Having established that ray tracing is a valid way to think about some aspects of reflection and refraction, we now set out to create a lens.

What is a lens? In the simplest case, it is an interface between two media that takes rays from one point and focuses them to another:

\[ n_1 \frac{SA}{n_2 \frac{AP}{C}} = C \]

What shape should the interface be? Applying Fermat's principle, a ray will follow the path of minimum time, i.e., the minimum OPL.

Q1: How can all the rays from the same location end up at the same point?

A: If all those paths take the same time, i.e., have the same optical path length, for the above situation, this means

\[ n_1 \frac{SA}{n_2 \frac{AP}{C}} = C \]

If you write it out, the above expression is the equation for a hyperbola.

Sadly, hyperbolas are extremely difficult to make. Lenses are usually ground, which essentially always yields a surface shaped like a sphere. So, the question then is whether spherical shapes can be used as lenses, and when?

Consider, then, a spherical interface:
The center line in such an arrangement is called the optic axis. We will start by considering the case where the ray crosses the optic axis at point \( P \) (we will relax this assumption shortly).

In this case, the optical path length is

\[
\text{OPL} = n_1 l_0 + n_2 l_i
\]

Now apply the law of cosines:

**For angle SCA:**

\[
l_0 = \left[ (s_0 + R)^2 + R^2 - 2R(s_0 + R) \cos \phi \right]^{1/2}
\]

**For angle ACP:**

\[
l_i = \left[ (s_i - R)^2 + R^2 + 2R(s_i - R) \cos \phi \right]^{1/2}
\]

where we have used the fact that

\[
\cos(\pi - \phi) = -\cos \phi
\]

what we would like to know which path the light takes from \( S \) to \( P \). In other words, we want to know which value of \( \phi \) minimizes the transit time, i.e., that satisfies the following condition:

\[
\frac{d\text{OPL}}{d\phi} = 0
\]

Doing this derivative gives

\[
\frac{n_1 R(s_0 + R) \sin \phi}{l_0(\phi)} - \frac{n_2 R(s_i - R) \sin \phi}{l_i(\phi)} = 0
\]

or

\[
\frac{n_1}{l_0} + \frac{n_2}{l_i} = \frac{1}{R} \left( \frac{n_2 s_i - n_1 s_0}{l_i} \right)
\]

In an ideal lens, this expression would be independent of \( \phi \). In the current case, different rays converge to different locations, and the imaging is imperfect. These imperfections are termed **spherical aberrations**.

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**Gaussian Optics**

Things simplify if we stay near the optic axis. In this case \( \phi \) is small; \( \cos \phi \approx 1 \), and

\[
l_0 \approx s_0
\]
\[ l_i \approx s_i \]

In this case, our expression reduces to

\[ \frac{n_1}{s_0} + \frac{n_2}{s_1} = \frac{n_2-n_1}{R} \]

Comments:

1. This expression is independent of \( \theta \).
2. This expression is reversible, i.e., it is valid for rays emerging from \( S \) and arriving at \( P \) (for small angles, at least).

And invert the curvature

\[ R \Rightarrow -R \]

It is useful to consider the two limiting cases of 1) the rays being

Focused at \( \infty \) and 2) parallel rays focusing to \( P \).

In the former case, \( s_i = \infty \) so

\[ \frac{n_1}{s_0} + \frac{n_2}{\infty} = \frac{n_2-n_1}{R} \]

This gives

\[ s_0 = \frac{n_1}{n_2-n_1} R \equiv f_o \text{ object focal length} \]

In the latter case, \( s_o = \infty \) and

\[ \frac{n_1}{\infty} + \frac{n_2}{s_i} = \frac{n_2-n_1}{R} \]

or

\[ s_i = \frac{n_2}{n_2-n_1} R \equiv f_i \text{ image focal length} \]

Earlier we made the assumption that the rays converge to point \( P \). But our expressions are valid even if the rays
never cross the optic axis:

\[ S_0 \rightarrow f_0 \rightarrow P \]

\[ n_1 \rightarrow n_2 \]

In the latter case, \( S_0 < f_0 \) and \( S_1 < 0 \). In this case we have a virtual focus at \( P \) on the left side.

Discussion topic: what is a virtual focus?

It is time to close the other side to make a real lens

\[ \text{(c)} \]

we will start by considering the specific case where \( S_{01} < f_0 \) for the left face. In this case the image is virtual and resides at \( P' \):

\[ \frac{n_m}{S_{01}} + \frac{n_s}{S_{01}} = \frac{n_s - n_m}{R_1} \]

\( P' \) acts like the object for the right face, at object distance

\[ S_{02} = |S_{z1}| + d = -S_{z1} + d \]
since $s_i < 0$. Now apply our formula to the right face:

\[
\frac{n_e}{-s_{i1} + d} + \frac{n_m}{s_{i2}} = \frac{n_m - n_e}{R_2}
\]

where we recall that $R_2 < 0$ the way the picture is drawn. Add the two expressions together:

\[
\frac{n_m}{S_0} + \frac{n_e}{-s_{i1} + d} + \frac{n_m}{s_{i2}} = \frac{n_e - n_m}{R_1} + \frac{n_m - n_e}{R_2}
\]

\[
\frac{n_m}{S_0} + \frac{n_m}{s_{i2}} = (n_e - n_m) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) + \frac{n_e d}{s_{i1}(d-s_{i1})}
\]

Finally, we will assume our lens is thin, i.e., $d < s_{i1}$, so we can neglect the last term. We are left with the Thin Lens Equation or Lensmaker's formula:

\[
\frac{n_m}{S_0} + \frac{n_m}{s_{i2}} = (n_e - n_m) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{f}
\]

Most of the time $n_m = 1$, so we will often just write

\[
\frac{1}{S_0} + \frac{1}{s_{i2}} = (n_2 - 1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

when we have also dropped the 1 and 2 subscripts, since we are not interested in the intermediate image.

**Comments:**

1. We derived this formula for the case $S_0 < S_0$, but - like before - this is not true.

2. The signs of $R_1$ and $R_2$ define the type of lens:

   - $R_1 > 0$, $R_2 > 0$: $R_1 < 0$, $R_2 < 0$, $R_1 > 0$, $R_2 > 0$, $R_1 = \infty$, etc.
Like before, it is useful to examine some limits:

$$s_0 = \infty$$

$$\frac{1}{s_i} = \left( n_e - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

$$s_i = \infty$$

$$\frac{1}{f_0} = \left( n_e - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

Evidently,

$$\frac{1}{f_0} = \frac{1}{f_i} = \frac{1}{\frac{1}{s_i}} = \left( n_e - 1 \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

**Extended Images**

$$\Theta = \frac{1}{f} \text{ magnification power}$$

The value of lenses, of course, is that they can make images. One can deduce how simply by recognizing that rays passing through the center of a thin lens do not bend (or more precisely, they bend, but bend right back):

Clearly, \( \frac{y_i}{y_0} = - \frac{s_i}{s_0} \)

or \( \frac{y_i}{y_0} = - \frac{s_i}{s_0} = M_T \text{ Transverse magnification.} \)

The - sign symbolizes the image being inverted.

**Mirrors**

Imaging can also be carried out...
with mirrors. Like lenses, ground mirrors are usually spherical, but can be used for imaging in the paraxial region.

\[ \frac{1}{s_0} + \frac{1}{s_c} = -\frac{2}{R} = \frac{1}{f} \]

Mirror formula

$s_c < 0 \Rightarrow$ real image on the left hand side of the mirror

$s_c > 0 \Rightarrow$ virtual image on the right