Lecture 9 - Light is Matter

Polarization

\[ \mathbf{dV} \quad \mathbf{dp} = \mathbf{r} \mathbf{dV} \]

\[ \mathbf{p} = \text{dipole moment per unit volume} \]

\[ \mathbf{p} = \varepsilon_0 \chi_e \mathbf{E} \]

\[ \varepsilon = \varepsilon_0 (1 + \chi_e) \quad \chi_e = \text{"electric susceptibility"} \]

Harmonic Oscillator Model

A surprisingly accurate model is thinking of the atom or element of the material as an electron attached to a nucleus by a spring.

\[ \mathbf{F} = -e \mathbf{E} \]

\[ P = -e \mathbf{x} \]

The force on this electron by an EM wave would be

If the oscillator is small,

\[ \mathbf{F} = -e \mathbf{E} = -e \mathbf{E}_0 \cos \omega t \hat{x} \]

\[ \mathbf{F} = \mathbf{Re} \left[ -e \mathbf{E}_0 e^{-i\omega t} \right] \hat{x} \]

Dynamics described by SHO equation in the x direction:

\[ m \frac{d^2 \mathbf{x}}{dt^2} + \mu \frac{d \mathbf{x}}{dt} + k \mathbf{x} = \mathbf{F}(t) \]

\[ = -e \mathbf{E}_0 e^{-i\omega t} \]

\[ \hat{x} \quad \text{damping} \]

It's convenient to divide out the mass:

\[ \frac{d^2 \mathbf{x}}{dt^2} + \xi \frac{d \mathbf{x}}{dt} + \omega_0^2 \mathbf{x} = \left( \frac{-e}{m} \right) \mathbf{E}_0 e^{-i\omega t} \]

where \( \xi = \mu/m \) and \( \omega_0 = \sqrt{k/m} \) is the resonance frequency.

The solution should have the form:

\[ x(t) = x_0 e^{-i\omega t} \]
where $\chi_0$ can be complex. Plugging in,

$$-\omega^2 \chi_0 - i\omega \chi_0 + \omega_0^n \chi_0 = \left( -\frac{e}{m} \right) E_0$$

where we have canceled the factors of $e^{i\omega t}$. Solving for $\chi_0$,

$$[-\omega^2 - i\omega + \omega_0^2] \chi_0 = \left( -\frac{e}{m} \right) E_0$$

$$\chi_0 = \frac{e/m}{\omega^2 - \omega_0^2 + i\alpha} \quad E_0 = \chi_0(\omega)$$

So

$$x(t) = \chi_0(\omega) e^{i\omega t} = \frac{eE_0/m}{\omega^2 - \omega_0^2 + i\alpha} e^{-i\alpha t}$$

So, we see that the amplitude is a very strong function of frequency, and becomes largest when $\omega = \omega_0$. This effect is normally called “resonance.”

Anyway, our main conclusion is that our atom has an induced dipole moment that oscillates with the same frequency as the $E$ field.

$$p(t) = -e\chi(t) = -\frac{e^2/m}{\omega^2 - \omega_0^2 + i\alpha} e^{i\omega t}$$

**Many Atoms**

We argued earlier that we are pursuing the microscopic origins of the index of refraction. To do this — whether it’s a solid, liquid, or gas — we need to consider the case of many atoms confined to a finite volume.

Now all the atoms respond to the driving $E$ field. So what we get is not an
oscillating dipole moment, but an oscillating electric polarization,

\[ P(t) = p(t) \cdot \mathbf{p} \]

where \( p = N/V = \text{number of atoms/unit volume} \).

To determine the index of refraction \( n = \sqrt{k_E} \), we need to determine the dielectric constant \( k_E = 1 + \kappa E \), which means we need to determine the susceptibility, \( \kappa E \), which means we need to determine the relationship between \( P(t) \) and \( E(t) \).

Written out, the above expression is

\[ P(t) = \rho \cdot \frac{e^2}{m} \cdot \mathbf{E} \cdot e^{ -i \omega t} = \frac{p e^2}{m} \cdot \mathbf{E}(t) \cdot e^{ -i \omega t} \]

Comparing to

\[ P = \varepsilon_0 \kappa E \cdot \mathbf{E} \]

we see that

\[ \kappa E = \frac{p e^2}{m \varepsilon_0} \cdot \frac{1}{\omega^2 - \omega^2 - i \omega \gamma} \]

and hence

\[ \varepsilon = \varepsilon_0 (1 + \kappa E) = \varepsilon_0 + \frac{p e^2}{m \varepsilon_0} \cdot \frac{1}{\omega^2 - \omega^2 - i \omega \gamma} \]

finally

\[ n(\omega) = \sqrt{1 + \kappa E} = \sqrt{1 + \frac{p e^2}{m \varepsilon_0} \cdot \frac{1}{\omega^2 - \omega^2 - i \omega \gamma}} \]

**General Comments**

1. All materials are made of atoms, so you can see that the index of refraction is almost always a strong function of frequency. For this reason, if white light is incident on a material (e.g. a piece of glass), different frequencies refract by different amounts, causing light to disperse into its different colors, creating a "rainbow". This phenomenon is
2. $n$, also defines the phase velocity of light in a material:

$$v = \frac{c}{n(\omega)}$$

Hence, the phase velocity is also frequency-dependent. So too is the relationship between frequency and wavenumber:

$$\omega = \frac{c}{n(\omega)} k \quad \text{or} \quad k = \frac{n(\omega) \omega}{c}$$

For this reason, we call this a *dispersion relation*.

3. If $\omega > \omega_0$, $n < 1$. This implies that the phase velocity is greater than $c$. This is the case for all materials at very high (x-ray) frequencies. We will see later, when we discuss group velocity, that this does not violate any principles of relativity.

4. If there is damping in our oscillator, i.e., if $\gamma \neq 0$, the index of refraction is complex. What does that mean?

A complex $n$ is usually written like so:

$$n(\omega) = N(\omega) + i \kappa(\omega)$$

The frequency $\omega$, is always real, which means the wave vector is complex:

$$k = \frac{n(\omega)}{c} \omega = N(\omega) \frac{\omega}{c} + i \kappa(\omega) \frac{\omega}{c}$$

Let’s consider, then, a harmonic wave propagating along the $z$ direction:

$$\mathbf{E}(z, t) = \mathbf{E}_0 \exp \left[ i \left( \frac{N \omega}{c} z + \frac{i \kappa \omega}{c} z - ct \right) \right]$$

$$= \mathbf{E}_0 \exp \left[ i \left( \frac{N \omega}{c} z - ct \right) - \frac{i \kappa \omega}{c} z \right]$$
\[ E_0 e^{i \left( \frac{N \omega}{c} z - ct \right)} e^{-\frac{K \omega z}{c}} \]

Oscillatory part with \( k = \frac{N \omega}{c} \)

\[ \uparrow \text{Decay term} \]

In other words, the wave loses amplitude (and, hence, energy) as it propagates. This is a consequence of the friction we put in our spring.

In a real atom there is, of course, no "friction." But there is always dissipation. In solids it is usually caused by decay into acoustic phonons.