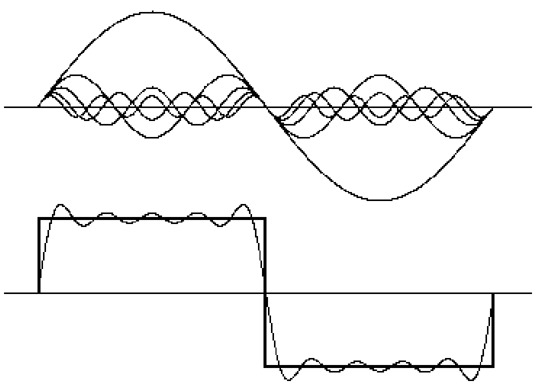
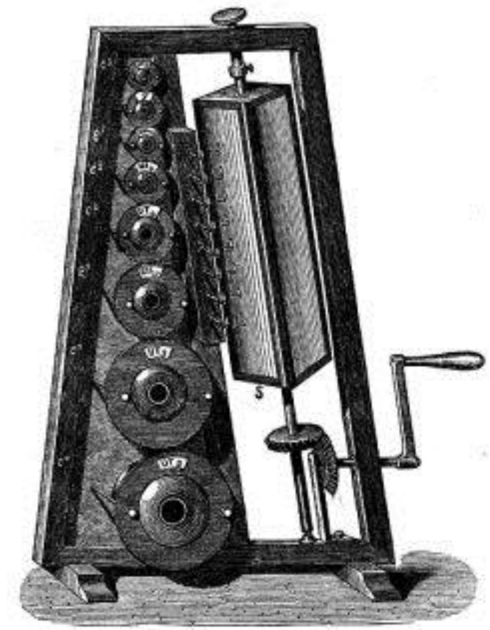
"Fourier's theorem is not only one of the most beautiful results of modern analysis, but it is said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics…

Fourier is a mathematical poem." **Lord Kelvin**

(March 21, 1768 – May 16, 1830)





$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}]$$

$$a_0 = \frac{2}{T}\int_0^T f(t)dt \qquad a_n = \frac{2}{T}\int_0^T f(t)\cos(\frac{2n\pi t}{T})dt \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin(\frac{2n\pi t}{T})dt$$

http://www.falstad.com/mathphysics.html

The universe stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth. **Joseph Fourier**

Every child is an artist. The problem is how to remain an artist after he(she) grows up. **Pablo Picasso**

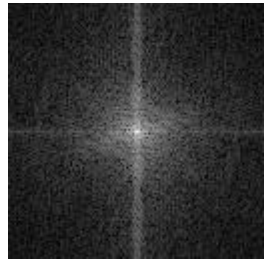
Mathematics is the abstract key which turns the lock of the physical universe. **- John Polkinghorne**

Figures don't lie, but liars figure. **- Mark Twain**

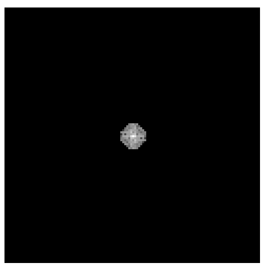
The calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more unequivocally than anything else the inception of modern mathematics; and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.
- John von Neumann

Image Processing using Fourier Transforms

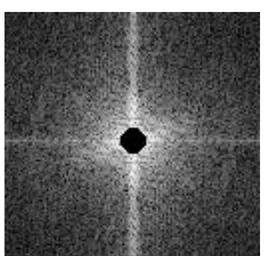
Brightness Image **Fourier Transform** **Inverse Transformed**



Low-Pass Filtered **Inverse Transformed**



High-Pass Filtered **Inverse Transformed**



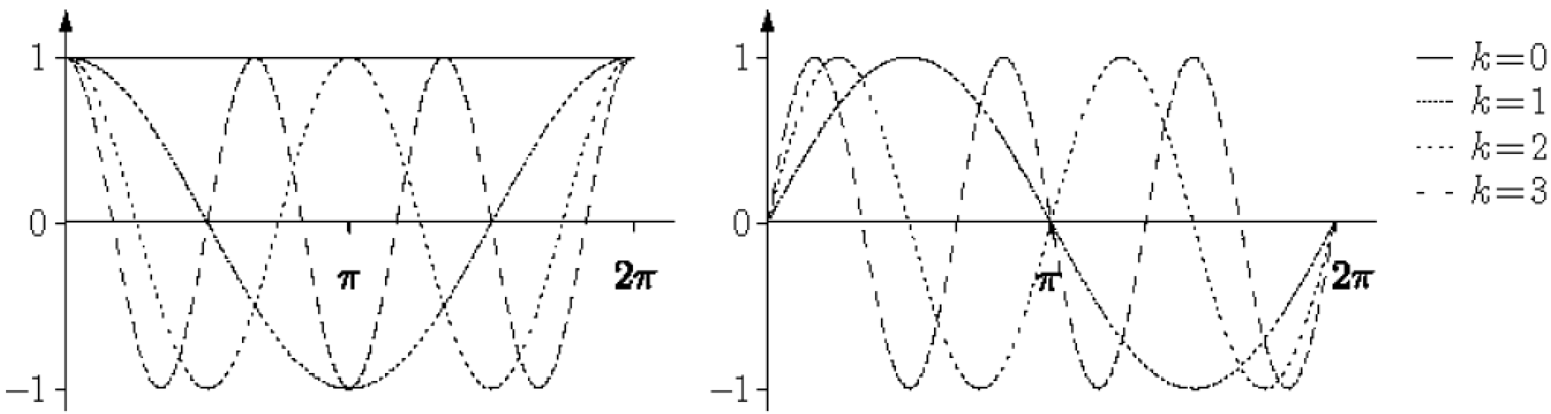
http://sharp.bu.edu/~slehar/fourier/fourier.html

Reminder/Review of Fourier Analysis (one dimension - repetitive)

Fourier series

$$f(t) = \sum_{k=0}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t)$$

$$\text{with } \omega_k = \frac{2\pi k}{T} \text{ and } B_0 = 0 \qquad \omega = 0, \frac{2\pi}{T}, \frac{4\pi}{T}, \frac{6\pi}{T}, \ldots$$



Basis functions of Fourier transformation: cosine (*left); sine (right)*

Orthogonality of the basis functions

$$\int_{-T/2}^{+T/2} \cos\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}\mathrm{d}t = \begin{cases} 0 & \text{for } n \neq m \\ T/2 & \text{for } n = m \neq 0 \\ T & \text{for } n = m = 0 \end{cases}$$

$$\int_{-T/2}^{+T/2} \sin\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}\mathrm{d}t = \begin{cases} 0 & \text{for } \begin{array}{l} n \neq m,\ n = 0 \\ \text{and/or } m = 0 \end{array} \\ T/2 & \text{for } n = m \neq 0 \end{cases}$$

$$\int_{-T/2}^{+T/2} \cos\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}\mathrm{d}t = 0\ .$$

Repetitive functions (with a fundamental period from –T/2 to +T/2)

Calculating the coefficients

$$f(t) = \sum_{k=0}^{\infty} (A_k \cos \omega_k t + B_k \sin \omega_k t)$$

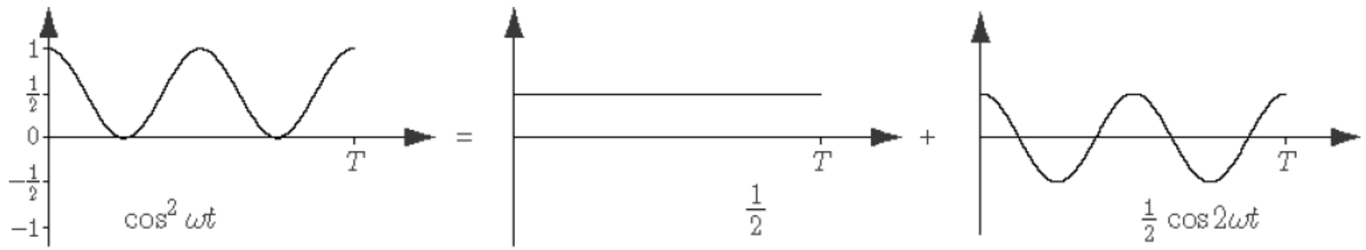
$$A_0 = \frac{1}{T} \int^{+T/2} f(t)\,\mathrm{d}t$$

$$A_k = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \cos \omega_k t\,\mathrm{d}t \qquad \text{for} \qquad k \neq 0$$

$$B_k = \frac{2}{T} \int_{-T/2}^{+T/2} f(t) \sin \omega_k t\,\mathrm{d}t \qquad \text{for all } k$$

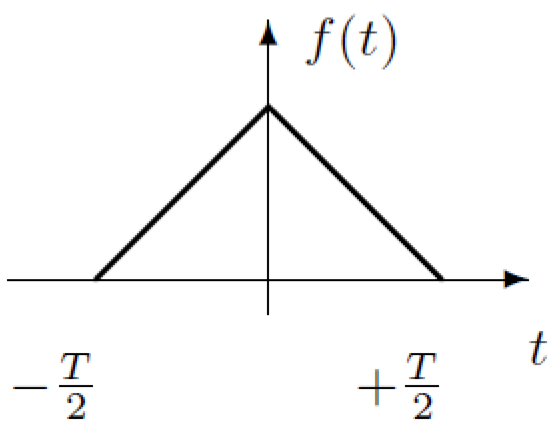
For instance, a Trigonometric identity is a Fourier expansion

$$f(t) = \cos^2 \omega t = \frac{1}{2} + \frac{1}{2}\cos 2\omega t$$



Second example (a triangular repetitive function)

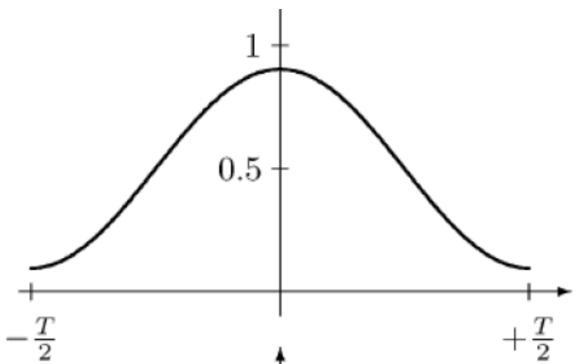
$$f(t) = \begin{cases} 1 + \dfrac{2t}{T} \text{ for } -T/2 \le t \le 0 \\ 1 - \dfrac{2t}{T} \text{ for } 0 \le t \le +T/2 \end{cases}$$



Second example (a triangular repetitive function)

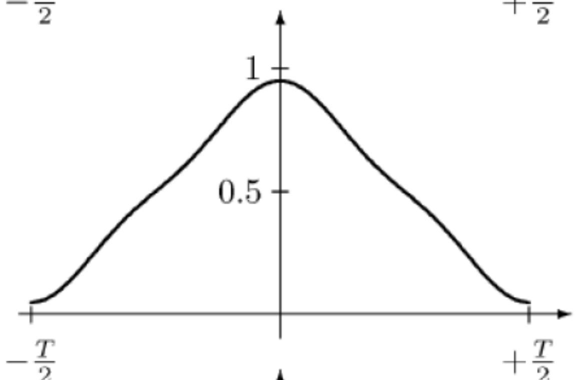
$$f(t) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \ldots\right)$$

$$A_k = \begin{cases} \frac{1}{2} & \text{for } k = 0 \\ \frac{4}{\pi^2 k^2} & \text{for k odd} \\ 0 & \text{for k even, } k \neq 0 \end{cases}$$



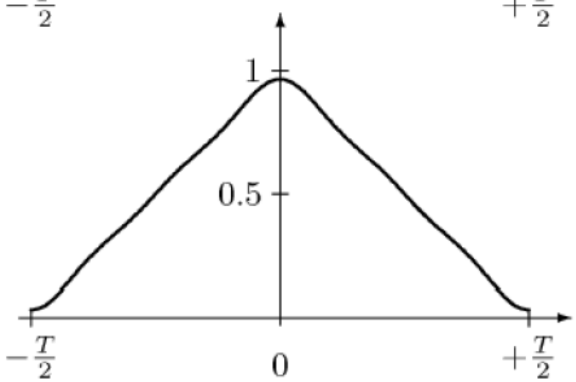
1st approximation:

$$\frac{1}{2} + \frac{4}{\pi^2}\cos\omega t$$



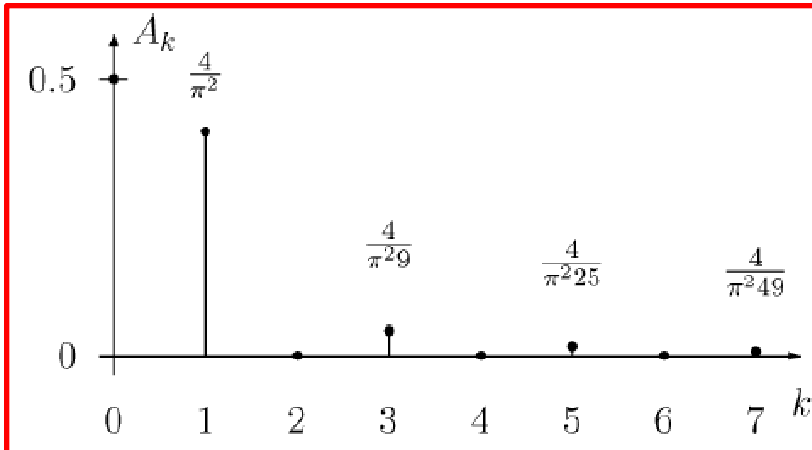
2nd approximation:

$$\frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega t + \frac{1}{9}\cos 3\omega t\right)$$



3rd approximation:

$$\frac{1}{2} + \frac{4}{\pi^2}\left(\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t\right)$$



Complex notation

$$f(t) = A_0 + \sum_{k=1}^{\infty} \left(\frac{A_k - \mathrm{i}B_k}{2} \mathrm{e}^{\mathrm{i}\omega_k t} + \frac{A_k + \mathrm{i}B_k}{2} \mathrm{e}^{-\mathrm{i}\omega_k t} \right)$$

$$C_0 = A_0,$$

$$C_k = \frac{A_k - \mathrm{i}B_k}{2},$$

$$C_{-k} = \frac{A_k + \mathrm{i}B_k}{2}, \qquad k = 1, 2, 3, \ldots,$$

$$\mathrm{e}^{\mathrm{i}\alpha t} = \cos \alpha t + \mathrm{i} \sin \alpha t$$

$$\cos \alpha t = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\alpha t} + \mathrm{e}^{-\mathrm{i}\alpha t}),$$

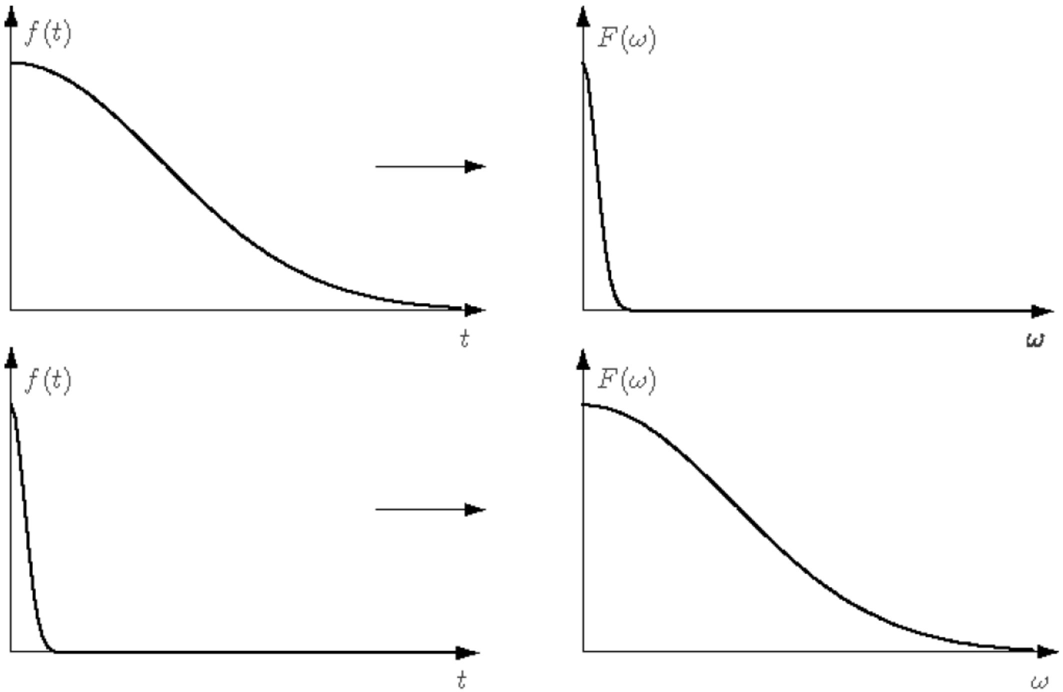
$$\sin \alpha t = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\alpha t} - \mathrm{e}^{-\mathrm{i}\alpha t})$$

$$f(t) = \sum_{k=-\infty}^{+\infty} C_k \mathrm{e}^{\mathrm{i}\omega_k t}, \qquad \omega_k = \frac{2\pi k}{T}$$

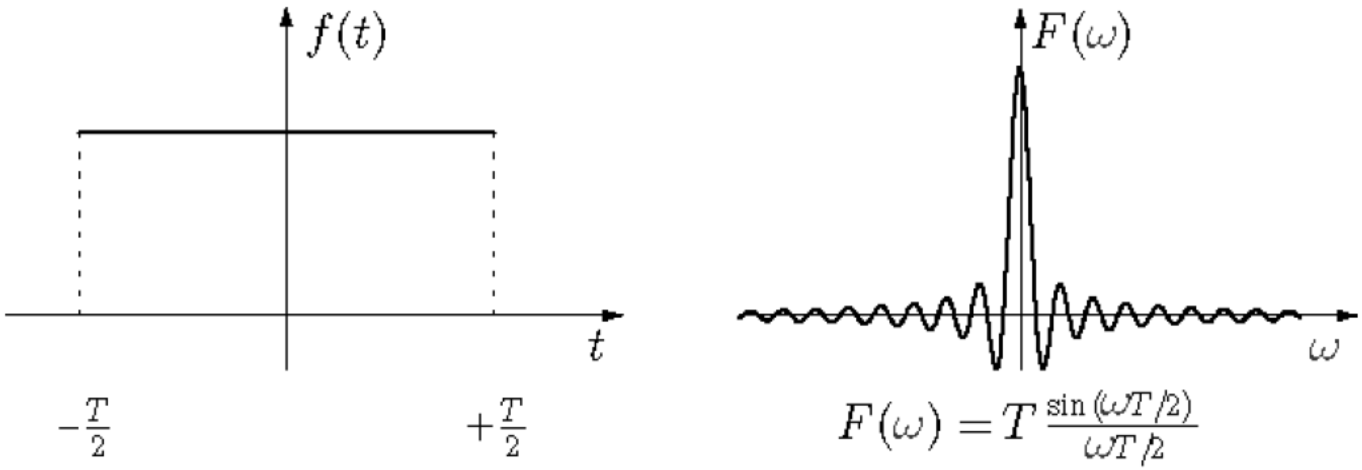
$$C_k = \frac{1}{T} \int_{-T/2}^{+T/2} f(t) \mathrm{e}^{-\mathrm{i}\omega_k t} \mathrm{d}t \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Fourier Transforms (one dimension; not repetitive)

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{+\mathrm{i}\omega t}\mathrm{d}\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$



A slowly-varying function has only low-frequency spectral components (*top*); *a rapidly-falling function has spectral components spanning a wide range of* frequencies (*bottom*)

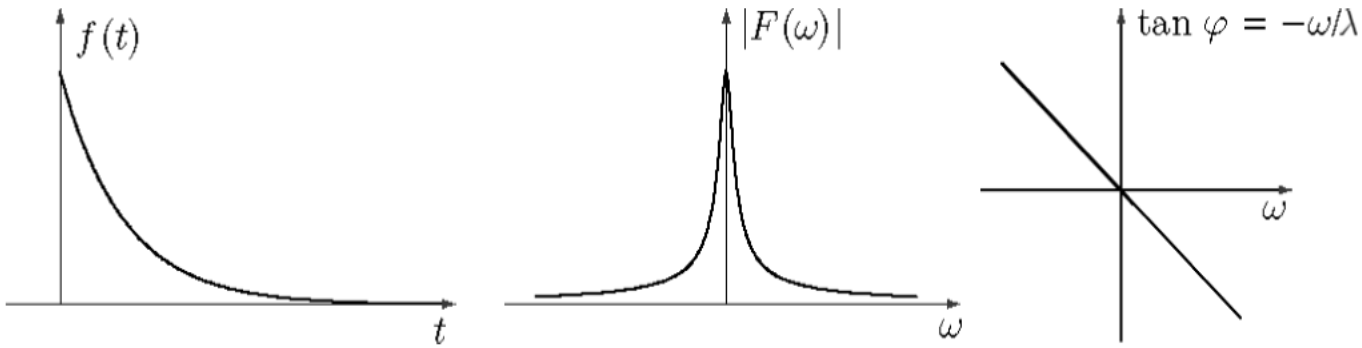


“Rectangular function” and Fourier transformation

Reminiscent of a point spread function of light focused after passing through a circular aperture.

Polar Representation of the Fourier Transform

$$z = a + \mathrm{i}b = \sqrt{a^2 + b^2}\, \mathrm{e}^{\mathrm{i}\varphi} \quad \text{with} \quad \tan\varphi = b/a$$



$$|F(\omega)|^2 = 1/(\lambda^2 + \omega^2)$$

power representation: $|F(\omega)|^2 = (\text{real part})^2 + (\text{imaginary part})^2$

Convolution

$$f(t) \otimes g(t) \equiv \int_{-\infty}^{+\infty} f(\xi) g(t-\xi) \mathrm{d}\xi$$

commutative : $\quad f(t) \otimes g(t) = g(t) \otimes f(t)$

$$f(t) \otimes g(t) = \int_{-\infty}^{+\infty} f(\xi) g(t-\xi) \mathrm{d}\xi = \int_{-\infty}^{+\infty} g(\xi') f(t-\xi') \mathrm{d}\xi'$$

Convolution Theorem

$$H(\omega) = \int \int f(\xi) g(t-\xi) \mathrm{d}\xi \, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \qquad = \int f(\xi) \mathrm{e}^{-\mathrm{i}\omega\xi} \mathrm{d}\xi \, G(\omega)$$

$$= \int f(\xi) \mathrm{e}^{-\mathrm{i}\omega\xi} \left[\int g(t-\xi) \mathrm{e}^{-\mathrm{i}\omega(t-\xi)} \mathrm{d}t \right] \mathrm{d}\xi \qquad = F(\omega) \, G(\omega)$$

$\uparrow$ expanded $\uparrow$