Last time we calculated the $E$ field on a screen due to diffraction of light from a single slit. It turns out that we can calculate the diffraction pattern from any, arbitrary slit by calculating its Fourier Transform (Lecture 12).

To see how let's look again at the single slit problem:

In the Fraunhofer or far field limit, we found that the $E$ field at the screen is given by:

$$E(y) = \frac{E_0}{L} \int_{-D/2}^{D/2} e^{-i k y y'} dy'$$

where we are now calling $y$ the location on the screen and $y'$ is the coordinate on the slit we are integrating over.

Let's define now a new position variable,

$$k_y = \frac{k y}{L}$$

This is basically the location on the screen in some weird units. In terms of this quantity,

$$E(y) = E(k_y) = \frac{E_0}{L} e^{-i k L} \int_{-D/2}^{D/2} e^{-i k y y'} dy'$$

we now introduce a new object called an aperture function, defined as

$$A(y') = \begin{cases} 1 & -D/2 \leq y' \leq D/2 \\ 0 & \text{otherwise} \end{cases}$$
In terms of $A$, we can now write the field like so:

$$E(k_y) = \frac{E_0}{L} e^{ik_L} \int_{-\infty}^{\infty} dy' A(y') e^{-ik_y y'}$$

Referring to Lecture 12, we recognize the integral on the RHS as the Fourier transform of $A$:

$$E(k_y) = \frac{E_0}{L} e^{ik_L} A(k_y)$$

where

$$A(k_y) = \int_{-\infty}^{\infty} dy' A(y') e^{-ik_y y'}$$

while we have shown this for a single slit, it turns out to be true for any aperture. Up to now we have been looking at 1D apertures, but it even works for apertures in two dimensions:

$$E(k_x, k_y) = \frac{E_0}{L} e^{ik_L} A(k_x, k_y)$$

where

$$A(k_x, k_y) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' A(x', y') e^{-ik_x x' -ik_y y'}$$

$$A(x', y') = \text{function describing a 2D aperture}$$

$$k_x = \frac{kx}{L}, \quad k_y = \frac{ky}{L}$$

**Superposition**

Why do we care? The reason is that we can simplify the calculation of diffraction patterns by making use of well-known properties of Fourier transforms.

For example, suppose we were diffracting light from two identical apertures displaced by $(\delta x, \delta y)$:
The aperture function for this situation is

\[ A(x, y) = A_0(x, y) + A_0(x-\Delta x, y-\Delta y) \]

where \( A_0(x, y) \) is the aperture function of one of the apertures. Computing the Fourier transform,

\[ A(k_x, k_y) = \int dx dy A(x, y) e^{-i(k_xx + k_yy)} \]

\[ = \int dx dy A_0(x, y) e^{-i(k_xx + k_yy)} \]

\[ + \int dx dy A_0(x-\Delta x, y-\Delta y) e^{-i(k_xx + k_yy)} \]

The first term is just the FT of \( A_0 \):

\[ A(k_x, k_y) = A_0(k_x, k_y) + \int dx dy A_0(x-\Delta x, y-\Delta y) e^{-i(k_xx + k_yy)} \]

To deal with the second integral, just shift the origin

\[ x \rightarrow x + \Delta x , \ y \rightarrow y + \Delta y \]

\[ A(k_x, k_y) = A_0(k_x, k_y) \]

\[ + \int dx dy A_0(x, y) e^{-i(k_xx + k_yy)} \]

\[ = A_0(k_x, k_y) + \frac{e^{-i(k_xx + k_yy)}}{\int dx dy A_0(x, y) e^{-i(k_xx + k_yy)}} \int dx dy A_0(x, y) e^{-i(k_xx + k_yy)} \]

\[ = A_0(k_x, k_y) + \frac{e^{-i(k_xx + k_yy)}}{A_0(k_x, k_y)} A_0(k_x, k_y) \]

\[ A(k_x, k_y) = A_0(k_x, k_y) \left[ 1 + \frac{e^{-i(k_xx + k_yy)}}{A_0(k_x, k_y)} \right] \]

The \( E \) field on the screen is given by

\[ E(\hat{r}) = \frac{E_0 e^{ik\hat{r}}}{\int \frac{1}{L}} \left[ A_0(\hat{r}) \left( 1 + e^{-i k \cdot \Delta \hat{r}} \right) \right] \]

where we have defined

\[ \hat{r} = (k_x, k_y) , \ \Delta \hat{r} = (\Delta x, \Delta y) \]

vector describing location on the screen.

The resulting diffraction pattern would be a sinusoidal function with periodicity \( \frac{1}{\Delta r} \), modulated by an envelope corresponding to \( A_0(k_x, k_y) \).
Evidently we can generate the diffraction pattern just by adding a suitable number of phase factors, \( \exp[-i \phi] \), where \( \phi \) is the location of the aperture.

**Example: Young's two-slit experiment**

We can use this framework to solve Young's two-slit experiment. Recall that, for a single slit, we had

\[
E(y) = \frac{D \Phi_0}{L} e^{-i \frac{kL \sin \left( \frac{kDy}{2L} \right)} \left( 1 + \frac{L}{kd} \right)}
\]

To generalize this to two apertures, we can just write

\[
E(y) = \frac{D \Phi_0}{L} e^{-i \frac{kL \sin \left( \frac{kDy}{2L} \right)} \left( 1 + \frac{L}{kd} \right)}
\]

Recall that \( ky = ky/L \), so

\[
E(y) = \frac{D \Phi_0}{L} e^{-i \frac{kL \sin \left( \frac{kDy}{2L} \right)} \left( 1 + \frac{L}{kd} \right)}
\]

Here's the result:

- \( L = 50 \text{ nm} \)
- \( D = 0.1 \text{ mm} \)
- \( b = 1 \text{ mm} \)
- \( \lambda = 633 \text{ nm} \)

\[
\begin{align*}
\text{L} & = 50 \text{ nm} \\
\text{D} & = 0.1 \text{ mm} \\
\text{b} & = 1 \text{ mm} \\
\text{\( \lambda \)} & = 633 \text{ nm} \\
\end{align*}
\]

- two finite slit diffraction pattern
- single, finite slit pattern
Periodic Arrays

Suppose we had a periodic array of such objects, where each is offset from its neighbor by $\Delta x$ and $\Delta y$:

\[
A(k_x, k_y) = A_0(k_x, k_y) \left[ \sum_{m,n} e^{-i(k_x \Delta x - i k_y \Delta y)} \right]
\]

This quantity is zero unless the exponent vanishes, in which case the sum is infinite. Hence

\[
A(k_x, k_y) = A_0(k_x, k_y) \sum_{j,k} \delta(k_x - \frac{2\pi j}{\Delta x}) \delta(k_y - \frac{2\pi k}{\Delta y})
\]

This statement is called the array theorem.

What appears on the screen is a periodic array of spots. These are called Brogg spots.

Discussion Topic: X-ray crystallography
Thoughts:
- Recalling the fundamental property of a Fourier transform: small apertures produce big diffraction patterns and big objects produce small diffraction patterns.
- We have examined the case of apertures, but scattering of physical objects can be described by appropriate aperture functions.
- Diffraction from a small shard of glass would be described by a complex $A(x,y)$ that describes the change in optical path length.
- In some cases, the aperture function of an unknown object may be determined from its diffraction pattern. This is hampered by the fact that we don't actually measure $E$ in most cases, but $I = E^2$. This problem is usually called the Phase Problem.

This phase problem is straightforward for the periodic case. This is the basis for protein crystallography, which is the basis of the field of Structural Genomics.