Time-Domain *vs*. Frequency-Domain Sound Field Measurements

<u>Introduction</u>:

In this lab handout, we discuss the physical meaning of, the relationship(s) between, and the experimental techniques associated with ***time-domain*** *vs*. ***frequency-domain*** measurements of over-pressure p and particle velocity $\vec{u}$ associated with an arbitrary sound field.

For sound waves propagating in air, the instantaneous over-pressure $p(\vec{r},t)$ (*n.b*. a scalar quantity) and the instantaneous 3-D vector particle velocity $\vec{u}(\vec{r},t)$ obey their respective wave equations, which are {neglecting/ignoring (small) dissipative/energy loss effects) and for normal/everyday sound pressure levels $(SPL \ll 134\,dB)$ }:

$$\boxed{\nabla^2 p(\vec{r},t)+\frac{1}{v^2}\frac{\partial^2 p(\vec{r},t)}{\partial t^2}=0} \quad \text{and:} \quad \boxed{\nabla^2\vec{u}(\vec{r},t)+\frac{1}{v^2}\frac{\partial^2\vec{u}(\vec{r},t)}{\partial t^2}=0}$$

where v is the wave propagation speed in the medium (= air, in this case, *<u>here</u>*).

The instantaneous over-pressure $p(\vec{r},t)$ and particle velocity $\vec{u}(\vec{r},t)$ are ***not*** independent quantities. For *e.g*. traveling-type sound waves in air, neglecting/ignoring (small) dissipative/ energy loss effects and normal/everyday sound pressure levels $(SPL \ll 134\,dB)$, the Euler equation for inviscid (*i.e*. dissipationless) fluid flow reasonably accurately describes the spatial-temporal relationship between these two instantaneous physical quantities:

$$\boxed{\frac{\partial\vec{u}(\vec{r},t)}{\partial t}=-\frac{1}{\rho_o}\vec{\nabla}p(\vec{r},t)} \quad \text{where for sound propagation in air: } \rho_o^{air}=1.204\,kg/m^3 \text{ @ NTP.}$$

<u>Sound Field Measurements in One Dimension</u>:

For the sake of simplicity and clarity, we first discuss 1-D sound fields, such as that associated with the propagation of a monochromatic (*i.e*. single-frequency) traveling plane wave in the $+\hat{z}$ direction, or *e.g*. the "far-field" $(r \gg \lambda)$ regime associated with the radial-outward propagation of {monochromatic/ single-frequency} spherical waves emanating from a point sound source, located at the origin. Please refer to the UIUC Physics 406 Lecture Notes XII and XII – Part 2 for discussion/details of the nature of these two 1-D type sound fields.

In order to completely describe an arbitrary, instantaneous monochromatic/single-frequency { $\omega \equiv 2\pi f$ }1-D sound field associated with a 1-D longitudinal sound wave (at least locally) propagating in the $+\hat{z}$ direction, we need to measure ***two*** physical quantities at the listener's position, *e.g*. $\vec{r}=z\hat{z}$:

a.) the instantaneous over-pressure, $p(\vec{r},t;\omega)$ and

b.) the instantaneous 1-D particle velocity $u_z(\vec{r},t;\omega)$.

The ***most*** ***general*** mathematical description – in the ***time***-***domain*** – for these two ***instantaneous physical*** quantities, associated with a monochromatic/single-frequency traveling 1-D sound wave (at least locally) propagating in the $+\hat{z}$ direction, for a listener's position at $\vec{r} = z\hat{z}$ are:

$$p(\vec{r},t;\omega) = p_0(\vec{r},\omega)\cos\Theta_p(\vec{r},t;\omega) = p_0(\vec{r},\omega)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right) = p_0(\vec{r},\omega)\cos\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right]$$

$$u_z(\vec{r},t;\omega) = u_{0_z}(\vec{r},\omega)\cos\Theta_{u_z}(\vec{r},t;\omega) = u_{0_z}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right) = u_{0_z}(\vec{r},\omega)\cos\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right]$$

Note that the ***amplitudes*** for over-pressure $p_0(\vec{r},\omega)$ and longitudinal particle-velocity $u_{0_z}(\vec{r},\omega)$ are ***time***-***independent*** quantities {for a constant-amplitude sound source}. However, depending on the detailed nature of the ***specific*** sound source under consideration in a given physics situation, the ***amplitudes*** $p_0(\vec{r},\omega)$ and $u_{0_z}(\vec{r},\omega)$ in general can be/are position- ***.and.*** frequency-dependent.

Note that the ***overall*** arguments of the cosine function(s), $\Theta_p(\vec{r},t;\omega)$ and $\Theta_{u_z}(\vec{r},t;\omega)$ in the above $+\hat{z}$ 1-D traveling-wave expressions for instantaneous over-pressure $p(\vec{r},t;\omega)$ and 1-D longitudinal particle velocity $u_z(\vec{r},t;\omega)$ are, in general ***not*** constants, due to {in general} possible position- and frequency-dependence of the overall phases $\varphi_p(\vec{r},\omega)$ and $\varphi_{u_z}(\vec{r},\omega)$, the {longitudinal} wavenumber $k_z(\vec{r},\omega)$ and the $(\vec{r}=0)$ phases $\varphi_p^o(\omega)$ and $\varphi_{u_z}^o(\omega)$:

$$\Theta_p(\vec{r},t;\omega) = \omega t + \varphi_p(\vec{r},\omega) = \omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)$$

$$\Theta_{u_z}(\vec{r},t;\omega) = \omega t + \varphi_{u_z}(\vec{r},\omega) = \omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)$$

The ***overall*** phase(s) associated with the generalized over-pressure and {longitudinal} 1-D particle velocity waves are:

$$\varphi_p(\vec{r},\omega) = \varphi_p^o(\omega) - k_z(\vec{r},\omega)z \quad \text{At the origin } (\vec{r}=0): \quad \varphi_p(0,\omega) = \varphi_p^o(\omega)$$

$$\varphi_{u_z}(\vec{r},\omega) = \varphi_{u_z}^o(\omega) - k_z(\vec{r},\omega)z \quad \text{At the origin } (\vec{r}=0): \quad \varphi_{u_z}(0,\omega) = \varphi_{u_z}^o(\omega)$$

At $(\vec{r}=0, t=0)$ we also see that: $\Theta_p(0,0;\omega) = \varphi_p^o(\omega)$ and: $\Theta_{u_z}(0,0;\omega) = \varphi_{u_z}^o(\omega)$.

Thus the instantaneous over-pressure and {longitudinal} 1-D particle velocity at $(\vec{r}=0, t=0)$ are:

$$p(0,0;\omega) = p_0(0,\omega)\cos\varphi_p^o(\omega)$$

$$u_z(0,0;\omega) = u_{0_z}(0,\omega)\cos\varphi_{u_z}^o(\omega)$$

Important Comment: The $(\vec{r}=0, t=0)$ phases $\varphi_p^o(\omega)$ and $\varphi_{u_z}^o(\omega)$ are defined ***relative*** *e.g*. to the sine-wave signal output from a sine-wave function generator that is used to produce the sound field in the first place. The sine-wave signal output from the function generator $V_{FG}(\omega) = V_{FG}^o \cos\omega t$ thus provides the ***reference*** signal needed for defining, and determining/ experimentally measuring the $(\vec{r}=0, t=0)$ ***relative*** phases $\varphi_p^o(\omega)$ and $\varphi_{u_z}^o(\omega)$.

Note that for a ***constant/fixed*** value of the ***overall*** phase(s), the position z and the time t are related to each other via $z(t) = v_\phi(\vec{r},\omega)\cdot t$ where $v_\phi(\vec{r},\omega)$ is the ***phase speed*** associated with the longitudinal propagation of the 1-D traveling wave: $v_\phi(\vec{r},\omega) = f \cdot \lambda_z(\vec{r},\omega) = \omega / k_z(\vec{r},\omega)$ {= the speed of propagation of ***surfaces*** of ***constant phase***}, which again, depending on the detailed nature of the ***specific*** sound source under consideration in a given physics situation, the phase speed $v_\phi(\vec{r},\omega)$, the {longitudinal} wavelength $\lambda_z(\vec{r},\omega)$ and the {longitudinal} wavenumber $k_z(\vec{r},\omega) \equiv 2\pi/\lambda_z(\vec{r},\omega)$ can be/are in general both position- ***.and.*** frequency-dependent. Thus, we see that:

$$\boxed{\Theta_p(\vec{r},t;\omega) = \omega t + \varphi_p(\vec{r},\omega) = \omega t - \omega z/v_\varphi(\vec{r},\omega) + \varphi_p^o(\omega) = \omega\left(t - z/v_\varphi(\vec{r},\omega)\right) + \varphi_p^o(\omega)}$$

$$\boxed{\Theta_{u_z}(\vec{r},t;\omega) = \omega t + \varphi_{u_z}(\vec{r},\omega) = \omega t - \omega z/v_\varphi(\vec{r},\omega) + \varphi_{u_z}^o(\omega) = \omega\left(t - z/v_\varphi(\vec{r},\omega)\right) + \varphi_{u_z}^o(\omega)}$$

The ***overall*** phase(s) associated with the generalized over-pressure and {longitudinal} 1-D particle velocity traveling waves are:

$$\boxed{\varphi_p(\vec{r},\omega) = \varphi_p^o(\omega) - k_z(\vec{r},\omega)z = \varphi_p^o(\omega) - \omega z/v_\varphi(\vec{r},\omega)}$$

$$\boxed{\varphi_{u_z}(\vec{r},\omega) = \varphi_{u_z}^o(\omega) - k_z(\vec{r},\omega)z = \varphi_{u_z}^o(\omega) - \omega z/v_\varphi(\vec{r},\omega)}$$

A modern digital oscilloscope or *e.g*. a digital recorder (*n.b*. both are manifestly ***time-domain*** instruments!) can be used to measure the time-dependent voltages output from omni-directional pressure and/or 1-D particle velocity microphones, *e.g*. located at the "listener" point $\vec{r} = z\hat{z}$ in a sound field associated with a monochromatic/single-frequency traveling 1-D sound wave (at least locally) propagating in the $+\hat{z}$ direction. The instantaneous ***voltage*** signals output from the *p*- and *u*-mics will be of the form:

$$\boxed{V_{p\text{-}mic}(\vec{r},t;\omega) = V_{p\text{-}mic}^0(\vec{r},\omega)\cos\left[\omega t - k_z z + \varphi_p^o(\omega)\right]}$$

$$\boxed{V_{u\text{-}mic}(\vec{r},t;\omega) = V_{u\text{-}mic}^0(\vec{r},\omega)\cos\left[\omega t - k_z z + \varphi_{u_z}^o(\omega)\right]}$$

The *p*- and *u*-mics must be ***absolutely*** calibrated to obtain their respective ***microphone sensitivity calibration constants*** $S_{p\text{-}mic}(mV/Pa)$ and $S_{u\text{-}mic}(mV/Pa^*)$ {or $S_{u\text{-}mic}(mV/mm/s)$} $(n.b.\ 1.0\ Pa^* = 2.42(mm/s))$, in order to {absolutely} convert their respective voltage signals $V_{p\text{-}mic}(\vec{r},t;\omega)$ and $V_{u\text{-}mic}(\vec{r},t;\omega)$ into $p(\vec{r},t;\omega)$ and $u_z(\vec{r},t;\omega)$ signals:

$$\boxed{p(\vec{r},t;\omega) = V_{p\text{-}mic}(\vec{r},t;\omega)/S_{p\text{-}mic}\ (Pa)}$$

$$\boxed{u_z(\vec{r},t;\omega) = V_{u\text{-}mic}(\vec{r},t;\omega)/S_{u\text{-}mic}\ (Pa^* \text{ or } mm/s)}$$

The ***absolute*** calibration of *p*- and *u*-mics that we routinely use in the UIUC Physics of Music/Musical Instruments Lab is discussed in detail in the Lab Handout "**Absolute Calibration of Pressure and Particle Velocity Microphones**" – available on the UIUC Physics 193POM/ P406POM Lab Handout Webpages.

We can gain some additional physical insight into the nature of these two instantaneous acoustic signals by using the trigonometric identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$. At the listener's location $\vec{r} = z\hat{z}$, for arbitrary time *t*:

$$\begin{aligned} p(\vec{r},t;\omega) &= p_0(\vec{r},\omega)\cos\left[\omega t + \varphi_p(\vec{r},\omega)\right] = p_0(\vec{r},\omega)\cos\left[\omega t - \left(k_z(\vec{r},\omega)z - \varphi_p^o(\omega)\right)\right] \\ &= p_0(\vec{r},\omega)\left\{\cos\omega t\cos\left(k_z(\vec{r},\omega)z - \varphi_p^o(\omega)\right) - \sin\omega t\sin\left(k_z(\vec{r},\omega)z - \varphi_p^o(\omega)\right)\right\} \\ &= \underbrace{\left\{p_0(\vec{r},\omega)\cos\left(k_z(\vec{r},\omega)z - \varphi_p^o(\omega)\right)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}\\ \text{in-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\cos\omega t - \underbrace{\left\{p_0(\vec{r},\omega)\sin\left(k_z(\vec{r},\omega)z - \varphi_p^o(\omega)\right)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}\\ 90^o\text{out-of-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\sin\omega t \end{aligned}$$

$$\begin{aligned} u_z(\vec{r},t;\omega) &= u_{0_z}(\vec{r},\omega)\cos\left[\omega t + \varphi_{u_z}(\vec{r},\omega)\right] = u_{0_z}(\vec{r},\omega)\cos\left[\omega t + \varphi_{u_z}(\vec{r},\omega)\right] \\ &= u_{0_z}(\vec{r},\omega)\left\{\cos\omega t\cos\left(k_z(\vec{r},\omega)z - \varphi_{u_z}^o(\omega)\right) - \sin\omega t\sin\left(k_z(\vec{r},\omega)z - \varphi_{u_z}^o(\omega)\right)\right\} \\ &= \underbrace{\left\{u_{0_z}(\vec{r},\omega)\cos\left(k_z(\vec{r},\omega)z - \varphi_{u_z}^o(\omega)\right)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}\\ \text{in-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\cos\omega t - \underbrace{\left\{u_{0_z}(\vec{r},\omega)\sin\left(k_z(\vec{r},\omega)z - \varphi_{u_z}^o(\omega)\right)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}\\ 90^o\text{out-of-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\sin\omega t \end{aligned}$$

If we choose the listener's position to be at the origin $(\vec{r} = 0)$, at arbitrary time *t*, using the fact that $\cos x$ $(\sin x)$ is an *even* (*odd*) function of *x*, *i.e.* $\cos(-x) = \cos x$ $(\sin(-x) = -\sin x)$ respectively, the above expressions become:

$$p(0,t;\omega) = \underbrace{\left\{p_0(0,\omega)\cos\varphi_p^o(\omega)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}=0\\ \text{in-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\cos\omega t + \underbrace{\left\{p_0(0,\omega)\sin\varphi_p^o(\omega)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}=0\\ 90^o\text{out-of-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\sin\omega t$$

$$u_z(0,t;\omega) = \underbrace{\left\{u_{0_z}(0,\omega)\cos\varphi_{u_z}^o(\omega)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}=0\\ \text{in-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\cos\omega t + \underbrace{\left\{u_{0_z}(0,\omega)\sin\varphi_{u_z}^o(\omega)\right\}}_{\substack{\text{Amplitude component @ }\vec{r}=0\\ 90^o\text{out-of-phase with}\\ V_{FG}(t;\omega)=V_{FG}^o\cos(\omega t)}}\sin\omega t$$

As discussed in detail in Physics 406 Lecture Notes XIII – Part 2, experimentally, we can *e.g.* use lock-in amplifier and/or spectral analysis cross-correlation techniques to obtain/measure/ determine the above in-phase and 90^o out-of-phase ***amplitude*** components of over-pressure and 1-D particle velocity, phase-referenced relative to the sine-wave signal $V_{FG}(\omega) = V_{FG}^o \cos\omega t$ output from the sine-wave function generator that is used to produce the monochromatic/single-frequency sound field in the first place.

Note that the above purely real mathematical expressions that describe the ***instantaneous*** over-pressure $p(\vec{r},t;\omega)$ and 1-D particle velocity $u_z(\vec{r},t;\omega)$ are manifestly ***time-domain*** quantities. These expressions can be related to their ***frequency-domain*** counterparts as follows:

First, we "***complexify***" the above ***instantaneous*** time-domain over-pressure $p(\vec{r},t;\omega)$ and 1-D particle velocity $u_z(\vec{r},t;\omega)$ expressions by adding an "imaginary", 90^o phase-shifted component to their purely real expressions. Defining $i \equiv \sqrt{-1}$, with complex conjugation $i^* \equiv -i = -\sqrt{-1}$ {hence $i^* \cdot i = i \cdot i^* = +1$}, the ***complex instantaneous*** over-pressure $\tilde{p}(\vec{r},t;\omega)$ and 1-D particle velocity $\tilde{u}_z(\vec{r},t;\omega)$ at the listener's location $\vec{r} = z\hat{z}$, for arbitrary time *t* are:

$$\begin{aligned}\tilde{p}(\vec{r},t;\omega) &= p_0(\vec{r},\omega)\cos\left[\omega t + \varphi_p(\vec{r},\omega)\right] + i\cdot p_0(\vec{r},\omega)\sin\left[\omega t + \varphi_p(\vec{r},\omega)\right] \\ &= p_0(\vec{r},\omega)\cos\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right] + i\cdot p_0(\vec{r},\omega)\sin\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right] \\ &= p_0(\vec{r},\omega)\left\{\cos\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right] + i\cdot\sin\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right]\right\}\end{aligned}$$

$$\begin{aligned}\tilde{u}_z(\vec{r},t;\omega) &= u_{0_z}(\vec{r},\omega)\cos\left[\omega t + \varphi_{u_z}(\vec{r},\omega)\right] + i\cdot u_{0_z}(\vec{r},\omega)\sin\left[\omega t + \varphi_{u_z}(\vec{r},\omega)\right] \\ &= u_{0_z}(\vec{r},\omega)\cos\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right] + i\cdot u_{0_z}(\vec{r},\omega)\sin\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right] \\ &= u_{0_z}(\vec{r},\omega)\left\{\cos\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right] + i\cdot\sin\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right]\right\}\end{aligned}$$

We then use the Euler relation $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ to equivalently write these expressions in complex exponential notation:

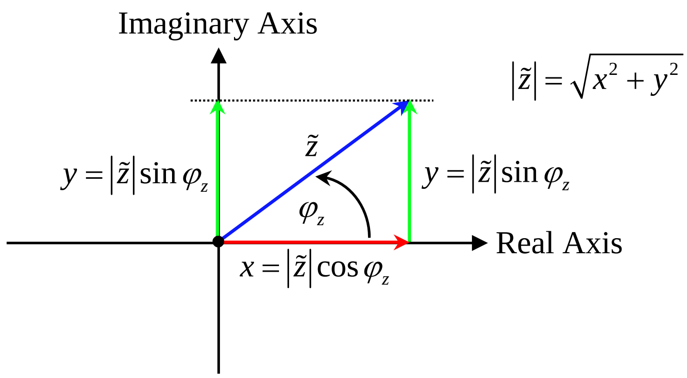
$$\tilde{p}(\vec{r},t;\omega) = p_0(\vec{r},\omega)e^{i\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right]} = p_0(\vec{r},\omega)\cdot e^{i\left[\varphi_p^o(\omega) - k_z(\vec{r},\omega)z\right]}\cdot e^{i\omega t} = p_0(\vec{r},\omega)\cdot e^{i\varphi_p(\vec{r},\omega)}\cdot e^{i\omega t}$$

$$\tilde{u}_z(\vec{r},t;\omega) = u_{0_z}(\vec{r},\omega)e^{i\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right]} = u_{0_z}(\vec{r},\omega)\cdot e^{i\left[\varphi_{u_z}^o(\omega) - k_z(\vec{r},\omega)z\right]}\cdot e^{i\omega t} = u_{0_z}(\vec{r},\omega)\cdot e^{i\varphi_{u_z}(\vec{r},\omega)}\cdot e^{i\omega t}$$

Now for ***any*** complex quantity $\tilde{z} = x + iy$, the ***magnitude*** of the complex quantity $\tilde{z}$ is $|\tilde{z}| = \sqrt{\tilde{z}\cdot\tilde{z}^*} = \sqrt{(x+iy)\cdot(x-iy)} = \sqrt{x^2+y^2}$ {*n.b.* a purely ***real*** quantity}, the phase $\varphi_z = \tan^{-1}(y/x)$, $x = |\tilde{z}|\cos\varphi_z$ and $y = |\tilde{z}|\sin\varphi_z$. Thus, we can equivalently write the complex quantity $\tilde{z}$ as:

$$\tilde{z} = x + iy = |\tilde{z}|\cos\varphi_z + i|\tilde{z}|\sin\varphi_z = |\tilde{z}|\left(\cos\varphi_z + i\sin\varphi_z\right) = |\tilde{z}|e^{i\varphi_z}$$

A ***phasor*** ***diagram*** of the "generic" complex quantity $\tilde{z} = x + iy = |\tilde{z}|\cos\varphi_z + i|\tilde{z}|\sin\varphi_z = |\tilde{z}|e^{i\varphi_z}$ in the complex plane is shown in the figure below:



The ***real*** part (or component) of $\tilde{z}$, $x = \text{Re}\{\tilde{z}\} = |\tilde{z}|\cos\varphi_z$ lies on the ***horizontal*** axis. The ***imaginary*** part (or component) of $\tilde{z}$, $y = \text{Im}\{\tilde{z}\} = |\tilde{z}|\sin\varphi_z$ lies on the ***vertical*** axis. Thus, ***complex*** $\tilde{z}$ lies somewhere in the complex plane, oriented at an angle $\varphi_z = \tan^{-1}(y/x)$, ***referenced*** to the ***horizontal*** axis. In an acoustical physics situation, the physical meaning of the real (imaginary) part of the complex quantity $\tilde{z}$ {*e.g.* complex $\tilde{p}(\vec{r},t;\omega)$ or $\tilde{u}_z(\vec{r},t;\omega)$ } is that component of $\tilde{z}$ which is in-phase (90° out-of-phase) with the {purely ***real***} ***reference*** signal {$V_{FG}(\omega) = V_{FG}^{o}\cos\omega t$ }, respectively.

Since $\tilde{z} = x + iy = |\tilde{z}|e^{i\varphi_z}$, we can write the (purely real amplitude)·(complex overall phase) amplitude products in the above ***time***-***domain*** expressions for ***complex*** ***instantaneous*** over-pressure $\tilde{p}(\vec{r},t;\omega)$ and 1-D particle velocity $\tilde{u}_z(\vec{r},t;\omega)$ as ***complex*** ***amplitudes***:

$$\boxed{\tilde{p}_0(\vec{r},\omega) = p_0(\vec{r},\omega)\cdot e^{i\varphi_p(\vec{r},\omega)} = p_0(\vec{r},\omega)\cdot e^{i\left[\varphi_p^o(\omega) - k_z(\vec{r},\omega)z\right]}}$$

$$\boxed{\tilde{u}_{0_z}(\vec{r},\omega) = u_{0_z}(\vec{r},\omega)\cdot e^{i\varphi_{u_z}(\vec{r},\omega)} = u_{0_z}(\vec{r},\omega)\cdot e^{i\left[\varphi_{u_z}^o(\omega) - k_z(\vec{r},\omega)z\right]}}$$

Note that the above ***complex*** ***amplitudes*** for over-pressure and 1-D particle velocity are ***time-independent*** {for a constant-amplitude sound source}, and in fact are none other than the ***frequency***-***domain*** representations of the ***time***-***domain*** expressions for ***complex*** ***instantaneous*** over-pressure $\tilde{p}(\vec{r},t;\omega)$ and 1-D particle velocity $\tilde{u}_z(\vec{r},t;\omega)$!

As discussed in Physics 406 Lecture Notes XIII – Part 2, the complex ***frequency***-***domain*** *vs.* complex ***time***-***domain*** representations of acoustical quantities such as complex over-pressure $\tilde{p}$ and/or complex particle velocity $\tilde{u}$ are related to each other by ***Fourier*** ***transforms*** of each other.

For any ***continuous*** complex ***time-domain*** function $\tilde{f}(t)$:

The Fourier transform of $\tilde{f}(t)$ to the frequency domain is: $\boxed{\tilde{f}(\omega) \equiv \int_{-\infty}^{+\infty} \tilde{f}(t) e^{-i\omega t} dt}$

The inverse Fourier transform of $\tilde{f}(\omega)$ to the time domain is: $\boxed{\tilde{f}(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{+i\omega t} d\omega}$

Thus, it should now be clear to the reader that for a ***harmonic/single-frequency sound field***, we can write the ***time-domain complex instantaneous*** over-pressure $p(\vec{r},t;\omega)$ and 1-D particle velocity $u_z(\vec{r},t;\omega)$ at the listener's location $\vec{r} = z\hat{z}$ for arbitrary time *t* in an elegant and compact manner - as the product of the (***complex frequency-domain amplitude***)·(complex $e^{i\omega t}$ factor):

$$\begin{aligned}\tilde{p}(\vec{r},t;\omega) &= \tilde{p}_0(\vec{r},\omega) \cdot e^{i\omega t} \\ &= p_0(\vec{r},\omega) \cdot e^{i\varphi_p(\vec{r},\omega)} \cdot e^{i\omega t} = p_0(\vec{r},\omega) \cdot e^{i\left[\varphi_p^o(\omega) - k_z(\vec{r},\omega)z\right]} \cdot e^{i\omega t} = p_0(\vec{r},\omega) \cdot e^{i\left[\omega t - k_z(\vec{r},\omega)z + \varphi_p^o(\omega)\right]}\end{aligned}$$

$$\begin{aligned}\tilde{u}_z(\vec{r},t;\omega) &= \tilde{u}_{0_z}(\vec{r},\omega) \cdot e^{i\omega t} \\ &= u_{0_z}(\vec{r},\omega) \cdot e^{i\varphi_{u_z}(\vec{r},\omega)} \cdot e^{i\omega t} = u_{0_z}(\vec{r},\omega) \cdot e^{i\left[\varphi_{u_z}^o(\omega) - k_z(\vec{r},\omega)z\right]} \cdot e^{i\omega t} = u_{0_z}(\vec{r},\omega) \cdot e^{i\left[\omega t - k_z(\vec{r},\omega)z + \varphi_{u_z}^o(\omega)\right]}\end{aligned}$$

Sound Field Measurements in Three Dimensions:

The mathematical description and experimental measurement of 3-D sound fields is a straight-forward generalization of the above-discussed 1-D situation. For a monochromatic/ single-frequency 3-D sound field, the ***physical*** {purely ***real***}, ***instantaneous time-domain*** scalar over-pressure $p(\vec{r},t;\omega)$ and 3-D instantaneous vector particle velocity $\vec{u}(\vec{r},t;\omega)$ are, *e.g*. in Cartesian coordinates:

$$\begin{aligned}p(\vec{r},t;\omega) = p_0(\vec{r},\omega)\cos\Theta_p(\vec{r},t;\omega) &= p_0(\vec{r},\omega)\cos\left(\omega t + \varphi_p(\vec{r},\omega)\right) \\ &= p_0(\vec{r},\omega)\cos\left[\omega t - \vec{k}(\vec{r},\omega)\bullet\vec{r} + \varphi_p^o(\omega)\right]\end{aligned}$$

$$\begin{aligned}\vec{u}(\vec{r},t;\omega) &= u_x(\vec{r},t;\omega)\hat{x} + u_y(\vec{r},t;\omega)\hat{y} + u_z(\vec{r},t;\omega)\hat{z} \\ &= u_{0_x}(\vec{r},\omega)\cos\Theta_{u_x}(\vec{r},t;\omega)\hat{x} + u_{0_y}(\vec{r},\omega)\cos\Theta_{u_y}(\vec{r},t;\omega)\hat{y} + u_{0_z}(\vec{r},\omega)\cos\Theta_{u_z}(\vec{r},t;\omega)\hat{z} \\ &= u_{0_x}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\hat{x} + u_{0_y}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_y}(\vec{r},\omega)\right)\hat{y} \\ &\qquad + u_{0_z}(\vec{r},\omega)\cos\left(\omega t + \varphi_{u_z}(\vec{r},\omega)\right)\hat{z} \\ &= u_{0_x}(\vec{r},\omega)\cos\left[\omega t - \vec{k}(\vec{r},\omega)\bullet\vec{r} + \varphi_{u_x}^o(\omega)\right]\hat{x} + u_{0_y}(\vec{r},\omega)\cos\left[\omega t - \vec{k}(\vec{r},\omega)\bullet\vec{r} + \varphi_{u_y}^o(\omega)\right]\hat{y} \\ &\qquad + u_{0_z}(\vec{r},\omega)\cos\left[\omega t - \vec{k}(\vec{r},\omega)\bullet\vec{r} + \varphi_{u_z}^o(\omega)\right]\hat{z}\end{aligned}$$

where the 3-D vector wavenumber $\vec{k}(\vec{r},\omega) = k_x(\vec{r},\omega)\hat{x} + k_y(\vec{r},\omega)\hat{y} + k_z(\vec{r},\omega)\hat{z}$,

with $k = \left|\vec{k}\right| = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

The listener's position vector {referenced to the sound source located at the origin} is $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Thus, the 3-D dot-product factor: $\vec{k}(\vec{r},\omega)\bullet\vec{r} = k_x(\vec{r},\omega)x + k_y(\vec{r},\omega)y + k_z(\vec{r},\omega)z$. Note that $\cos\Theta_x, \cos\Theta_y, \cos\Theta_z$ are the 3-D direction cosines. Then, since $\hat{k}_x\bullet\hat{r} = \cos\Theta_x$, $\hat{k}_y\bullet\hat{r} = \cos\Theta_y$ and $\hat{k}_z\bullet\hat{r} = \cos\Theta_z$ the above 3-D dot-product factor can also equivalently be written as:
$\vec{k}(\vec{r},\omega)\bullet\vec{r} = k_x(\vec{r},\omega)r\cos\Theta_x + k_y(\vec{r},\omega)r\cos\Theta_y + k_z(\vec{r},\omega)r\cos\Theta_z$.

Thus, we see that for a monochromatic/single-frequency 3-D sound field, the ***physical***, ***instantaneous*** ***time***-***domain*** scalar over-pressure $p(\vec{r},t;\omega)$ is similar in form to that for the monochromatic/single-frequency 1-D sound field case, whereas for the ***physical***, ***instantaneous*** ***time***-***domain*** 3-D vector particle velocity $\vec{u}(\vec{r},t;\omega) = u_x(\vec{r},t;\omega)\hat{x} + u_y(\vec{r},t;\omega)\hat{y} + u_z(\vec{r},t;\omega)\hat{z}$, we have three {orthogonal/*x*-*y*-*z*} components to deal with/measure for the 3-D sound field case *vs*. only one component for the 1-D case. Three independent, orthogonally-oriented particle velocity microphones, located at the listener's position $\vec{r}$ are thus needed to measure/completely specify the 3-D vector particle velocity $\vec{u}(\vec{r},t;\omega)$ at that location.

For the monochromatic/single-frequency 3-D sound field we again "***complexify***" the ***physical***, ***instantaneous*** ***time***-***domain*** scalar over-pressure $p(\vec{r},t;\omega)$ and 3-D vector particle velocity $\vec{u}(\vec{r},t;\omega)$ following the above-described prescription for the 1-D sound field case. Then, the relations between ***complex*** ***time***-***domain*** scalar over-pressure $\tilde{p}(\vec{r},t;\omega)$, ***complex*** ***time***-***domain*** 3-D vector particle velocity $\tilde{\vec{u}}(\vec{r},t;\omega)$ and their ***complex*** ***frequency***-***domain*** counterparts, $\tilde{p}(\vec{r},\omega)$ and $\tilde{\vec{u}}(\vec{r},\omega)$ are also elegantly/compactly given by:

$$\boxed{\tilde{p}(\vec{r},t;\omega) = \tilde{p}(\vec{r},\omega)\cdot e^{i\omega t}}$$

$$\boxed{\begin{aligned}\vec{u}(\vec{r},t;\omega) &= u_x(\vec{r},t;\omega)\hat{x} + u_y(\vec{r},t;\omega)\hat{y} + u_z(\vec{r},t;\omega)\hat{z} = \tilde{\vec{u}}(\vec{r},\omega)\cdot e^{i\omega t}\\ &= \tilde{u}_x(\vec{r},\omega)\cdot e^{i\omega t}\hat{x} + \tilde{u}_y(\vec{r},\omega)\cdot e^{i\omega t}\hat{y} + \tilde{u}_z(\vec{r},\omega)\hat{z}\cdot e^{i\omega t}\end{aligned}}$$