Your text spends a great deal of time making use of the Huygens principle, which is a geometric construct that can be used to get some insight into how light will diffract from an object. For example, if we illuminate an aperture from behind, Huygens' principle tells us that we should think of it as a superposition of many small sources ("wavelets") that emit light isotropically.

Huygens' principle dates back to the early 1600s—long before the development of wave mechanics—and at first glance appears to have no physical connection to Maxwell's equations. So it may seem odd that it is used so extensively.

In this lecture we will discuss the Kirchhoff Diffraction Theory, which shows how Huygens' principle follows from basic wave mechanics, and forms the basis for Fourier Optics.

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**Kirchhoff Diffraction Theory**

\[ \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \]

Assume the polarization doesn't change

\[ \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \quad E = \text{scalar} \]

\[ E(\vec{r},t) = E(\vec{r}) \ e^{-ict} \]

\[ \nabla^2 E(\vec{r}) + k^2 E(\vec{r}) = 0 \quad \text{Helmholtz Equation} \]

Solve with the help of Green's theorem, which states that

\[ \int_{\mathcal{V}} (u_1 \nabla^2 u_2 - u_2 \nabla^2 u_1) \, dv = \oint_{\partial \mathcal{V}} (u_1 \nabla u_2 - u_2 \nabla u_1) \cdot d\vec{s} \]
Suppose \( u_1 \) or \( u_2 \) are both solutions to Helmholtz eq. with the same eigenvalue, \( k^2 \):

\[
-u_1 \nabla^2 u_2 - u_2 \nabla^2 u_1 = -k^2 u_1 u_2 + k^2 u_1 u_2 = 0
\]

Hence

\[
\oint_S (u_1 \nabla u_2 - u_2 \nabla u_1) \cdot d\mathbf{n} = 0
\]

We will now make use of a special sol'n (see lecture #8).

\[
G(r) = \frac{e^{ikr}}{r} = u_2, \quad r = |\mathbf{r}|
\]

This solves Helmholtz everywhere except \( r = 0 \) where it is singular. This quantity is called a Greens Function. Note that it is also a spherical outgoing wave.

Apply Green's theorem to the following weird volume:

\[
\oint_S u_1 = E(s)
\]

volume that excludes the singularity at the origin.

\[ S' = \text{small sphere} \]

\[
\oint_S \left[ E \nabla \left( \frac{e^{ikr}}{r} \right) - \frac{e^{ikr}}{r} \nabla E \right] \cdot d\mathbf{n} = 0
\]

Considering the second integral:

\[
\nabla \left( \frac{e^{ikr}}{r} \right) = \left( -\frac{e^{ikr}}{r^2} + \frac{ik e^{ikr}}{r} \right) \hat{r}
\]

\[ = - \left( \frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr} \hat{r} \]
Greens theorem gives

$$\oint_{s} \left[ \mathbf{E} \cdot \left( \frac{e^{i k r}}{r} \right) - \frac{e^{i k r}}{r} \mathbf{v} \right] \cdot d\mathbf{a}$$

$$+ \oint_{s'} \left[ -\varepsilon \left( \frac{1}{r^2} - \frac{i k}{r} \right) \mathbf{E} \cdot \mathbf{r} - \frac{e^{i k r}}{r} \mathbf{v} \right] \cdot d\mathbf{a} = 0$$

$$d\mathbf{a} = -d\mathbf{a} \mathbf{r}$$

$$d\Sigma = \sin \theta d\theta d\phi$$

Now

$$\mathbf{E} \cdot d\mathbf{a} = -\frac{\partial \mathbf{E}}{\partial r} r^2 d\Sigma$$

This leaves

$$\oint_{s} \left[ \mathbf{E} \cdot \left( \frac{e^{i k r}}{r} \right) - \frac{e^{i k r}}{r} \mathbf{v} \right] \cdot d\mathbf{a}$$

$$+ \oint_{s'} \left[ E - i k r \mathbf{E} + r \frac{\partial \mathbf{E}}{\partial r} \right] e^{i k r} d\Sigma = 0$$

This should be true for any diameter of $s$ we choose. So let's take it to be small. In this case

$$i k r \approx 1$$

$$i k r \mathbf{E} \approx 0 \quad \mathbf{E} \approx \text{const}$$

$$r \frac{\partial \mathbf{E}}{\partial r} \approx 0$$

as long as $\mathbf{E}$ itself is not singular.

The result is

$$\oint_{s} \left[ \mathbf{E} \cdot \left( \frac{e^{i k r}}{r} \right) - \frac{e^{i k r}}{r} \mathbf{v} \right] \cdot d\mathbf{a} + 4 \pi i \mathbf{E} = 0$$

or

$$\mathbf{E} = \frac{1}{4 \pi} \left[ \oint_{s} \mathbf{E} \cdot \mathbf{r} d\mathbf{a} - \oint_{s} \mathbf{E} \cdot \frac{e^{i k r}}{r} d\mathbf{a} \right]$$

This expression is called the Kirchhoff integral theorem. It relates the
value of $E$ to its value integrated over a surface surrounding the location of interest. We have taken this location to be the origin, but the relationship applies everywhere.

To see why this is significant, let's apply this expression to a known case, namely a spherical outgoing wave emanating from some source at point $s$.

$$E(r, t) = \frac{A}{r} e^{i(kr' - \omega t)} \quad A = \text{scalar}$$

$$r' = |\mathbf{r}'|$$

The integral theorem assumes we have removed the harmonic time dependence, i.e.

$$E(r) = \frac{A}{r'} e^{ikr'}$$

The theorem should give us a relationship between the field at the origin and its value on the boundary of $S$.

$$E_0 = \frac{1}{4\pi} \oint_S \frac{e^{ikr}}{r} \nabla \left( \frac{A}{r'} e^{ikr'} \right) \cdot d\mathbf{a}$$

$$= -\frac{1}{4\pi} \oint_S \frac{A}{r'} e^{ikr'} \nabla \left( \frac{e^{ikr}}{r} \right) \cdot d\mathbf{a}$$

Notice that

$$\nabla \left( \frac{A}{r'} e^{ikr'} \right) \cdot d\mathbf{a} = \frac{\partial}{\partial r'} \left( \frac{A}{r'} e^{ikr'} \right) \cdot \hat{r}' \cdot \hat{n} \, d\mathbf{a}$$

rate of change in the $r'$ direction

projection onto $\hat{n}$ direction

$$\nabla \left( \frac{e^{ikr}}{r} \right) \cdot d\mathbf{a} = \frac{1}{3r^2} \left( e^{ikr} \right) \hat{r} \cdot \hat{n} \, d\mathbf{a}$$

Now let's do the derivatives:
\[ \frac{d}{dr} \left( \frac{A}{r} e^{ikr'} \right) = A e^{ikr'} \left( \frac{ik}{r'} - \frac{1}{r'} \right) \]

\[ \frac{d}{dr} \left( \frac{A}{r} e^{ikr} \right) = e^{ikr} \left( \frac{ik}{r} - \frac{1}{r^2} \right) \]

Let's now suppose we are "far" from point O, i.e., \( r' \gg \lambda \), the wavelength. For visible light, this could be a few tens of microns. In this case, we can neglect the \( 1/r' \) and \( 1/(r')^2 \) terms. The result is

\[ E_o = \frac{ikA}{4\pi} \int \frac{e^{ikr'}}{r'} \hat{r} \cdot \hat{n} \, da \]

\[ -\frac{ikA}{4\pi} \int \frac{e^{ikr}}{r} \hat{r} \cdot \hat{n} \, da \]

\[ E_o = \frac{ikA}{2\pi} \int \frac{e^{ik(r+r')}}{rr'} \left( \frac{\hat{r} \cdot \hat{n} - \hat{r}' \cdot \hat{n}}{2} \right) \, da \]

This much simpler expression - which applies specifically to the case of a spherical outgoing wave - is called the Fresnel-Kirchhoff diffraction formula. The quantity

\[ K = \frac{\hat{r} \cdot \hat{n} - \hat{r}' \cdot \hat{n}}{2} \]

is called the obliquity factor. It was originally introduced phenomenologically by Fresnel but follows from basic principles here.

So what's the point? The point is that I can think of the field at \( E_o \) as being composed of a superposition of little sources, each of which produces an \( E \) field

\[ dE_o = \frac{iA}{\lambda} K(\hat{r} \cdot \hat{r}', \hat{n}) \frac{e^{ik(r+r')}}{rr'} \, da \]

Putting in the harmonic time dependence and taking the real part

\[ dE_o = \frac{A}{\lambda m} K(\hat{r} \cdot \hat{r}', \hat{n}) \cos \left[ k(r+r') - \omega t + \pi/2 \right] \, da \]
This is a rigorous justification of the Huygens principle discussed earlier.

**Discussion: Physical meaning of $K$**

The place where this principle is most useful is in determining the diffraction from an aperture:

Instead of integrating over all of $S$, we now just integrate over the opening in the aperture.

If $r'$ is large, its value is basically constant across the aperture. So the field at $Q$ is just a superposition of spherical outgoing waves from every point on the aperture.

The obliquity factor, $K$, is zero on the left side of the aperture, assuring that we only get a diffraction pattern on the right.

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**Fraunhofer Diffraction**

We will consider diffraction of a plane wave from a single, square aperture.

\[
dE = \frac{E_0}{r} \ e^{-ikr} \ dy,
\]

where we have buried all the effects of the source location, which we have assumed is far away, inside $E_0$.

\[
r = \left[ L^2 + (y_2-y_1)^2 \right]^{1/2}
\]
If the distance \( L \) is large, i.e., \( L \gg D \), the arithmetic becomes simpler. This limit is called the Fraunhofer or Far Field limit. In this case

\[
    r = \left( L^2 + y_2^2 + y_1^2 - 2y_1y_2 \right)^{1/2}
\]

\[
    \approx \left( L + y_2^2 - 2y_1y_2 \right)^{1/2}
\]

\[
    = \left( L + y_2^2 \right)^{1/2} \left( 1 - \frac{2y_1y_2}{L^2+y_2^2} + \cdots \right) \text{ binomial expansion}
\]

Assuming \( L \gg y_2 \),

\[
    r \approx L \left( 1 - \frac{y_2^2}{L^2} y_1 \right)
\]

Hence,

\[
    dE = \frac{E_0}{L} e^{-iky_2y_1/L}
\]

Now let's do the integral

\[
    E = \frac{E_0}{L} e^{ikL} \int_{-D/2}^{D/2} dy_1 e^{-iky_2y_1/L} - D/2
\]

\[
    = \frac{E_0}{L} e^{ikL} \frac{1}{-iky_2/L} \left. \right|_{-D/2}^{D/2}
\]

\[
    = \frac{E_0}{L} e^{ikL} \left( \frac{e^{-iky_2D/2L} - e^{iky_2D/2L}}{iky_2} \right)
\]

\[
    = \frac{2E_0}{ky_2} e^{ikL} \left( \frac{e^{iky_2D/2L} - e^{-iky_2D/2L}}{2i} \right)
\]

\[
    E(y_2) = \frac{DEL}{L} e^{ikL} \frac{\sin \left( \frac{kDy_2}{2L} \right)}{\left( kDy_2/2L \right)}
\]

The quantity \( \sin x = \frac{\sin x}{x} \) is called a sinc function.
This is the (familiar?) single slit diffraction pattern.

Square aperture:

\[ I \propto \left( \frac{\sin \beta x}{\beta x} \right)^2 \left( \frac{\sin \beta y}{\beta y} \right)^2 \]