Multi-lens systems

Lenses really become useful when we use combinations of them to create complex optical systems.

Let's start by considering a pair of lenses with focal lengths $f_1$ and $f_2$, separated by $d < f_1, f_2$.

For this situation

$$\frac{1}{s_{1i}} = \frac{1}{f_1} - \frac{1}{s_{0i}}$$

$$s_{1i} = \frac{s_{0i}f_1}{s_{0i} - f_1}$$

Now let's add the second lens and see what happens to these rays.

To see how this images, first consider the principal rays from the first lens:

Note: Take $s_{0i} > f$
For the second lens, the object distance is negative:

\[ s_{o2} = d - s_{c1} < 0 \]

This is because the object is virtual and on the RHS of the lens. Of course, the formula still works:

\[ \frac{1}{s_{i2}} = \frac{1}{f_2} - \frac{1}{s_{o2}} \quad \text{or} \quad s_{i2} = \frac{s_{o2} f_2}{s_{o2} - f_2} \]

Putting it all together,

\[ s_{i2} = \frac{(d - s_{c1}) f_2}{d - s_{c1} - f_2} \]

or

\[ s_{i2} = \frac{[d(s_{o1} - f_1) - s_{o1} f_1] f_2}{(d - f_2)(s_{o1} - f_1) - s_{o1} f_1} \]

The result is a real image. The overall transverse magnification

\[ M_f = M_{f1} \cdot M_{f2} = \frac{s_{i2}}{s_{o1}} \]

\[ = \frac{f_1 f_2}{(s_{o1} - f_1)(d - f_2) - s_{o1} f_1} \]

For the above arrangement, \( M_f < 0 \), i.e., the image is inverted.

Note: For large \( d \), \( M_f \) can be positive. So, with multi-lens systems, it is possible to generate upright real images.

As before, it is useful to examine the limits:

\[ s_{o1} \to \infty \quad s_{i2} = \frac{f_2 (d - f_1)}{d - (f_1 + f_2)} \approx f_2 \]

This is called the “back” or “image” focal length.

\[ s_{i2} \to \infty \quad s_{o1} = \frac{f_1 (d - f_2)}{d - (f_1 + f_2)} \approx f_1 \]

This is called the “front” or “object” focal length.
Optical Systems: Magnifying Glass

We now proceed to the practical matter of how lenses can be used to help us see things better. We will start by examining the simplest such system.

In optics, the human eye is normally conceptualized as an angle detector. The apparent size of the object is given by the angular range of the rays, \( \alpha_u \) (\( \alpha \) stands for "unaided").

\[
\alpha_u = \frac{y_0}{d}
\]

in the paraxial regime.

What happens if we put a lens in between?

We will place the object closer than the focal length.

The lens creates an image that is magnified, i.e., \( M > 1 \). On the other hand, the image is now farther away than the viewer. Does this help?

To find out, we need to know the angular magnification,

\[
M_A = \frac{\alpha_i}{\alpha_u}
\]

where \( \alpha_i \) is the apparent angular size of the image created by the lens. Again,
for paraxial rays,

\[
\alpha = \frac{y_i}{L} \quad \text{so} \quad M = \frac{y_i}{y_0} \frac{d}{L}
\]

To determine this we need to know \( s_i \):

\[
\frac{1}{s_i} = \frac{1}{f} - \frac{1}{s_0} \quad \text{or} \quad s_i = \frac{s_0 f}{s_0 - f} < 0
\]

Image size

\[
y_i = -\frac{s_i}{s_0} \quad y_0 = \frac{f}{f - s_0}
\]

Image distance

\[
L = d - s_0 + s_i
\]

\[
L = d - s_0 - s_i = d - s_0 - \frac{s_0 f}{s_0 - f}
\]

\[
\frac{1}{L} = \frac{1}{d - s_0 - s_0 f} = \frac{s_0 - f}{(d - s_0)(s_0 - f)}
\]

\[
\quad = \frac{s_0 - f}{d s_0 - s_0^2 - d f + s_0 f - s_0 - f} \quad \text{[circular]} \quad + \quad \text{[circular]}
\]

\[
= \frac{s_0 - f}{t_0^2 + d(f - s_0)} > 0
\]

Hence,

\[
M = \frac{d}{y_0} \frac{f y_0}{s_0} \frac{f - s_0}{s_0^2 + d(f - s_0)}
\]

\[
M = \frac{d f}{s_0^2 + d(f - s_0)}
\]

So, does it work? Well, it depends on where you hold it. To get a large magnification you want the denominator to be small. At fixed \( d \) this happens when \( s_0 = d/2 \). In this case

\[
M = \frac{4f}{4f - d} > 1
\]

\[Q: \text{what happens when } d = 4f?\]
Optical Systems: Compound Microscope

There are two problems with single lens magnifiers:

1. It requires adjusting d. This is not always practical.
2. It causes eye strain. At large magnification the rays are rapidly diverging, which the eye may struggle to image on the retina.

A better solution is the compound microscope, the simplest of which is just a combination of two lenses:

\[
M = M_1 \cdot M_2
\]

The first lens is called the objective lens and makes a real image of the object inside the body of the microscope.

The second lens, called the eyepiece, acts like a magnifying glass whose object is the image created by the objective, creating a virtual image for the eye to see.

The relevant overall magnification is the product of the transverse magnification of the first lens and the angular magnification of the second:

Often, the image is placed at \( \infty \), which is optimal for comfort.
In a microscope there is always a trade-off between sensitivity and image quality. To get sharp, high-resolution images, one needs to stay in the paraxial regime, accepting only rays along the optical axis. To see dim, faint objects, on the other hand, one needs to collect as much light as possible, which may require opening the aperture to allow rays at steeper angles to enter. This will brighten the image but will introduce aberrations, making the image blurry.

If one is unwilling to compromise, one can fork out the cash and buy a microscope with aspherical aberration-corrected optics, that can produce sharp images, even accepting rays that are far from the optic axis. The figure of merit of such a microscope is the numeric aperture

\[ NA = n_m \cdot \sin \theta_{\text{max}} \]

where \( \theta_{\text{max}} \) is the largest angle accepted by the objective lens of the instrument.

Reading: Hecht 5.7.7 on the Telescope.