as shown in the figure, where the ray is partially reflected and partially transmitted. The ray is polarized in the scattering plane, as shown.

a) Is it possible for the reflected ray to be completely extinguished? If so, at what value of $\theta$ does this happen? (Give your answer in terms of $n_d$).

b) Is it possible for the transmitted ray to be completely extinguished? If so, under what conditions (again, in terms of $n_d$) does this happen?

2) [35 pts.] A small, dipole antenna of length $d$, carrying an alternating current $I(t) = I_0 \cos \omega t$, lies flat on the ground, as shown. A spherical, Styrofoam bead with radius $R$ and weight $mg$ resides a distance $h$ above the antenna. $R << h$.

a) Compute the radiation pressure on the bead.

b) At what value of the current amplitude, $I_0$, will the bead float?

3) [40 pts.] Two thin lenses, each with focal length $f$, lie on the optic axis as shown in the figure (top panel). The two lenses are one focal length apart, i.e., they are separated by a distance $f$.

For this problem, you may assume all the rays are paraxial, i.e., $\sin \theta = \tan \theta = \theta$.

a) Compute the system matrix, $M$, for this situation.

b) A light source is placed on the optic axis at a distance $2f$ from the first lens (middle panel). Using the matrix formalism, determine the location of the image of this source (do not use the thin lens formula, unless it is to check your answer).

c) A third lens is now placed a distance $f$ to the right of the second lens (bottom panel). What is the system matrix of this three-lens system?

d) Where is the image now?
1) The reflected ray will be extinguished at the Brewster angle, given by the condition \( \theta_i = \theta_t = \frac{\pi}{2} \).

\[
\eta_d \sin \theta_i = \sin \theta_t = \sin \left( \frac{\pi}{2} - \theta_i \right) = \cos \theta_i
\]

\[\tan \theta_i = \frac{1}{\eta_d} \quad \theta_i = \arctan \left( \frac{1}{\eta_d} \right)\]

b) The transmitted ray is extinguished when \( \theta_t \) becomes complex, i.e., when \( \sin \theta_t = \eta_d \sin \theta_i > 1 \)

\[\sin \theta_i > \frac{1}{\eta_d} \quad \theta_i \geq \arcsin \left( \frac{1}{\eta_d} \right)\]

2) Because \( h \) is large, we can consider radiation from the antenna to have dipole form:

\[
\mathbf{E} = -\frac{\mu_0 \nu_0 c^2}{4 \pi} \left( \sin \theta \right) \sin \left[ \nu \left(t - \frac{r}{c}\right) \right] \hat{\theta}
\]

where \( \rho_0 = q_0 d = \frac{I_0 d}{c} \)

\[\mathbf{B} = \frac{\mathbf{E}}{c} \hat{\phi}\]

\[\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{Poynting vector}\]

\[\mathbf{p} = \frac{1}{c^2} \mathbf{S} \quad \text{Momentum per unit volume}\]

a) The radiation pressure is just

\[P = \frac{1}{c^2} |S|\]

at the location of the bead, given by \( \theta = \pi/2 \), \( r = h \):

\[P = \frac{1}{c} |S|\]
\[ \frac{1}{\mu_0 c} \left\{ \frac{M_0}{c^2} \omega^2 \left( \frac{\sin \theta}{\sin \frac{\theta}{2}} \right) \sin \left[ \omega \left( t - \frac{1}{c} \right) \right] \right\} \]

\[ \left\{ -\frac{M_0}{4\pi i c} \omega^2 \left( \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right) \sin \left[ \omega \left( t - \frac{1}{c} \right) \right] \right\} \]

\[ = \frac{M_0 e^2 \omega^4}{16 \pi^2 c^2} \sin^2 \left[ \omega \left( t - \frac{1}{c} \right) \right] \]

It is acceptable to express the answer in terms of the time-averaged value,

\[ \langle P \rangle = \frac{M_0 e^2 \omega^4}{32 \pi^2 c^2} \]

b) For the bead to float, the time-averaged radiation force on the bead must be equal to mg. Hence

\[ \langle P \rangle \Omega R^2 = \frac{M_0 e^2 \omega^4}{32 \pi^2 c^2} \left( \frac{\Omega R^2}{c^2} \right) = mg \]

\[ \Omega R^2 = \frac{32 \pi^2 c^2 mg}{M_0 e^2 \omega^2} \]

\[ \Omega = \left( \frac{32 \pi^2 c^2 mg}{M_0 e^2 \omega^2} \right)^{1/2} \]

The system matrix for two lenses consists of a thin lens followed by a transfer matrix followed by another lens:

\[ M = \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & -\frac{1}{f} \\ f & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & -\frac{1}{f} \\ f & 0 \end{pmatrix} \]
b) For this part, the first step is to determine the vector of a ray at the point at which it enters the system. Assuming it enters the first lens at height $y_1$:

$$\mathbf{v}_1 = \begin{pmatrix} y_1 \\ -y_1/2f \end{pmatrix}$$

The ray departing the system is given by:

$$\mathbf{v}_2 = M \mathbf{v}_1 = \begin{pmatrix} 0 & -1/f \\ f & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ -y_1/2f \end{pmatrix} = \begin{pmatrix} -y_1/2f \\ y_1/2 \end{pmatrix}$$

Note that, because $\theta_2 < 0$, this ray will cross the optic axis, so the image will be real. The location of the image is given by:

$$\tan(-\theta_2) = \frac{y_2}{s_i} = -\frac{y_1}{s_i}$$

$$s_i = -\frac{y_2}{\theta_2} = -\frac{y_1}{\theta_2} \left(-\frac{f}{y_1} \right) = \frac{f}{2}$$

c) Just add another transfer matrix and then a thin lens:

$$M = \begin{pmatrix} 1 & \frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} -y_1/2f \\ y_1/2 \end{pmatrix}$$

d) The light source did not move, so $\mathbf{v}_1$ is the same as in part b. Hence:

$$\mathbf{v}_2 = \begin{pmatrix} f & -1 \\ 0 & f \end{pmatrix} \begin{pmatrix} -y_1/2f \\ y_1/2 \end{pmatrix} = \begin{pmatrix} y_1/2f \\ -y_1/2 \end{pmatrix}$$

Like in part b), $s_i = -\frac{y_2}{\theta_2}$
\[ s_i = -\frac{\frac{1}{2}}{f} \]

The image is now virtual and at the center of the system of lenses.