Lecture 6 - Electromagnetic waves

Last time we discussed that waves in a medium can be described by the wave equation

\[ \frac{\nabla^2 f(x,t)}{\nabla^2 x^2} = \frac{1}{v^2} \frac{\partial^2 f(x,t)}{\partial t^2} \]

where \( v \) is called the phase velocity. In more than one dimension, this generalizes to

\[ \nabla^2 f(\vec{r},t) = \frac{1}{c^2} \frac{\partial^2 f(\vec{r},t)}{\partial t^2} \]

where \( \nabla^2 \) is the Laplacian.

Today, we will apply these concepts to light, which we will see is an electromagnetic wave.

How can we show this? We will have to back up, review basic electrodynamics, and then obtain Maxwell’s Equations, from which electromagnetic waves follow naturally.

E&M Review

Electricity and magnetism are based on two fields,

\[ E = E(\vec{r},t) \quad \text{Electric field} \]
\[ B = B(\vec{r},t) \quad \text{Magnetic field} \]

Note that, unlike \( f(x,t) \) above, \( E \) and \( B \) are vector fields, i.e., they have a direction, not just a magnitude.

Lorentz Force

\[ F = qE + qv \times B \]
\[ q (\vec{E} + \vec{v} \times \vec{B}) \]

where \( \vec{v} \) is the velocity of the charge.

\( \vec{E} \) just exerts a force in its own direction. It only acts on charges that are moving and only perpendicular to its direction of motion.

**Faraday's Law**

In the prevailing view taught in school, Faraday's law is really a statement of two (seemingly) different facts:

1. A wire moving in a magnetic field will acquire a current, because of the Lorentz force (above)

2. A changing \( \vec{E} \) field will create a \( \vec{E} \) field.

Let's consider these two things separately.

Suppose I drag a loop of wire into a region with a \( \vec{B} \) field:

The charges in the segment of wire on the left will experience a Lorentz force, which will drive a current around the loop.

Once both ends of the loop have entered the field, the current stops. The reason is that there is an equal, compensating force on the charges on the opposite
The same thing happens if I move the magnet. If the wire is fixed and I translate the field region, I get the same current. (This is, after all, the same process viewed in the rest frame of the loop.) However, there is no Lorentz force. This happens because a changing magnetic field creates an electric field.

Both of these phenomena were captured by Michael Faraday in his law of magnetic induction:

\[ \mathcal{M} = \int_B \mathbf{E} \cdot d\mathbf{a} \quad \text{magnetic flux} \]

The induction law states that a changing flux (whatever the mechanism) creates an emf ("electromotive force") around the loop,

\[ \text{emf} = - \frac{d\Phi_m}{dt} \]

where

\[ \text{emf} = \oint_{\partial A} \mathbf{E} \cdot d\mathbf{l} \]

emf has units of volts, and is essentially the voltage drop around the loop.

Using a vector identity called Stokes' theorem, the induction law can be written in differential form as

\[ \nabla \times \mathbf{E} = -\frac{d}{dt} \mathbf{B} \]
\[ \nabla \times \vec{E} = \hat{x} \left( \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y \right) + \hat{y} \left( \frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z \right) + \hat{z} \left( \frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \right) \]

is the curl of \( \vec{E} \).

- A changing \( \vec{B} \) creates an \( \vec{E} \) that is looping around on itself.
- Both the integral and differential forms are true everywhere, whether there is a physical loop or not.
- The differential form is one of Maxwell's equations.

**Gauss's Law**

Gauss's law is essentially a statement about electric flux. As discussed above, one way to create an electric field is to change a \( \vec{B} \) field. But the \( \vec{B} \) never acts like a true source of \( \vec{E} \). Let me explain. The field lines created by a

\[ \frac{d\vec{B}}{dt} \]

always close on themselves:

\[ \text{Changing } \vec{B} \]

\[ \text{resulting } \vec{E} \text{ field lines.} \]

Hence, it is never possible for a \( \vec{B} \) field to act as a source of \( \vec{E} \) field lines:

A changing \( \vec{B} \) field could never for example lead to a net flux through a closed surface:

\[ \Phi_E = \oint \vec{E} \cdot d\vec{a} = 0 \]

The only thing that can do this is electric charge. Gauss's law states this quantitatively.
\[ \oint \mathbf{E} \cdot d\mathbf{a} = \frac{\mathcal{Q}_{\text{enc}}}{\varepsilon_0} \]

where \( A \) is a closed surface and \( \mathcal{Q}_{\text{enc}} \) is the charge contained in that surface:

\[ \mathcal{Q}_{\text{enc}} = \int \rho \, dv \]

where \( \rho \) is the charge density, i.e., the charge per unit volume \( \Omega \), and \( \varepsilon_0 \) is the permittivity of free space.

\[ \varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{Nm}^2 \]

Like above, this integral equation

\[ \oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int \rho \, dv \]

\( \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \)

\( \nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \) is the divergence of \( \mathbf{E} \).

This is another one of Maxwell's equations.

**Magnetic Monopoles**

Are there similar sources of magnetic flux? i.e., are there magnetic charges?

\[ \oint \mathbf{B} \cdot d\mathbf{a} \propto \mathcal{Q}_m \]

There is nothing in Maxwell's equations that forbids magnetic
charge. But for whatever reason it appears not to exist. Hence,
\[ \oint \mathbf{B} \cdot d\mathbf{a} = 0 \]
for a closed surface. Using the divergence theorem,
\[ \nabla \cdot \mathbf{B} = 0 \]

**Ampere's Law and Displacement Current**

If magnetic charges do not exist, how do we even make a \( \mathbf{B} \) field?
It turns out that there are two ways:
- Moving charges (i.e. currents)
- Changing \( \mathbf{E} \) field

If either of these two things is present, it gives rise to a circulating \( \mathbf{B} \) field.

\[ \oint \mathbf{B} \cdot d\mathbf{a} = \mu_0 \int \left( \mathbf{J} + \varepsilon_0 \frac{d\mathbf{E}}{dt} \right) \cdot d\mathbf{a} \]

where \( \mu_0 \) is the permeability of free space,
\[ \mu_0 = \frac{4\pi \times 10^{-7} \text{ Ns}^2/\text{C}^2} \]
\( j \) is the current density (current per unit area).

The quantity
\[
\varepsilon_0 \frac{d\vec{E}}{dt} = \vec{J}_p
\]
is often called the displacement current.

In differential form,
\[
\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}
\]

Maxwell's Equations

These four equations make up Maxwell's equations:
\[
\begin{align*}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{E} &= \rho / \varepsilon_0 \\
\nabla \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \\
\nabla \cdot \vec{B} &= 0
\end{align*}
\]

Media

Inside materials, the values of \( \varepsilon_0 \) and \( \mu_0 \) may change:
\[
\begin{align*}
\varepsilon &= \varepsilon_0 \kappa_E \\
\mu &= \mu_0 \kappa_m
\end{align*}
\]

\( \kappa_m \) = relative permeability
\( \kappa_E \) = relative permittivity or "dielectric constant"

EM Waves

The existence of electromagnetic waves follows directly from Maxwell's equations. To see how, let's start by assuming we're in vacuum:
\[
\begin{align*}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{E} &= 0 \\
\nabla \times \vec{B} &= \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt} \\
\nabla \cdot \vec{B} &= 0
\end{align*}
\]
Now let's take the curl of Faraday's law:
\[ \nabla \times (\nabla \times \vec{E}) = -\frac{1}{\varepsilon_0} \frac{\partial^2 \vec{E}}{\partial t^2} \]
\[ = -\frac{1}{\varepsilon_0} \left( \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \right) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

Next, we will use this vector product rule:
\[ \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \]

This gives:
\[ \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

\[ \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \]

where
\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.997 \times 10^8 \text{ m/s} \]

Following the same procedure for Ampere's law, we get:
\[ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} - \frac{\mu_0}{c^2} \frac{\partial \vec{E}}{\partial t} \]

Like any wave eq., these have harmonic solutions:
\[ \vec{E}(r, t) = \vec{E}_0 \left( e^{i(k \cdot r - \omega t)} \right) \]
\[ \vec{B}(r, t) = \vec{B}_0 \left( e^{i(k \cdot r - \omega t)} \right) \]

where we remember that the actual fields are the real parts of these quantities.

**Discussion:** What makes an EM wave propagate?