Calculation of Inharmonicities in Musical Instruments

As we have discussed in the POM lectures, ***real*** 1-dimensional musical instruments do ***not*** have a ***perfect/ideal*** harmonic sequence of overtones – *i.e.* $f_n^{thy} = n \cdot f_1^{thy}, \quad n = 1, 2, 3, \ldots$, in contrast to the predictions of the ***perfect/ideal*** "simple" first-order theory of how the instrument works – such first-order theories ***neglect/ignore*** ***higher-order*** ***real-world*** ***effects*** such as finite string stiffness, viscous damping/dissipative effects that do indeed change/shift/perturb the resonant frequencies from their "ideal" "simple" first-order theory predictions.

For any given 1-dimensional musical instrument, if we ***measure*** the frequencies of the ***individual*** ***harmonics*** of the instrument f_n^{expt}, we can compare them with the ***perfect/ideal*** "simple" first-order theory prediction $f_n^{thy} = n \cdot f_1^{thy}, \quad n = 1, 2, 3, \ldots$ of the ***harmonic sequence of overtones*** to see how close to ***perfect/ideal*** the musical instrument actually is. In general, the experimental results will be ***close*** to, but not ***precisely*** ***identical*** with the simple ***perfect***/***ideal*** theory predictions. The ***deviation*** of each harmonic f_n^{expt} from the ***perfect***/***ideal*** prediction $f_n^{thy} = n \cdot f_1^{thy}, \quad n = 1, 2, 3, \ldots$ is a measure of the ***inharmonicity*** of the musical instrument.

I.) Stringed Instrument Inharmonicities:

For ***stringed*** instruments, the ***fundamental*** frequency f_1^{expt} of an ***open*** vibrating string is ***not significantly*** perturbed from its ***perfect/ideal*** theory value of $f_1^{thy} = v/\lambda_1 = 2\sqrt{T/\mu}/L$, thus we can use it as a ***reference*** for the higher harmonics $f_n^{thy} = n \cdot f_1^{thy}, \quad n = 1, 2, 3, \ldots$.

We can calculate the ***frequency*** ***ratios*** $R_n^{expt} \equiv \left(f_n^{expt} / f_1^{expt} \right) = n_{expt} \quad (\neq n = 1, 2, 3\ldots)$ and compare these results to the "simple" ***perfect***/***ideal*** theory predictions $R_n^{thy} \equiv \left(f_n^{thy} / f_1^{thy} \right) = n, \quad n = 1, 2, 3, \ldots$

The ***measured*** frequency ratios $R_n^{expt} \equiv \left(f_n^{expt} / f_1^{expt} \right) = n_{expt}$ should be ***close*** to, but will ***not*** be ***precisely*** ***equal*** to $n, \quad n = 1, 2, 3, \ldots$ for stringed instruments, *e.g.* due to finite string ***stiffness*** and, to a lesser extent, due to the effect(s) of viscous damping/dissipation effects of the surrounding air in proximity of the vibrating string.

The ***deviation*** of a given ***measured*** frequency ratio $R_n^{expt} \equiv \left(f_n^{expt} / f_1^{expt} \right)$ from its ***perfect/ideal*** theory value $R_n^{thy} \equiv \left(f_n^{thy} / f_1^{thy} \right) = n, \quad n = 1, 2, 3, \ldots$ is given by the ***experiment*** *vs*. ***theory difference*** between these two quantities:

$$\Delta R_n^{expt-thy} \equiv R_n^{expt} - R_n^{thy} = \left(\frac{f_n^{expt}}{f_1^{expt}} \right) - \left(\frac{f_n^{thy}}{f_1^{thy}} \right) = n_{expt} - n$$

If we then ***normalize*** the above expression to the theory ratio $R_n^{thy} \equiv \left(f_n^{thy} / f_1^{thy} \right) = n, \quad n = 1, 2, 3, \ldots$ then we obtain the ***fractional*** ***deviation*** of a given ***measured*** frequency ratio $R_n^{expt} \equiv \left(f_n^{expt} / f_1^{expt} \right)$ from its ***perfect/ideal*** theory value $R_n^{thy} \equiv \left(f_n^{thy} / f_1^{thy} \right) = n, \quad n = 1, 2, 3, \ldots$:

$$\frac{\Delta R_n^{expt-thy}}{R_n^{thy}} \equiv \frac{R_n^{expt} - R_n^{thy}}{R_n^{thy}} = \frac{\left(\frac{f_n^{expt}}{f_1^{expt}}\right) - \left(\frac{f_n^{thy}}{f_1^{thy}}\right)}{\left(\frac{f_n^{thy}}{f_1^{thy}}\right)} = \frac{n_{expt} - n}{n} = \frac{\Delta n_{expt-thy}}{n}$$

Next, if we ***multiply*** this expression by the ***theory*** ***value*** of the ***frequency*** of this ***harmonic*** f_n^{thy}, then this is equal to the ***shift/departure*** of this harmonic's frequency from its ***perfect/ideal*** theory value $\Delta f_n^{expt-thy}$ (in *Hz*):

$$\Delta f_n^{expt-thy}\ (Hz) = f_n^{thy}\left(\frac{\Delta R_n^{expt-thy}}{R_n^{thy}}\right) = f_n^{thy}\left[\frac{\left(\frac{f_n^{expt}}{f_1^{expt}}\right) - \left(\frac{f_n^{thy}}{f_1^{thy}}\right)}{\left(\frac{f_n^{thy}}{f_1^{thy}}\right)}\right] = f_n^{thy}\left(\frac{n_{expt} - n}{n}\right) = f_n^{thy}\left(\frac{\Delta n_{expt-thy}}{n}\right)$$

Stated another way:

$$\left(\frac{\Delta f_n^{expt-thy}}{f_n^{thy}}\right) = \left(\frac{\Delta R_n^{expt-thy}}{R_n^{thy}}\right) = \left(\frac{R_n^{expt} - R_n^{thy}}{R_n^{thy}}\right) = \left[\frac{\left(\frac{f_n^{expt}}{f_1^{expt}}\right) - \left(\frac{f_n^{thy}}{f_1^{thy}}\right)}{\left(\frac{f_n^{thy}}{f_1^{thy}}\right)}\right] = \left(\frac{n_{expt} - n}{n}\right) = \left(\frac{\Delta n_{expt-thy}}{n}\right)$$

Then the ***% deviations*** $\Delta_n^{expt-thy}(\%)$ of the ***actual/measured*** harmonics of a stringed instrument from their ***perfect/ideal*** theory values are obtained by ***multiplying*** the above expression by 100%:

$$\Delta_n^{expt-thy}(\%) \equiv 100\left(\frac{\Delta f_n^{expt-thy}}{f_n^{thy}}\right) = 100\left(\frac{\Delta R_n^{expt-thy}}{R_n^{thy}}\right) = 100\left(\frac{R_n^{expt} - R_n^{thy}}{R_n^{thy}}\right)$$

$$= 100\left[\frac{\left(\frac{f_n^{expt}}{f_1^{expt}}\right) - \left(\frac{f_n^{thy}}{f_1^{thy}}\right)}{\left(\frac{f_n^{thy}}{f_1^{thy}}\right)}\right] = 100\left(\frac{n_{expt} - n}{n}\right) = 100\left(\frac{\Delta n_{expt-thy}}{n}\right)$$

Now ***one semi-tone*** in the ***tempered scale*** is equal to 100 ***cents***, and since there are 12 notes in the ***tempered scale***, then one ***octave*** = 1200 cents. The frequency ratio between two adjacent notes (i.e. a semitone) in the tempered scale is $a^{1/12} = 2^{1/12} = 1.0594631$.

Thus, one semitone ***low*** from the frequency f_n^{thy} is given by the formula:

$$f_{low}^{semitone} = f_n^{thy} / a^{1/12} = f_n^{thy} / 2^{1/12} = f_n^{thy} / 1.0594631 = 0.943874\, f_n^{thy}$$

and one semitone ***high*** from the frequency f_n^{thy} is given by the formula:

$$f_{high}^{semitone} = a^{1/12} f_n^{thy} = 2^{1/12} f_n^{thy} = 1.0594631\, f_n^{thy}$$

The frequency ***difference*** $\Delta f_{low}^{semitone}$ for ***<u>one</u> semitone <u>low</u>*** from f_n^{thy} is given by:

$$\begin{aligned}\Delta f_{low}^{semitone} &= f_n^{thy} - f_{low}^{semitone} = f_n^{thy} - f_n^{thy}/a^{1/12} = f_n^{thy}\left(1-1/a^{1/12}\right) = f_n^{thy}\left(1-1/2^{1/12}\right)\\ &= f_n^{thy}\left(1-0.943874\right)\\ &= 0.0561257 f_n^{thy} \quad (Hz)\end{aligned}$$

The frequency ***difference*** $\Delta f_{low}^{semitone}$ for ***<u>one</u> semitone <u>high</u>*** from f_n^{thy} is given by:

$$\begin{aligned}\Delta f_{high}^{semitone} &= f_{high}^{semitone} - f_n^{thy} = a^{1/12} f_n^{thy} - f_n^{thy} = f_n^{thy}\left(a^{1/12}-1\right) = f_n^{thy}\left(2^{1/12}-1\right)\\ &= f_n^{thy}\left(1.0594630-1\right)\\ &= 0.0594630 f_n^{thy} \quad (Hz)\end{aligned}$$

Notice that $\left(\Delta f_{low}^{semitone} = 0.0561257 f_n^{thy}\right) \neq \left(\Delta f_{high}^{semitone} = 0.0594630 f_n^{thy}\right)$.
They're close, but they are ***<u>not</u> <u>precisely</u>*** equal to each other.

Suppose the frequency of a note on the musical scale, *e.g.* A5 is $f_n^{thy} = 880.00\ Hz$.
The frequency of A*b*5 (one semitone ***<u>low</u>***) is 830.61 Hz, ***differing*** from A5 by 49.39 Hz.
The frequency of B5 (one semitone ***<u>high</u>***) is 932.33 Hz, ***differing*** from A5 by 52.33 Hz.

In order to express our ***measured experimental*** *vs*. ***theory*** frequency ***differences*** $\Delta f_n^{expt-thy}$ in ***<u>cents</u>***, we need to a.) first ***normalize*** $\Delta f_n^{expt-thy}$ to $\Delta f_{low}^{semitone}$ (if $\Delta f_n^{expt-thy} < 0$) or ***normalize*** $\Delta f_n^{expt-thy}$ to $\Delta f_{high}^{semitone}$ (if $\Delta f_n^{expt-thy} > 0$), then b.) ***multiply <u>each</u>*** of these ***<u>ratios</u>*** by <u>100 ***cents***</u>:

$$\text{If } \Delta f_n^{expt-thy} < 0: \ \#\,cents\ low = 100\left(\frac{\Delta f_n^{expt-thy}}{\Delta f_{low}^{semitone}}\right)\ cents$$

$$\text{If } \Delta f_n^{expt-thy} > 0: \ \#\,cents\ high = 100\left(\frac{\Delta f_n^{expt-thy}}{\Delta f_{high}^{semitone}}\right)\ cents$$

Then, inserting the formulas for $\Delta f_n^{expt-thy}$, $\Delta f_{low}^{semitone}$ and $\Delta f_{high}^{semitone}$ in these expressions, we have:

$$\text{If } \Delta f_n^{expt-thy} < 0: \ \boxed{\#\,cents\ low = 100\left(\frac{\cancel{f_n^{thy}}}{\cancel{f_n^{thy}}\left(1-1/2^{1/12}\right)}\right)\left(\frac{n_{expt}-n}{n}\right) = \frac{100}{\left(1-1/2^{1/12}\right)}\left(\frac{n_{expt}}{n}-1\right)\ cents}$$

$$\text{If } \Delta f_n^{expt-thy} > 0: \ \boxed{\#\,cents\ high = 100\left(\frac{\cancel{f_n^{thy}}}{\cancel{f_n^{thy}}\left(2^{1/12}-1\right)}\right)\left(\frac{n_{expt}-n}{n}\right) = \frac{100}{\left(2^{1/12}-1\right)}\left(\frac{n_{expt}}{n}-1\right)\ cents}$$

where: $\boxed{n_{expt} \equiv \left(f_n^{expt}/f_1^{expt}\right)}$ and: $\boxed{n \equiv \left(f_n^{thy}/f_1^{thy}\right), \quad n = 1, 2, 3, \ldots}$

Numerically, these formulas, for ***stringed instrument inharmonicities*** are:

If $\Delta f_n^{expt-thy} < 0$:

$$\#\,cents\ low\ = \frac{100}{\left(1-1/2^{1/12}\right)}\left(\frac{n_{expt}}{n}-1\right) = \frac{100}{0.0561257}\left(\frac{n_{expt}}{n}-1\right) = 1781.715\left(\frac{n_{expt}}{n}-1\right)\ cents$$

If $\Delta f_n^{expt-thy} > 0$:

$$\#\,cents\ high = \frac{100}{\left(2^{1/12}-1\right)}\left(\frac{n_{expt}}{n}-1\right) = \frac{100}{0.0594630}\left(\frac{n_{expt}}{n}-1\right) = 1681.718\left(\frac{n_{expt}}{n}-1\right)\ cents$$

where: $n_{expt} \equiv \left(f_n^{expt} / f_1^{expt} \right)$ and: $n \equiv \left(f_n^{thy} / f_1^{thy} \right), \quad n = 1, 2, 3, \ldots$

II.) Wind/Brass Lip-Reed Instrument Inharmonicities:

For ***wind/brass lip-reed*** instruments, the ***fundamental*** frequency f_1^{expt} of an ***open*** vibrating string ***is significantly*** perturbed from its ***perfect/ideal*** theory value of $f_1^{thy} = v/\lambda_1$, because of a variety of higher-order effects not taken into account in the simple perfect/ideal theory – e.g. frequency-dependent wavelength effects at the bell end of the instrument; the (longitudinal) speed of sound propagation in the confined/internal space of the instrument is also formally frequency dependent, i.e. $v = v(f)$ and especially so a lower frequencies, due to frequency-dependent viscous damping/dissipation effects. In wind/brass lip-reed instruments, the fundamental frequency f_1^{expt} v(*aka* the "pedal note" tends to be systematically pulled ***low***, and significantly so. Thus, we ***cannot*** use the fundamental as a ***reference*** for the higher harmonics $f_n^{thy} = n \cdot f_1^{thy}, \quad n = 1, 2, 3, \ldots$. The pedal note/fundamental is also extremely difficult to play on wind/brass lip-reed instruments, and also not with any great accuracy. The second harmonic of wind/brass lip-reed instruments is much less adversely affected than the fundamental (= the first harmonic), and also easy to play, accurately so. Hence we can/will use the ***second harmonic*** as a ***reference*** for the higher harmonics in wind/brass lip-reed instruments.

We can then calculate the ***frequency ratios*** $R_n^{expt} \equiv \left(f_n^{expt} / f_2^{expt} \right) = n_{expt}/2 \ \left(\neq n = 1, 2, 3 \ldots \right)$ and compare these results to the "simple" ***perfect/ideal*** theory predictions $R_n^{thy} \equiv \left(f_n^{thy} / f_2^{thy} \right) = n/2, \quad n = 1, 2, 3, \ldots$

The ***measured*** frequency ratios $R_n^{expt} \equiv \left(f_n^{expt} / f_2^{expt} \right) = n_{expt}/2$ should be ***close*** to, but will ***not*** be ***precisely equal*** to $n/2, \quad n = 1, 2, 3, \ldots$ for wind/brass lip-reed instruments, due to the effect(s) of viscous damping/dissipation effects of the air inside the bore of and in contact with the inside surface(s) of the instrument.

The ***deviation*** of a given ***measured*** frequency ratio $R_n^{expt} \equiv \left(f_n^{expt} / f_2^{expt} \right)$ from its ***perfect/ideal*** "simple" theory value $R_n^{thy} \equiv \left(f_n^{thy} / f_2^{thy} \right) = n/2, \quad n = 1, 2, 3, \ldots$ is given by the ***experiment*** *vs*. ***theory*** ***difference*** between these two quantities:

$$\Delta R_n^{expt-thy} \equiv R_n^{expt} - R_n^{thy} = \left(\frac{f_n^{expt}}{f_2^{expt}} \right) - \left(\frac{f_n^{thy}}{f_2^{thy}} \right) = \frac{n_{expt}}{2} - \frac{n}{2} = \frac{\Delta n_{expt-thy}}{2}$$

If we then ***normalize*** the above expression to the theory ratio $R_n^{thy} \equiv \left(f_n^{thy} / f_2^{thy} \right) = n/2, \quad n = 1, 2, 3, \ldots$ then we obtain the ***fractional*** ***deviation*** of a given ***measured*** frequency ratio $R_n^{expt} \equiv \left(f_n^{expt} / f_2^{expt} \right)$ from its ***perfect/ideal*** theory value $R_n^{thy} \equiv \left(f_n^{thy} / f_2^{thy} \right) = n/2, \quad n = 1, 2, 3, \ldots$:

$$\frac{\Delta R_n^{expt-thy}}{R_n^{thy}} \equiv \frac{R_n^{expt} - R_n^{thy}}{R_n^{thy}} = \frac{\left(\frac{f_n^{expt}}{f_2^{expt}} \right) - \left(\frac{f_n^{thy}}{f_2^{thy}} \right)}{\left(\frac{f_n^{thy}}{f_2^{thy}} \right)} = \frac{\frac{n_{expt}}{2} - \frac{n}{2}}{\frac{n}{2}} = \frac{\frac{\Delta n_{expt-thy}}{2}}{\frac{n}{2}} = \frac{\Delta n_{expt-thy}}{n}$$

Next, if we ***multiply*** this expression by the ***theory*** ***value*** of the ***frequency*** of this ***harmonic*** f_n^{thy}, then this is equal to the ***shift/departure*** of this harmonic's frequency from its ***perfect/ideal*** theory value $\Delta f_n^{expt-thy}$ (in *Hz*):

$$\Delta f_n^{expt-thy} \; (Hz) = f_n^{thy} \left(\frac{\Delta R_n^{expt-thy}}{R_n^{thy}} \right) = f_n^{thy} \left[\frac{\left(\frac{f_n^{expt}}{f_2^{expt}} \right) - \left(\frac{f_n^{thy}}{f_2^{thy}} \right)}{\left(\frac{f_n^{thy}}{f_2^{thy}} \right)} \right] = f_n^{thy} \left(\frac{n_{expt} - n}{n} \right) = f_n^{thy} \left(\frac{\Delta n_{expt-thy}}{n} \right)$$

Stated another way:

$$\left(\frac{\Delta f_n^{expt-thy}}{f_n^{thy}} \right) = \left(\frac{\Delta R_n^{expt-thy}}{R_n^{thy}} \right) = \left(\frac{R_n^{expt} - R_n^{thy}}{R_n^{thy}} \right) = \left[\frac{\left(\frac{f_n^{expt}}{f_2^{expt}} \right) - \left(\frac{f_n^{thy}}{f_2^{thy}} \right)}{\left(\frac{f_n^{thy}}{f_2^{thy}} \right)} \right] = \left(\frac{n_{expt} - n}{n} \right) = \left(\frac{\Delta n_{expt-thy}}{n} \right)$$

Then the ***% deviations*** $\Delta_n^{expt-thy} (\%)$ of the ***actual/measured*** harmonics of a stringed instrument from their ***perfect/ideal*** theory values are obtained by ***multiplying*** the above expression by 100%:

$$\Delta_n^{expt-thy} (\%) \equiv 100 \left(\frac{\Delta f_n^{expt-thy}}{f_n^{thy}} \right) = 100 \left(\frac{\Delta R_n^{expt-thy}}{R_n^{thy}} \right) = 100 \left(\frac{R_n^{expt} - R_n^{thy}}{R_n^{thy}} \right)$$

$$= 100 \left[\frac{\left(\frac{f_n^{expt}}{f_2^{expt}} \right) - \left(\frac{f_n^{thy}}{f_2^{thy}} \right)}{\left(\frac{f_n^{thy}}{f_2^{thy}} \right)} \right] = 100 \left(\frac{n_{expt} - n}{n} \right) = 100 \left(\frac{\Delta n_{expt-thy}}{n} \right)$$

Now ***one*** ***semi-tone*** in the ***tempered*** ***scale*** is equal to 100 ***cents***, and since there are 12 notes in the ***tempered*** ***scale***, then one ***octave*** = 1200 cents. The frequency ratio between two adjacent notes (i.e. a semitone) in the tempered scale is $a^{1/12} = 2^{1/12} = 1.0594631$.

Thus, one semitone ***low*** from the frequency f_n^{thy} is given by the formula:

$$f_{low}^{semitone} = f_n^{thy} / a^{1/12} = f_n^{thy} / 2^{1/12} = f_n^{thy} / 1.0594631 = 0.943874\, f_n^{thy}$$

and one semitone ***high*** from the frequency f_n^{thy} is given by the formula:

$$f_{high}^{semitone} = a^{1/12} f_n^{thy} = 2^{1/12} f_n^{thy} = 1.0594631 f_n^{thy}$$

The frequency ***difference*** $\Delta f_{low}^{semitone}$ for ***one*** ***semitone*** ***low*** from f_n^{thy} is given by:

$$\begin{aligned}\Delta f_{low}^{semitone} &= f_n^{thy} - f_{low}^{semitone} = f_n^{thy} - f_n^{thy} / a^{1/12} = f_n^{thy}\left(1 - 1/a^{1/12}\right) = f_n^{thy}\left(1 - 1/2^{1/12}\right)\\ &= f_n^{thy}\left(1 - 0.943874\right)\\ &= 0.0561257 f_n^{thy} \quad (Hz)\end{aligned}$$

The frequency ***difference*** $\Delta f_{low}^{semitone}$ for ***one*** ***semitone*** ***high*** from f_n^{thy} is given by:

$$\begin{aligned}\Delta f_{high}^{semitone} &= f_{high}^{semitone} - f_n^{thy} = a^{1/12} f_n^{thy} - f_n^{thy} = f_n^{thy}\left(a^{1/12} - 1\right) = f_n^{thy}\left(2^{1/12} - 1\right)\\ &= f_n^{thy}\left(1.0594630 - 1\right)\\ &= 0.0594630 f_n^{thy} \quad (Hz)\end{aligned}$$

Notice that $\left(\Delta f_{low}^{semitone} = 0.0561257 f_n^{thy}\right) \neq \left(\Delta f_{high}^{semitone} = 0.0594630 f_n^{thy}\right)$.
They're close, but they are ***not*** ***precisely*** equal to each other.

Suppose the frequency of a note on the musical scale, *e.g.* A5 is $f_n^{thy} = 880.00\ Hz$.
The frequency of A♭5 (one semitone ***low***) is 830.61 Hz, ***differing*** from A5 by 49.39 Hz.
The frequency of B5 (one semitone ***high***) is 932.33 Hz, ***differing*** from A5 by 52.33 Hz.

In order to express our ***measured experimental*** *vs*. ***theory*** frequency ***differences*** $\Delta f_n^{expt-thy}$ in ***cents***, we need to a.) first ***normalize*** $\Delta f_n^{expt-thy}$ to $\Delta f_{low}^{semitone}$ (if $\Delta f_n^{expt-thy} < 0$) or ***normalize*** $\Delta f_n^{expt-thy}$ to $\Delta f_{high}^{semitone}$ (if $\Delta f_n^{expt-thy} > 0$), then b.) ***multiply*** ***each*** of these ***ratios*** by 100 ***cents***:

If $\Delta f_n^{expt-thy} < 0$: $\#\, cents\ low = 100\left(\dfrac{\Delta f_n^{expt-thy}}{\Delta f_{low}^{semitone}}\right)\ cents$

If $\Delta f_n^{expt-thy} > 0$: $\#\, cents\ high = 100\left(\dfrac{\Delta f_n^{expt-thy}}{\Delta f_{high}^{semitone}}\right)\ cents$

Then, inserting the formulas for $\Delta f_n^{expt-thy}$, $\Delta f_{low}^{semitone}$ and $\Delta f_{high}^{semitone}$ in these expressions, we have:

If $\Delta f_n^{expt-thy} < 0$: $\boxed{\#\,cents\ low\ = 100\left(\frac{f_n^{thy}}{f_n^{thy}\left(1-1/2^{1/12}\right)}\right)\left(\frac{n_{expt}-n}{n}\right) = \frac{100}{\left(1-1/2^{1/12}\right)}\left(\frac{n_{expt}}{n}-1\right)\ cents}$

If $\Delta f_n^{expt-thy} > 0$: $\boxed{\#\,cents\ high = 100\left(\frac{f_n^{thy}}{f_n^{thy}\left(2^{1/12}-1\right)}\right)\left(\frac{n_{expt}-n}{n}\right) = \frac{100}{\left(2^{1/12}-1\right)}\left(\frac{n_{expt}}{n}-1\right)\ cents}$

where: $\boxed{n_{expt} \equiv 2\left(f_n^{expt}/f_2^{expt}\right)}$ and: $\boxed{n \equiv 2\left(f_n^{thy}/f_2^{thy}\right),\quad n = 1, 2, 3, ...}$

Numerically, these formulas, for ***wind/brass lip-reed instrument inharmonicities*** are:

If $\Delta f_n^{expt-thy} < 0$:

$$\boxed{\#\,cents\ low\ = \frac{100}{\left(1-1/2^{1/12}\right)}\left(\frac{n_{expt}}{n}-1\right) = \frac{100}{0.0561257}\left(\frac{n_{expt}}{n}-1\right) = 1781.715\left(\frac{n_{expt}}{n}-1\right)\ cents}$$

If $\Delta f_n^{expt-thy} > 0$:

$$\boxed{\#\,cents\ high = \frac{100}{\left(2^{1/12}-1\right)}\left(\frac{n_{expt}}{n}-1\right) = \frac{100}{0.0594630}\left(\frac{n_{expt}}{n}-1\right) = 1681.718\left(\frac{n_{expt}}{n}-1\right)\ cents}$$

where: $\boxed{n_{expt} \equiv 2\left(f_n^{expt}/f_2^{expt}\right)}$ and: $\boxed{n \equiv 2\left(f_n^{thy}/f_2^{thy}\right),\quad n = 1, 2, 3, ...}$

Legal Disclaimer and Copyright Notice:

Legal Disclaimer:

The author specifically disclaims legal responsibility for any loss of profit, or any consequential, incidental, and/or other damages resulting from the mis-use of information contained in this document. The author has made every effort possible to ensure that the information contained in this document is factually and technically accurate and correct.

Copyright Notice:

The contents of this document are protected under both United States of America and International Copyright Laws. No portion of this document may be reproduced in any manner for *commercial* use without prior *written* permission from the author of this document. The author grants permission for the use of information contained in this document for *private*, *non-commercial* purposes only.