Thin Film Interference

Suppose the rays meet at some distant point, p.

In principle this is a problem of multiple-beam interference.

Consider two of these rays. What governs the interference is the phase difference, which is determined by the optical path length together with whatever phase shifts are incurred upon reflection.

Using old-fashioned geometry notation,

\[
\Delta OPL = n_f (\overline{AB} + \overline{BC}) - n \overline{AD}
\]

\[
\overline{AB} = \overline{BC} = \frac{d}{\cos \theta_t}
\]

As for the other ray,

\[
\overline{AD} = \overline{AC} \sin \theta_c = \overline{AC} \frac{n_f \sin \theta_t}{n}
\]

\[
\overline{AC} = 2 \times d \tan \theta_c
\]

So,

\[
\Delta OPL = \frac{2n_f d}{\cos \theta_t} - \frac{2n_f d \tan \theta_t \sin \theta_t}{n}
\]
\[ \frac{2 \pi n f d}{\cos \theta_t} \left( 1 - \sin^2 \theta_t \right) \]

\[ = 2 \pi n f d \cdot \cos \theta_t \]

The phase difference between the two rays at \( P \) is

\[ \Delta \phi_{0PL} = k \Delta \phi_{PL} \quad \text{where} \quad k = \frac{\omega}{c} = \frac{2 \pi}{\lambda} \]

Of course, this is not the whole story. Whether \( n > n_f \) or \( n < n_f \), one of the two reflections will incur a \( \pi \) phase shift. Recall from Lecture 11.

\[ \Gamma_s = \frac{-\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)} \]

\[ t_s = \frac{2 \sin \theta_i \cos \theta_i}{\sin(\theta_i + \theta_t)} \]

Hence

\[ \Delta \phi = \Delta \phi_{0PL} - \pi \]

\[ = \frac{4 \pi n f d}{\lambda} \cos \theta_t - \pi \]

The angle we know is \( \theta_i \), so

\[ \Delta \phi = \frac{4 \pi n f d}{\lambda} \left( 1 - \sin^2 \theta_t \right)^{1/2} - \pi \]

\[ \Delta \phi = \frac{4 \pi n f d}{\lambda} \left[ 1 - \left( \frac{n}{n_f} \right)^2 \sin^2 \theta_i \right]^{1/2} - \pi \]

Constructive interference occurs when \( \Delta \phi \) is an integer multiple of \( 2 \pi \).

Continuing in terms of \( \theta_i \), strong reflection will occur when

\[ \frac{4 \pi n f d}{\lambda} \cos \theta_t - \pi = m \cdot 2 \pi \]
Multiple wave interference

Things get even more interesting if we include all the waves:

\[ d \cos \theta + = (2m + 1) \frac{\lambda}{4n_f} = (2m + 1) \frac{\lambda_f}{4} \]

where \( \lambda \) is the vacuum wavelength and \( \lambda_f \) is the wavelength in the film.

Similarly, minima occur when

\[ d \cos \theta - = 2m \cdot \frac{\lambda}{4n_f} = \frac{m \lambda}{2n_f} = m \frac{\lambda_f}{2} \]

**Discussion topic:** Fringes from an extended source.

**Discussion topic:** Normal incidence and “Newton’s rings”

\[ r = \text{reflectance coefficient for beam entering the film from the medium.} \]

\[ t = \text{transmission coefficient for beam entering the film from the medium.} \]

\[ t’ = \text{transmission coefficient for beam departing the film.} \]

\[ r’ = \text{reflectance for beam departing the film.} \]

\[ E_{or} = E_r + E_r’ + E_r” + ... \]
We will assume we are nowhere near the critical angle, which requires some different arithmetic. Note that the presence or absence of a phase shift is contained in the sign of $r$ or $r'$. A few observations:

1. $r = -r'$
2. $tt' = 1 - r^2$. This is easy to show applying angle-addition formulas to the above expressions.

We will start by considering the special case

$$2nfd \cdot \cos \theta = m\lambda$$

Recall that this was the condition for destructive interference in the two-beam case. That is, all the waves above are in-phase. In this case, all the waves have exactly the same phase, so the amplitude of the reflected wave is just the sum of the amplitudes of the individual waves:

$$E_{0r} = E_0 r + E_0 r't' + E_0 t r' t' + \ldots$$

This is a geometric series in $(r')^2$:

$$E_{0r} = E_0 r + E_0 t r' t' \left[ 1 + (r')^2 + (r')^4 + \ldots \right]$$

Because $|r'| < 1$, we can sum the series.

$$E_{0r} = E_0 r + E_0 t r' t' \cdot \frac{1}{1 - (r')^2}$$

$$= E_0 r - E_0 tt' \frac{r}{1 - r^2}$$

Substituting for $tt'$, we get

$$E_{0r} = 0$$
This means that, in the condition of destructive interference, the reflected ray is completely extinguished.

Suppose now we change $\delta_1$ such that

$$2nf \cos \theta_t = (m + \frac{1}{2}) \lambda$$

This was the condition of constructive interference previously.

Now we get an extra $\pi$ phase shift on each traversal through the film. Now we get

$$E_{or} = E_{or} - E_{ot} r_t \left[ 1 - (r_t')^2 + (r_t')^4 + \ldots \right]$$

$$= E_{or} - E_{ot} r_t \left( -\frac{1}{1 + (r_t')^2} \right)$$

$$= E_{or} + E_{ot} r_t \left( 1 - r_t^2 \right) \frac{1}{1 + r_t^2}$$

$$= E_{or} \left( 1 + \frac{1 - r_t^2}{1 + r_t^2} \right)$$

$$= E_{or} \left( \frac{1 + r_t^2 + 1 - r_t^2}{1 + r_t^2} \right)$$

$$= \frac{2\sqrt{r_t}}{1 + r_t^2} E_0$$

$$I = \left( \frac{2r_t}{1 + r_t^2} \right)^2 \frac{\varepsilon \nu - \frac{\nu_0^2}{2}}{E_0^2}$$

**General Case**

We now wish to consider the general case, for which neither of

$$2nf \cos \theta_t = m \lambda$$

$$2nf \cos \theta_t = (m + \frac{1}{2}) \lambda$$

is satisfied. In this case the reflected waves have travelled different optical path lengths, so are not in phase. Hence we cannot just add the amplitudes but must account for the phase differences.

To do this, we have no choice but to use complex notation, without which the problem is intractable.

For each trip through the film the wave picks up a phase shift of $\pi$, ...
where
\[ \delta = k \Delta \text{opt} \]
\[ k = \frac{2\pi}{\lambda} \]
\[ = k \cdot 2nf \cos \theta \]

Recall that in complex notation, a phase shift is imposed simply by multiplying by a factor of \( e^{i\delta} \); we simply need to include a factor of \( e^{i\delta} \) for each pass through the film. Our infinite sum becomes

\[ E_{or} = E_0r' + E_0 \sum_{n=0}^{\infty} n^3 \left( r' \right)^{2n} e^{i\delta} \]

\[ = E_0r + E_0 \sum_{n=0}^{\infty} \frac{n^3 (r')^2 e^{i\delta}}{1 - (r')^2 e^{i\delta}} \]

Again, we use the fact that (assuming \( n \) and \( r' \) are real)
\[ r = -r' \]
\[ t = 1 - r^2 \]

This gives

\[ E_{or} = E_0r - E_0 \frac{(1 - r^2) e^{i\delta}}{1 - r^2 e^{i\delta}} \]

\[ = E_0r \left[ 1 - \frac{(1 - r^2) e^{i\delta}}{1 - r^2 e^{i\delta}} \right] \]

\[ = E_0r \left[ \frac{1 - r^2 e^{i\delta} - e^{i\delta} - r^2 e^{i\delta}}{1 - r^2 e^{i\delta}} \right] \]
Using the same reasoning, the transmitted beam can be computed to be:

\[ E_{t\theta} = E_0 \frac{1 - e^{i\delta}}{1 - r^2 e^{i\delta}} \]

\[ I_r = \frac{2r^2(1 - \cos\delta)}{(1 + r^4) - 2r^2 \cos\delta} I_0 \]

\[ I_t = \frac{(1 - r^2)^2}{(1 + r^4) - 2r^2 \cos\delta} I_0 \]

Note that \( I_r + I_t = I_0 \).

\( I_r \) and \( I_t \) are periodic functions of \( \delta \), so as in the two wave case, we expect a pattern of light and dark fringes. However, the function is now not sinusoidal:

Note that this function is called an Airy Function.

Evidently, the maximum transmission occurs when \( \delta = 2\pi m \), in which case \( I_r/I_0 = 1 \) and \( I_r = 0 \).

The minimum transmission occurs when \( \delta = 2\pi (m + \frac{1}{2}) \), in which case

\[ I_{t_{\text{min}}} = \frac{(1 - r^2)^2}{(1 + r^4)} I_0 \quad I_{r_{\text{max}}} = \frac{4r^2}{(1 + r^4)} I_0 \]
The angular "sharpness" of the fringes depends on the size of $r$. The more times the beam reflects, the sharper will be the angular maxima. To describe this effect, it is customary to define a coefficient of finesse,

$$F = \left( \frac{2r}{1-r^2} \right)^2$$

In which case

$$\frac{I_r}{I_0} = \frac{F \sin^2 \delta/2}{1 + F \sin^2 \delta/2} \quad \frac{I_t}{I_0} = \frac{1}{1 + F \sin^2 \delta/2}$$

The full width at half maximum (FWHM), defined by $I_t/I_0 = 1/2$,

$$\gamma = 4 \alpha \sin^2 \frac{\gamma}{2} = \frac{4}{\sqrt{F}}$$

at large $F$.

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**Fabry-Perot Interferometer**

What we have described above is an example of a resonant cavity. At normal incidence

$$S = k \cdot 2 \pi f d \cos \theta = 2k \pi f d = \frac{4 \pi \pi f d}{\lambda}$$

So maxima in transmission occur for specific wavelengths.

$$S = \frac{4 \pi \pi f d}{\lambda} = 2 \pi m \quad \lambda_m = \frac{2 \pi f d}{m} \quad m = \text{integer}$$

Acts like a tunable wavelength filter. Quality of the filter defined by the finesse, $F$. Tunable cavity is a Fabry-Perot interferometer.

Large $F$ => "high finesse cavity".

A cavity of fixed length built to function at a pre-defined $\lambda$, is called an etalon. This is the core part of a laser.
Like a Michelson, can work in angular mode with an extended source or imaging mode with a point source and a screen.

Most common use is with a collimated beam, in spectroscopy.

Scan the cavity width. Measure the spectrum of light from the source.

1. Fluorescence analysis
2. Brillouin scattering, etc.