Lecture 10 - Scattering and Fresnel Formulas

with a framework for handling radiation, we are now in a position to deal with the subject of scattering which could be defined colloquially as the non-directional propagation of light in a medium that is inhomogeneous.

In lectures 8-9, we saw that an atom interacting with light becomes an oscillating dipole which in a solid creates an oscillating polarization, $\mathbf{P}(t)$. In addition - like an antenna - this atom should also radiate:

$$\mathbf{E} = \frac{\mu_0 \rho_o \omega^2}{4\pi} \left( \frac{\sin \theta}{r^2} \right) \cos (kr - \omega t) \hat{\theta}$$

$$\mathbf{B} = \frac{1}{c} \mathbf{E} \times \hat{\phi}$$

(lecture 8)

Recall that the radiated power density is given by

$$S = \frac{1}{m_0} \mathbf{E} \times \mathbf{B}$$

$$I = |<S>|^2$$

$$I = \frac{\mu_0 \rho_o \omega^4}{32 \pi^2 c} \frac{\sin^2 \theta}{r^2}$$

where

$$\rho_o = -e \chi_0$$

$$\chi_0 = \frac{e E_0 / m}{\omega^2 - \omega_0^2 + i \omega \kappa}$$

Incidentally, if we compute the total power passing through a sphere of radius $R$,
\[ P = \frac{e^2}{\varepsilon_0 c^3} \int d\Omega \, I(\theta) \]

where \( \alpha = \chi_0 c^2 \) and we have used the fact that \[ \int_0^\pi \sin^2 \theta d\theta = \frac{\pi}{3} \]

The above expression is the famous Larmor formula for dipole radiation.

**Rayleigh Scattering**

To understand scattering from complex media, we can do a great deal just superposing multiple point dipoles. For example, light propagating through a dilute gas, such as the upper atmosphere, can be modeled well as scattering from random dipoles:

\[ E = \frac{\mu_0 \rho d c^2}{4\pi} \sum_{n=1}^{\infty} \left( \sin \theta_n \right) \cos \left[ k \left| \mathbf{r} - \mathbf{r}_n \right| - \omega t \right] \delta_n \]

This expression simplifies greatly if we assume the positions of the dipoles are distributed randomly. To see why, let's consider the case where we have two dipoles:

\[ E = E_1 + E_2 \]

\[ E_1 = \frac{\mu_0 \rho d c^2}{4\pi} \frac{\sin \theta_1}{|\mathbf{r} - \mathbf{r}_1|} \cos \left[ k \left| \mathbf{r} - \mathbf{r}_1 \right| - \omega t \right] \theta_1 \]

\[ E_2 = \frac{\mu_0 \rho d c^2}{4\pi} \frac{\sin \theta_2}{|\mathbf{r} - \mathbf{r}_2|} \cos \left[ k \left| \mathbf{r} - \mathbf{r}_2 \right| - \omega t \right] \theta_2 \]

where \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) are the locations of the two dipoles, and \( \mathbf{r} \) is the location of the observation point.
of the observer. $\theta_1, \theta_2, \hat{\theta}_1,$ and $\hat{\theta}_2$
describe the location $\vec{r}$ in the coordinate systems centered on $\vec{r}_1$ and $\vec{r}_2$.

The intensity at the location of the observer, $I$, is

$$I = c \varepsilon_0 \langle |E|^2 \rangle = c \varepsilon_0 \langle |\vec{E}_1 + \vec{E}_2|^2 \rangle$$

we have to add the two fields together

before squaring.

Suppose the two dipoles are both near the origin, and the observer is

far away. In this case,

$$|\vec{r} - \vec{r}_1| \approx |\vec{r} - \vec{r}_2| \approx |\vec{r}| \equiv r$$

$$\theta_1 \approx \theta_2 \equiv \theta$$

$$\hat{\theta}_1 \approx \hat{\theta}_2 \equiv \hat{\theta}$$

The sum then reduces to

$$\vec{E}_1 + \vec{E}_2 = \frac{\mu_0 \rho_2}{4\pi} \frac{\varepsilon_0^2}{r} \left[ \cos \left( k |\vec{r} - \vec{r}_1| - \omega t \right) + \cos \left( k |\vec{r} - \vec{r}_2| - \omega t \right) \right] \hat{\theta}$$

Crucially, we cannot assume the two
cosines are equal, because these are
oscillatory functions and may be very
different even if $\vec{r}_1$ and $\vec{r}_2$ are a
fraction of a wavelength apart.

Instead, we will consider what
happens to the intensity when averaged
over time:

$$|\vec{E}_1 + \vec{E}_2|^2 = E_1^2 + E_2^2 + 2 \vec{E}_1 \cdot \vec{E}_2$$

where the cross term

$$2 \vec{E}_1 \cdot \vec{E}_2 =$$

$$2 \left( \frac{\mu_0 \rho_2}{4\pi} \frac{\varepsilon_0^2}{r} \right) \cos \left( k |\vec{r} - \vec{r}_1| - \omega t \right) \cos \left( k |\vec{r} - \vec{r}_2| - \omega t \right)$$

Apply this trig identity:
\[ 2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta) \]

This gives

\[ 2 \vec{E}_1 \cdot \vec{E}_2 = (\text{stuff}) \left[ \cos \left( k |\vec{r} - \vec{r}_1| + k |\vec{r} - \vec{r}_2| - 2 \omega t \right) \right. \]

\[ + \cos \left( k |\vec{r} - \vec{r}_1| - k |\vec{r} - \vec{r}_2| \right) \]

This term averages to zero in time.

\[ (\text{stuff}) = \frac{\mu_0 \omega^2}{4 \pi} \frac{\sin \theta}{r} \]

Furthermore,

\[ \langle \cos^2(\ldots) \rangle = \frac{1}{2}. \] Hence

\[ I = c e_0 \langle |E_1 + E_2|^2 \rangle = c e_0 \frac{\frac{\pi^3}{3} \lambda^4}{16 \pi^2} \frac{\sin^2 \theta}{r^2} \]

which is double the intensity from a single dipole. This illustrates a very general rule in physics, which is that in the absence of coherent interference one adds intensities rather than amplitudes.

Applying this to the case of \( N \) dipole scatterers, the total power radiated is

\[ P_{\text{tot}} = N \frac{e^2 \pi \omega^4}{6 \pi \varepsilon_0 c^3} \]

where \( N \) is the number of scatterers. As we have seen, \( \chi_0 \) is highly frequency dependent. But, ignoring this, we can see - generically - that higher frequency light scatters more than low frequency light.
white light \rightarrow \text{dilute gas} \rightarrow \text{red light} \rightarrow \text{blue light}

This type of scattering is called Rayleigh scattering and is why the sky is blue and the sunset is red.

**Thoughts:**

1. What if the atoms are ordered? In this case the interference terms become
   possible to get strong rays of light coherently diffracted in other directions. This is called Bragg scattering and is important for understanding scattering of x-rays from crystals. (More on this in lecture 19.)

2. If the atoms are too close together — meaning much less than a wavelength — the path length difference will be too small for constructive interference to take place in anything other than the forward direction. In this situation scattering doesn't take place and the scatterers are best thought of as a continuous medium with an index of refraction, as described in lecture 9. Generally speaking scattering can only take place when the index of refraction of a system varies in space over distances of the order of a wavelength.

**Fresnel Equations**

The simplest scattering problem — which may be solved exactly — is reflection from a dielectric interface. Here we define an interface to be an abrupt discontinuity across which the index of refraction changes.
At such a boundary, bound charge may accumulate. This is because each material comprises little dipoles, which in general will not be compensated at the interface. This charge leads to a discontinuity in the perpendicular component of $\mathbf{D}$.

Using only this fact, we can solve the reflectance problem exactly.

Incident wave:

$$v_1 = \frac{c}{n_1}$$

$$E_I = E_{0I} e^{i(k_I \cdot \hat{r} - ct)}$$

$$\mathbf{B}_I = \frac{1}{n_1^2} \hat{k}_I \times \mathbf{E}_I$$ (into the page)

Reflected wave:

$$E_R = E_{0R} e^{i(k_R \cdot \hat{r} - ct)}$$

$$\mathbf{B}_R = \frac{1}{n_1} \hat{k}_R \times \mathbf{E}_R$$

$$|k_I| = |k_R| = \frac{\omega}{n_1}$$

Transmitted wave:

$$E_T = E_{0T} e^{i(k_T \cdot \hat{r} - ct)}$$

$$\mathbf{B}_T = \frac{1}{n_2} \hat{k}_T \times \mathbf{E}_T$$

$$|k_T| = \frac{\omega}{n_2}$$

$$v_2 = \frac{c}{n_2}$$

Now we have to match the boundary conditions. Again, they are,

1) $k_1 E^I_1 - k_2 E^I_2 = 0$

$K \equiv K_E$
\[ \hat{\tau}_x = B_1 - B_2 = 0 \]
\[ E''_1 - E''_2 = 0 \]