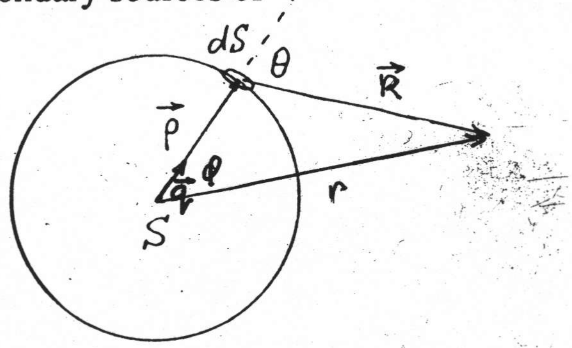
Fresnel Diffraction.

The most general diffraction conditions observe the diffraction at a point that is not remote, with the light source also not remote. The pattern is no longer the FT of the scattered amplitude. Fresnel nevertheless gives the result as an expression similar to Fraunhofer, summing over scattered waves. The formalism starts with a spherical wave from a point source:

$$E(\rho, t) = (E / \rho)\, e^{i(q\rho - \omega t)}.$$

It is postulated that all areas of a wave front act as secondary sources of wavelets like:



$$dE(\mathbf{r} + \boldsymbol{\rho}, t) = E(\rho, t)\, e^{i\pi/4} \frac{K}{r} e^{iqr}\, dS$$

$$= \frac{E\,K}{\rho\, r} e^{i\pi/4}\, e^{i[q(\rho + r) - \omega t]}\, dS$$

Here dS measures the size of the wavefront element and the 'obliquity factor' K is something new:

$$K(\theta) = \tfrac{1}{2}(1 + \cos\theta)$$

$$\begin{cases} e^{ikR}/R = ik \sum_n (2n+1) P_n(\cos\varphi)\, j_n(kr)\, h_n(k\rho), \quad * \\ R^2 = r^2 + \rho^2 - 2r\rho\cos\varphi, \\ h_n(z) \Rightarrow i^{-(n+1)} e^{iz}/z. \end{cases}$$

is inserted so that forward waves have full amplitude (K = 1, θ = 0) while backward waves are suppressed (K = 0, θ = π).

This all works surprisingly well. For example, after summing up the contributions from all the spherical surface r, one obtains the correct wave

$$E(\mathbf{r}, t) = (E / r')\, e^{i(q|\boldsymbol{\rho} + \mathbf{r}| - \omega t)}$$

$\hookrightarrow r' = |\vec{\rho} + \vec{r}|.$

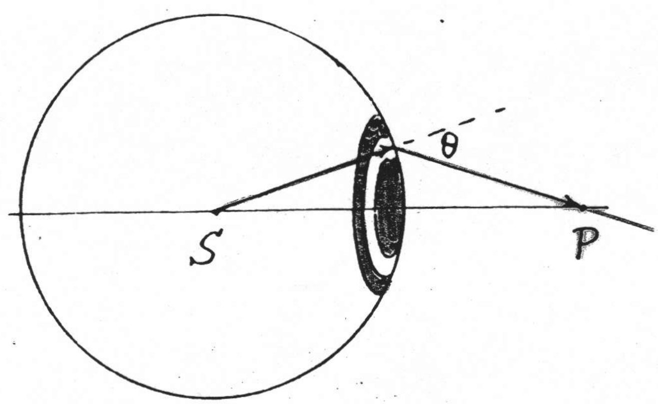
at the position $\boldsymbol{\rho} + \mathbf{r}$, which is what the wave equation say must be the case, given the wave at $\boldsymbol{\rho}$. For actual diffraction problems the sum is only over open areas of the wavefront but the calculation then takes the same form.

Correction: * $r \leftrightarrow \rho$; $(\rho > r)$

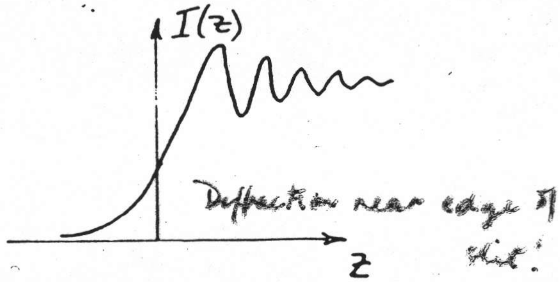
Some comments:

The calculations are tricky.

In the case of the spherical wave front, for example, the integral oscillates badly. The wave front can be broken into successive rings that cause the phase of the wave to differ by π. The successive areas contribute about equally, with only $K(\theta)$ gradually getting smaller, so the sum mostly just oscillates about an average to which it slowly converges. This turns out to be about half the contribution of the first zone.



Problems of diffraction by an infinite half plane, by apertures when the source, the screen or both are close to the diffracting obstructions, are all of the same character.



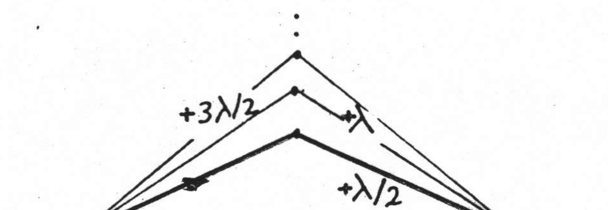
Note that the Fresnel approach includes Fraunhofer diffraction as a special case.

Fresnel zone plates.

Suppose we can make a mask that lets through only light with the phase differences $0 \to \lambda/2$; $\lambda \to 3\lambda/2$; $2\lambda \to 5\lambda/2$,... etc.

Questions: where are the boundaries?
what are the resulting properties?

The conditions of this situation are that the light comes from a point source, distance s_0 before the mask, and the image is viewed a distance s_i after the mask.



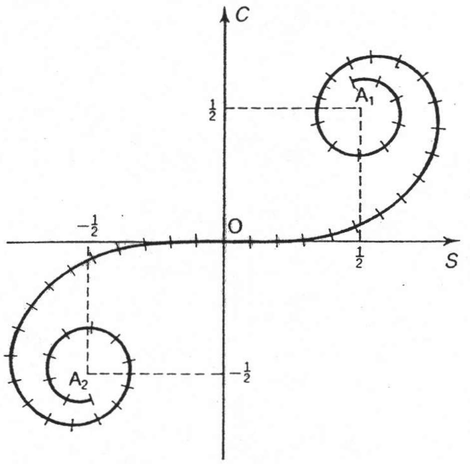
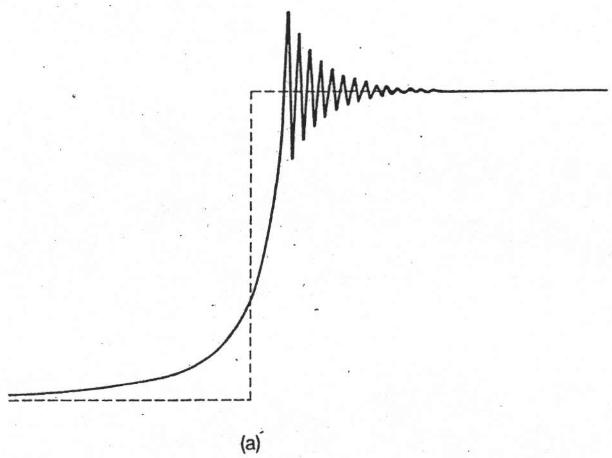


Figure 6.21 Cornu's spiral



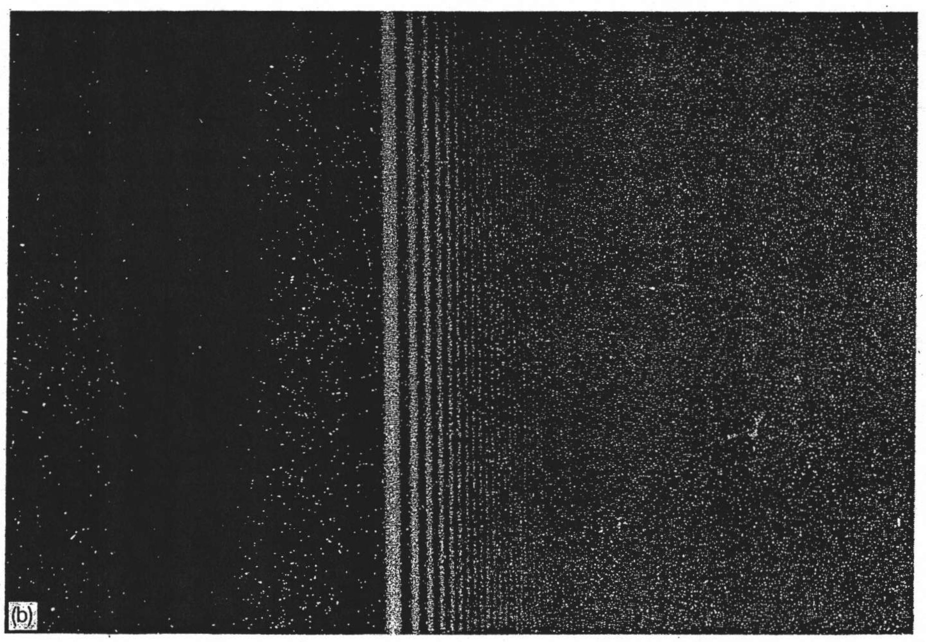


Figure 6.22 Intensity distribution near the shadow edge: (a) calculated, (b) experimental

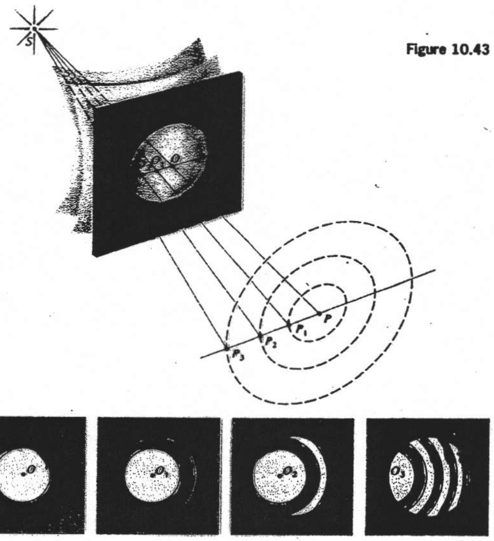
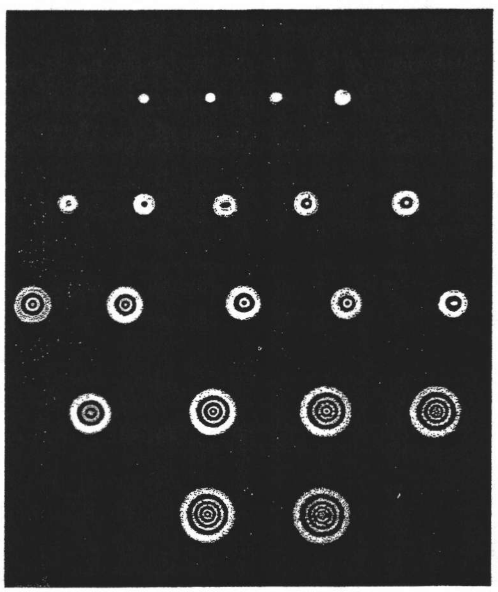


Figure 10.43 Zones in a circular aperture.

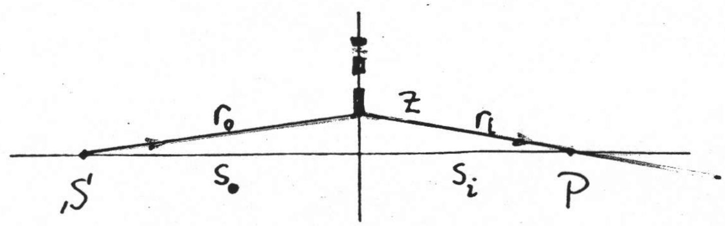
cf Babinet
central spot



Diffraction patterns for circular apertures of increasing size.
(Francis Weston Sears, *Optics*, ©1949, Addison-Wesley Reading, MA.

Geometry of zone plate:

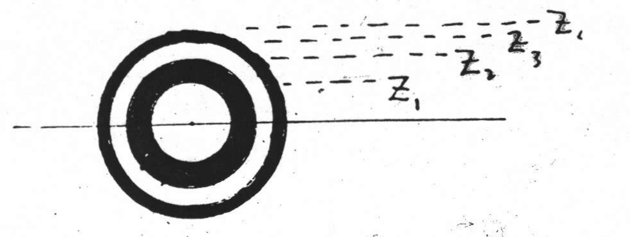
Given Pythagoras' theorem:



$$r^2 = x^2 + z^2,$$

then $\quad r_o^2 = s_o^2 + z^2\,; \qquad r_o \approx s_o + \dfrac{z^2}{2s_o}\,;$

similarly $\qquad r_i \approx s_i + \dfrac{z^2}{2s_i}\,.$



We want to make

$$s_o - r_0 + (s_i - r_i) = m\lambda/2;$$

so that

$$\frac{z^2}{2s_o} + \frac{z^2}{2s_i} = \frac{m\lambda}{2}.$$

This will be true for values z_m of z such that

$$\frac{1}{s_o} + \frac{1}{s_i} = \pm\frac{m\lambda}{z_m^2} = \frac{1}{f},$$

with the choice

$$z_m = \sqrt{m\lambda f}.$$

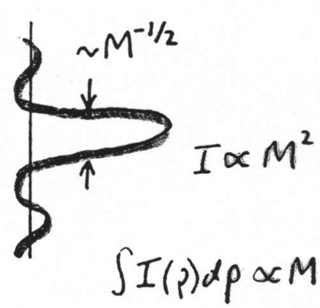
For this value of the summed reciprocal distances that satisfy the thin lens equation, successive open zones on the zone plate supply amplitude components that add in phase. Note also that successive zones have equal areas since, by definition of f,

$$\pi z_m^2 - \pi z_{m-1}^2 = \pi\lambda f. \qquad (\text{period } 2\lambda f)^*$$

What is the diffraction pattern?

M half zones each contribute u to the intensity at the focus so

$$I(0) \sim |Mu|^2 = M^2u^2.$$



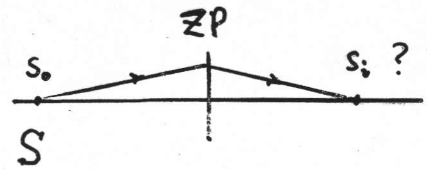
The maximum transverse wavevector is $\kappa \sim q\sqrt{M\lambda f}/f$.
so the spot size is $\sim \kappa^{-1} \sim (2\pi)\sqrt{\lambda f/M}$. This is Airy disk!

$(I/\kappa^2 \sim M$ because M zones)

Broader view of zone plate action.

If the phase from radius ρ is δ(ρ), and the transmission f(ρ), then the sum is:

$$E \sim \int_{ap} f(\rho)\, e^{i\delta(\rho)}\, 2\pi\rho\, d\rho.$$



Now we use $\Delta d = s_o - r_o + (s_i - r_i) = \dfrac{\rho^2}{2s_o} + \dfrac{\rho^2}{2s_i}$,

to write the phase as

$$\delta = \frac{2\pi\Delta d}{\lambda} = \frac{2\pi}{\lambda}\left[\frac{\rho^2}{2s_o} + \frac{\rho^2}{2s_i}\right],$$

and hence

$$E \sim \int_{ap} f(\rho)\, e^{iq\rho^2/2f}\, 2\pi\rho\, d\rho \qquad \text{with} \qquad \frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i}.$$

Change variables to $x = \rho^2$ so that $dx = 2\rho\, d\rho$ and

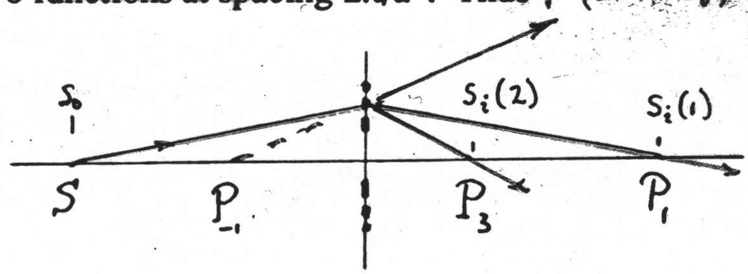
$$E \sim \int_{-\infty}^{\infty} f(x)\, e^{i\kappa x}\, \pi dx \qquad \text{with} \qquad \kappa = q/2f.$$

Now f(x) (of ZP) is periodic in $x = \rho^2$ with period $a^2 = 2\lambda/(s_o^{-1} + s_i^{-1})$: ($= 2\lambda f$ see p117)

$$f(x) = \Sigma_p A_p\, e^{i2\pi px/a^2}. \quad \text{(most general)}$$

Since E is the FT of f(x), κ is a set of δ functions at spacing $2\pi/a^2$. Thus f (and s_i) imaging occurs where

$$\kappa_p = \frac{2\pi}{2\lambda f_p} = \frac{2\pi p}{a^2},$$

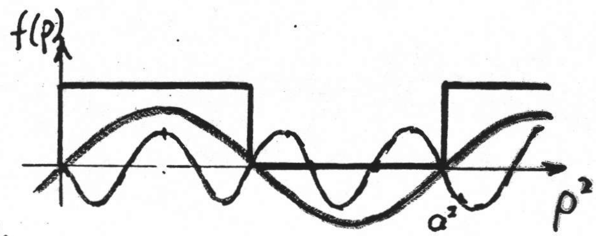


or

$$\frac{1}{f_p} = p\left[\frac{1}{s_o} + \frac{1}{s_i}\right].$$

$= p\cdot\frac{1}{f_1}$) ? order action

This shows that the zone plate has foci at ± f/p. They arise from the Fourier components of the zone plate transmission function. The zone plate acts like a lens but with strong chromatic aberration.



Holographic Methods: 1 Introductory conmments

Why is it hard to make a picture (or a movie) that look absolutely real? Two reasons are binocular vision and parallax.

Binocular vision: Two eyes give 'depth'. We can tell things are nearer because our two eyes see different things from their different positions.

Parallax: Motion makes near and far objects move differently.

Conclude: The wave field contains information about where the light comes from that only appears when an object is viewed from different places.

Holography tries to reconstruct enough of the wave field to make the view from (slightly) different positions realistic. There are some relevant points to be made:

- 1. An image made by an optical system retains the information about 3-D spatial structure, although in generally distorted form ($M_t \neq M_l$).
- 2. How will we store the information about the light field? A photograph responds at best to $|E|^2$, and all phase information about $E = E\, e^{i\phi}$ is lost.
- 3. Suppose we wish to correct errors in the image $f'(x)$ of an object $f(x)$ introduced by aberrations, presumed to arise as a convolution:

$$f'(x) = \int_{-\infty}^{\infty} f(x_1)\; g(x - x_1)\, dx_1,$$

with $g(x)$ the apparatus response given a δ-function input. Then

$$\tilde{f}'(q) = \tilde{f}(q)\, \tilde{g}(q),$$

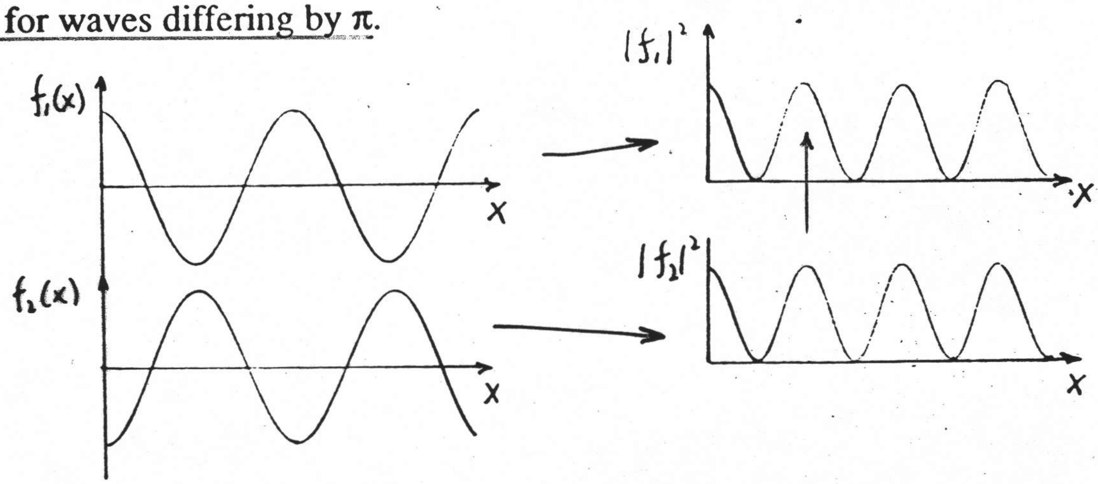
and the exact image can be recovered by inserting a screen with transmission $\tilde{g}^{-1}(q)$ in the back focal plane of the lens: $\tilde{f}(q) = \tilde{f}'(q)/\tilde{g}(q)$.

This does not work in practice. $\tilde{g}^{-1}$ is generally complex and difficult to simulate by a screen because it has zeros, for example.

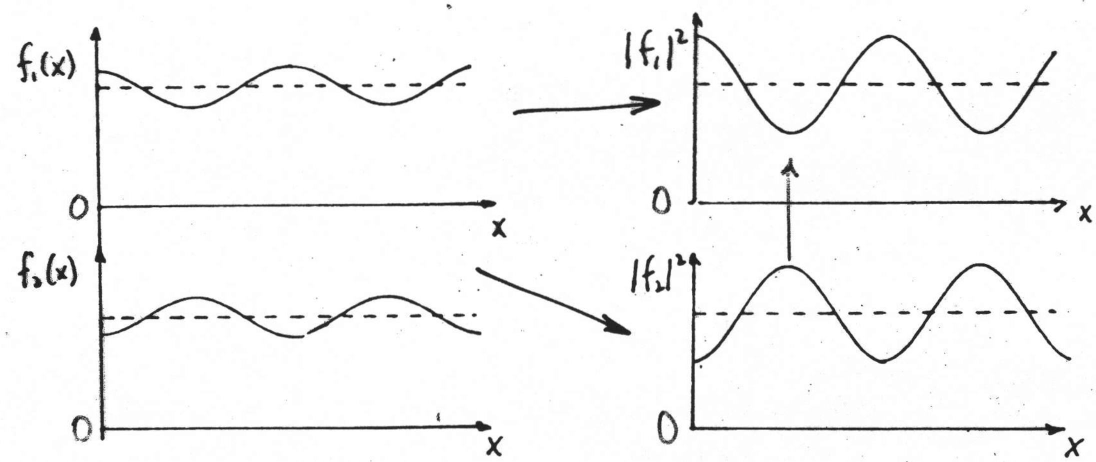
Holographic methods 2: nature of the solution (Gabor).

A major part of the answer to the problem is the ideas that the (weak) diffracted information in each Fourier component be allowed to interfere with the (strong) main beam; this restores the information about the phase of the wave fields. For example:

$|E|^2$ for waves differing by π.

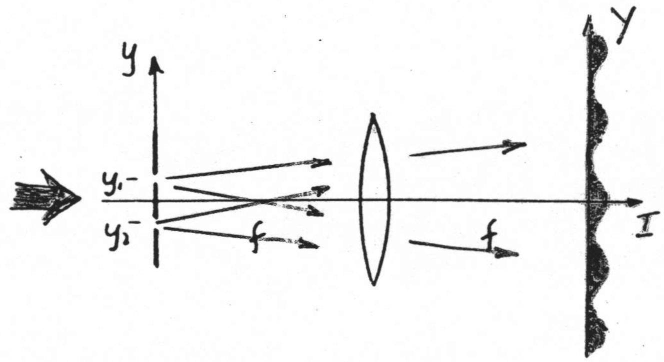


$|E|^2$ for waves differing by π and interfering with the main beam.



In the second case the phase of the weak beam remains apparent from the character even of the mod square of the summed diffracted and main beams. To retain phase information even in photographic plates it is necessary to take a photograph of the diffracted beam beating against the main beam. This is the essence of Gabor's idea.

1-D example: reconstruction of double slit.



Take a screen S with two narrow slits at distance f from a lens, and illuminate them from behind. Place a second screen S' at f behind the lens.

What appears on the second screen? Answer: it is the Fourier transform of the two slit transmission.

The two slits are δ-functions at y_1 and y_2, namely $\delta(y - y_1)$ and $\delta(y - y_2)$. Then

$$f(y) = \delta(y - y_1) + \delta(y - y_2).$$

Also,

$$\tilde{f}(\kappa) = \int_{-\infty}^{\infty} [\delta(y - y_1) + \delta(y - y_2)\}\, e^{-i\kappa y}\, dy,$$

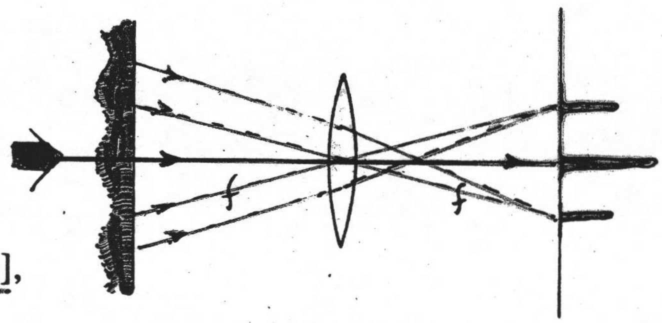
$$= \quad e^{-i\kappa y_1} + e^{-i\kappa y_2}.$$

$$= e^{-i\kappa(y_1+y_2)/2}\left(e^{-i\kappa(y_1-y_2)/2} + e^{i\kappa(y_1-y_2)/2} \right),$$

or,

$$\tilde{f}(\kappa) = 2\, e^{-i\kappa(y_1+y_2)/2} \cos \kappa(y_1 - y_2)/2 \qquad \kappa = q \cdot Y/f$$

A photograph gives $|\tilde{f}(Y)|^2 = 4\cos^2 [q\,(y_1 - y_2)Y/2f]$ --Young's fringes (instead of E(Y))



Now use the photograph to make a diffraction pattern. With $\cos^2\theta = (1+\cos 2\theta)/2$ we have

$$f'(y) = a + b\ \cos [q\,(y_1 - y_2)Y/f],$$

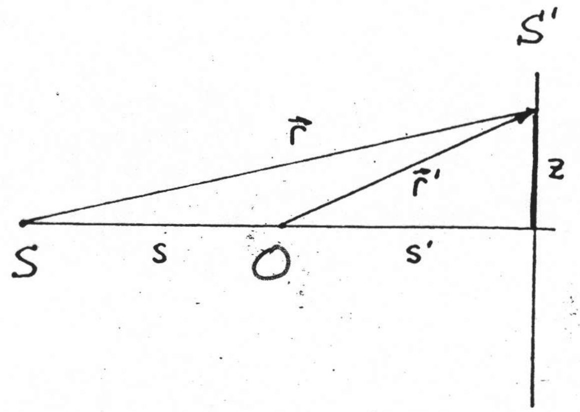
$$\tilde{f}'(\kappa) = a'\,\delta(\kappa) + b'\,[\delta(\kappa - \kappa_1) + \delta(\kappa + \kappa_1)],$$

with $\kappa_1 = q(y_1 - y_2)/f$, or lines at $Y = \pm(y_1 - y_2)$. The result is that the original two-slit transmission is reproduces from the photograph, but with twice the spacing of the two slits. This gives an example of how the diffraction process can be employed to reconstruct a faithful image of a transmission pattern. For example, more slits with selected fs can be added.

Holography with a point source.

We need to beat the scattered wave from the object against the main beam in order to preserve all information including phase information. The geometry shown accomplishes this for a weakly scattering point object and a point source S of light. (We can later add other object points with varying strengths, illuminated by the same beam, to represent the scattering from an entire object).

The path difference between the direct ray from S along r, and the scattered wave along s and r' is



$$d = r - r' - s,$$

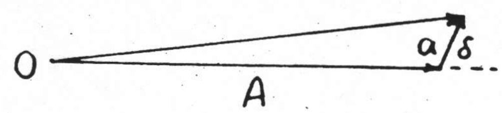
$$= s + s' + \frac{z^2}{2(s + s')} - s - s' - \frac{z^2}{2s'},$$

or

$$d = \frac{z^2}{2}\left[\frac{1}{s + s'} - \frac{1}{s'}\right].$$

Now take the form of the superposed waves at the screen S' to have a large term A from the direct beam and a small part a exp $i\delta$ from the scattered wave. Then

$$I = A^2 + a^2 + 2Aa \cos \delta,$$



or,

$$I \approx (A^2 + a^2)\left[1 + \frac{2Aa \cos \delta}{A^2 + a^2}\right].$$

Now using $\delta = qd$ we have

$$I \approx (A^2 + a^2)\left[1 + \frac{2a}{A} \cos \frac{qz^2}{2}\left[\frac{1}{s + s'} - \frac{1}{s'}\right]\right].$$

This is the darkening of a photographic negative located in the position of the screen S' and subsequently exposed to the two beams.

Comments on the exposure distribution:

1. There is a transmitted wave which is generally strong and a diffracted signal which is weak.

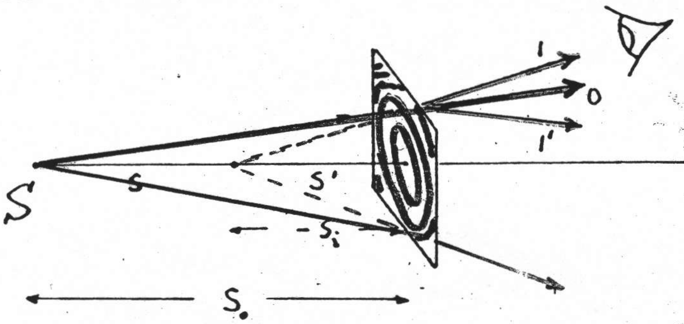
2. The portion from diffraction is a cosine zone plate which when employed to reconstruct the wave field gives just two diffracted beams, added to the direct beam.

3. The zone plate may be characterized on writing

$$s_o = s + s';$$

$$s_i = -s',$$

and
$$\frac{1}{f} = \frac{1}{s_o} + \frac{1}{s_i}.$$



4. The multiples of 2π in the phase difference from the zone plate apertures occur when $qd = \pm 2m\pi$:

$$\frac{qz_m^2}{2}\left[\frac{1}{s+s'} - \frac{1}{s'}\right] = \frac{qz_m^2}{2f} = 2m\pi,$$

so that with $q = 2\pi/\lambda$, the bright rings have exposure proportional to the scattering amplitude a and occur at radii given by

$$z_m^2 = 2m\lambda f,$$

5. The resulting diffracted wave has an amplitude proportional to a (ie, to the modulation), just like that from the original point scatterer. It emerges from the virtual object point s_i that satisfies the thin lens equation for this f. But this is precisely the point where the scattering object was formerly located. Thus the photograph reproduces the wave field containing the main beam plus the wave of amplitude a apparently emanating from the position the point source occupied when the photograph was taken.

Diffraction calculation.

We can redo the zone plate calculation for an on-axis point with the specific transmission function f(ρ) of the hologram: (light at S')

$$I \approx (A^2 + a^2)\left[1 + \frac{2a}{A}\cos\frac{qz^2}{2f}\right].$$

gives a transmission

$$f(\rho) = b + ca\cos\frac{q\rho^2}{2f}.$$

with b and c constants and f the value deduced above. The phase difference between the axial path from the point at s_o to the point s_i and the path through radius ρ is, as before,

$$\delta = qd = q\frac{\rho^2}{2}\left[\frac{1}{s+s'} - \frac{1}{s'}\right].$$

so the amplitudes at the observation point sum to

$$f(s') = \int_{-\infty}^{\infty} f(\rho)\, e^{i\delta(\rho)}\, 2\pi\rho\, d\rho,$$

$$= \int_{-\infty}^{\infty}\left(b + ca\cos\frac{q\rho^2}{2f}\right)\exp\left\{iq\frac{\rho^2}{2}\left[\frac{1}{s+s'} - \frac{1}{s'}\right]\right\}2\pi\rho\, d\rho$$

$$= 2\pi\int_{-\infty}^{\infty}\left(b + ca\cos\frac{qx}{2f}\right)\exp\left\{iq\frac{x}{2}\left[\frac{1}{s+s'} - \frac{1}{s'}\right]\right\}dx$$

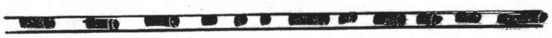
The result of the integration gives from the scattered wave a result

$$f(s_i') = ca\,\delta\left(\pm\frac{1}{2f} - \frac{1}{2(s+s')} + \frac{1}{2s'}\right) = ca\,\delta\left(\pm\frac{1}{f} - \frac{1}{s_o} - \frac{1}{s_i}\right).$$

This says that there is a bright beam comes out apparently from the point s_i that the object formerly occupied. One further point is that the phase factor works out the same for off-axis points. We can obtain an apparent 3-D source by the superposition of many such weak object points.

Some practical matters.

Transmission holograms are 2D in structure, much thinner than the interference fringes.



Reflection holograms are thick and have fringes perpendicular to the light path: the diffraction is then Bragg reflection.

A practical set-up.