Lecture 7 - Energy and Momentum

EM Waves are Transverse

Last time we saw that, as a consequence of Maxwell's equations, $E$ and $B$ satisfy wave equations:

$$\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}$$

Note that, unlike when we were discussing the classical wave, $E$ and $B$ are vector quantities.

$E = (E_x, E_y, E_z)$, $B = (B_x, B_y, B_z)$

Hence, each of these wave equations really is three equations. For the case of $E$,

$$\nabla^2 E_x = \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2}$$

$$\nabla^2 E_y = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2}$$

The consequence of this is that $E$ waves are polarized, e.g., the harmonic solutions have the form:

$$E(r, t) = E_0 \ e^{i(k \cdot r - \omega t)}$$

$$B(r, t) = B_0 \ e^{i(k \cdot r - \omega t)}$$

where we keep in mind that we really mean the real part of these quantities.

Anyway, $k$ defines the propagation direction, and, by convention, we usually say that the amplitude $E_0$ defines the direction of the polarization.

Note that both $E$ and $B$ are constant on any plane perpendicular to the propagation direction, $k$.

Which way does $E_0$ point? Recall that in a vacuum

$$\nabla \cdot E = 0 \quad (\text{Gauss's law})$$

Hence

$$\nabla \cdot E(r, t) = i(E_0 x k_x + E_0 y k_y + E_0 z k_z) \ e^{i(k \cdot r - \omega t)}$$
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)} \]

Plug in the harmonic solutions:
\[ \nabla \times \left[ \mathbf{E}_0 e^{i(k \cdot \mathbf{r} - \omega t)} \right] = -\frac{\partial}{\partial t} \left[ \mathbf{B}_0 e^{i(k \cdot \mathbf{r} - \omega t)} \right] \]
\[ i k \times \mathbf{E}_0 e^{i(k \cdot \mathbf{r} - \omega t)} = -i \omega \mathbf{B}_0 e^{i(k \cdot \mathbf{r} - \omega t)} \]

We can already see that \( \mathbf{B}_0 \) is always perpendicular to both \( \mathbf{E}_0 \) and \( k \).

\[ k \mathbf{E}_0 (\hat{z} \times \hat{z}) = \mathbf{B}_0 \]

Recall that \( \omega = ck \), so
\[ \mathbf{B}_0 = \frac{1}{c} \mathbf{E}_0 \hat{z} \]

In other words, \( \hat{E} \), \( \hat{B} \), and \( \hat{k} \) for a plane wave form a mutually orthogonal basis set, with \( \hat{E} \times \hat{B} = \hat{k} \).

Taking the real part to get the physical fields, we get our cartoon picture from Lecture 1.

\[ \mathbf{E}_0 \cdot \mathbf{k} = 0 \]

where I have skipped a few algebra steps that you should fill in yourself.

The conclusion is that
\[ \mathbf{E}_0 \cdot \mathbf{k} = 0 \]

so
\[ \mathbf{E}_0 \perp \mathbf{k} \]

The polarization is always perpendicular to the propagation direction (in vacuum).

Because \( \nabla \cdot \mathbf{B} = 0 \), the same is true of \( \mathbf{B}_0 \).

But how are \( \mathbf{E}_0 \) and \( \mathbf{B}_0 \) related? For this, it is useful to take
\[ \mathbf{k} = k \hat{z} \]
\[ \mathbf{E}_0 = E_0 \hat{x} \]

i.e., a wave polarized in the \( x \) direction, propagating along \( z \).

Recall that
Energy in EM Fields

Electric and magnetic fields contain energy and momentum. If a region of space contains EM fields, its total energy is

\[ W = \int \left( \frac{\varepsilon_0}{2} \vec{E}^2 + \frac{1}{2\mu_0} \vec{B}^2 \right) dV \]

We will use \( W \) to denote the total energy, to avoid confusion with the \( E \) field. \( W \) is essentially the work that must be done to assemble the system.

The integrand itself

\[ u(r,t) = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} = u_E + u_B \]

is called the energy density. It has separate electric and magnetic contributions, \( u_E \) and \( u_B \).

\[ [u] = \frac{\text{energy}}{\text{volume}} \]

So it's literally the energy per unit volume stored in the fields.

To compute \( u \), it is important to revert to the physical fields. For example, for a plane wave

\[ u = \frac{\varepsilon_0 E_0^2}{2} \cos^2(kz - ct) + \frac{B_0^2}{2\mu_0} \cos^2(kz - ct) \]

\[ = \left( \frac{\varepsilon_0 E_0^2}{2} + \frac{E_0^2}{2\mu_0 c^2} \right) \cos^2(kz - ct) \]

Since \( E \) and \( B \) are related, we can write the energy purely in terms of \( E_0 \).
This function oscillates very rapidly. For visible light, for example,
\[ v = \frac{\omega}{2\pi} \sim 10^{15} \text{ Hz} \]

So, it is much more useful to discuss the mean value, averaged over many cycles.

So, what's the average value of \( (\cos^2 x) \)?

\[ \langle \cos^2 x \rangle = \frac{1}{2} + \frac{1}{2} \langle \cos 2x \rangle = \frac{1}{2} \]

Hence

\[ \langle H \rangle = \frac{\varepsilon_0 E^2}{2} \quad \text{for a plane wave} \]

**Momentum in EM Fields**

Since EM fields contain energy, it should come as no surprise that EM waves can transmit energy and, hence, can carry momentum. The quantity that describes this (which you will derive if you take Phys 436) is called the *Poynting vector*:

\[ \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \]

The units of \( \vec{S} \) are

\[ \frac{\text{Energy}}{\text{Area \cdot time}} \sim \text{watts/m}^2 \]

So it is basically a power density.

For the case of our plane wave,

\[ S = \frac{1}{\mu_0} (E_0 \vec{x}) \times (\frac{1}{c} \vec{E}_0 \vec{y}) \cos^2(kz - \omega t) \]

\[ = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t) \hat{z} \]

So, the power flows in the direction of propagation, i.e., the \( \hat{z} \) direction.
Notice that, for a plane wave,
\[ \vec{S} = c \vec{U} \]
which makes sense: the power density is just the energy density times the speed at which it is flowing.

Again, for visible light ($\sim 10^{15}$/s) it is more useful to look at the average:

\[ <15> = \frac{c \varepsilon_0 E_0^2}{2} = I \]

Depending upon what field you work in, $I$ is called either the Intensity or the Irradiance.

For sunlight, $I = 1000 \text{ W/m}^2$.

Total power hitting the earth

\[ P = I \cdot \pi R^2 \sim 10^{17} \text{ W} \]

= 10000 terawatts

\[ \approx 15 \text{ TW} \]

World power consumption

Finally, \( \vec{S} \) also describes the momentum carried by the fields

\[ \vec{p} = \frac{1}{c^2} \vec{S} \]

This quantity has units of momentum/volume and is called the momentum density. The total momentum contained in some finite field volume is

\[ \vec{p} = \int p \, d\mathbf{v} \]

Sunlight absorbed by the earth results in a force

\[ F = \frac{d\vec{p}}{dt} = \mu c \cdot \pi R^2 \sim 10 \text{ tons} \]

Appreciable forces can, however, be created with lasers, a principle that is exploited in optical tweezers.

* Rigorous proof of this expression requires use of the Maxwell stress tensor.