

## Theory of Distortion I

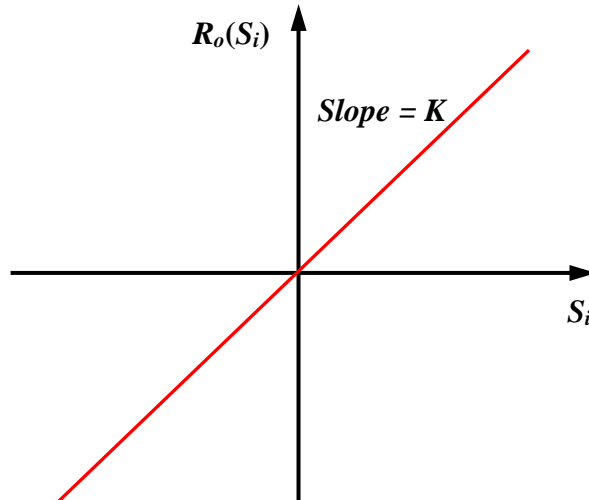
### Output Response of a Purely Linear System

In a wide variety of physical situations, it is highly desirable that the output *response*,  $R_o$  of an arbitrary system to an input *stimulus*,  $S_i$  be *perfectly* linear. Mathematically, a perfectly linear output response to an input stimulus is described by the relation:

$$R_o(S_i) = K S_i$$

where  $K$  is the constant of proportionality of the output response to the input stimulus, i.e.  $K = R_o(S_i) / S_i$ . In a perfectly linear system the output response,  $R_o(S_i)$  is proportionally the same, no matter how large or small the magnitude of the input stimulus,  $S_i$ . Such a linear relation can be described graphically, as shown in the figure below:

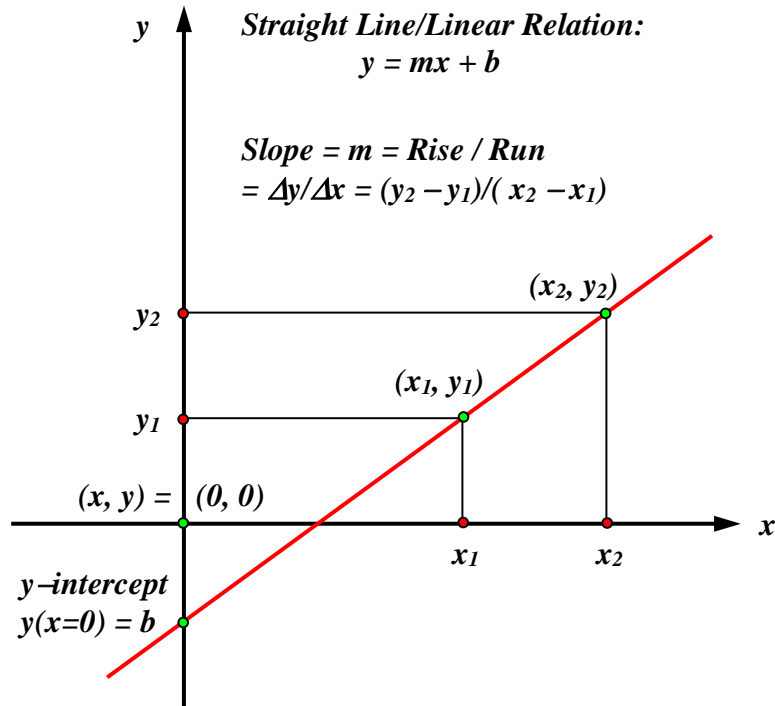
$$R_o(S_i) \text{ vs. } S_i: \quad R_o(S_i) = K S_i$$



Physically, the constant of proportionality,  $K$  is the *slope* of the straight line of the above graph. The quantity on the vertical, or  $y$ -axis is the *dependent* quantity ( $R_o$ ), the quantity on the horizontal, or  $x$ -axis is the *independent* quantity ( $S_i$ ). Thus, generically speaking, for a perfectly linear relationship, most generally we have the equation for a straight line:

$$y(x) = mx + b$$

where the slope of the line is  $m = \text{change in } y / \text{change in } x = \text{“rise”} / \text{“run”} = \Delta y / \Delta x = (y_2 - y_1) / (x_2 - x_1)$ , and the  $y$ -intercept, when  $x = 0$ , is  $y(0) = b$ . Thus, in the above example, with  $R_o(S_i) = K S_i$ , we see that  $b = 0$ . Thus, physically, the output response,  $R_o(S_i = 0)$  is zero when there is no input stimulus ( $S_i = 0$ ). The most general case for a straight line/linear relation is shown in the figure below:



Some simple physical examples of linear relationships are:

1. **Ohm's Law:** Electrical current,  $I$  flowing in a resistor (resistance,  $R$ ) with a potential difference,  $V$  across the resistor:  $I = V/R$  (slope,  $K = 1/R$ ).
2. **Hooke's Law:** Linear displacement (extension/compression),  $x$  from equilibrium position in a spring (spring constant  $k$ ) for a force,  $F$  applied to the spring:  $x = F/k$  (slope,  $K = 1/k$ ).
3. **Fresnel Relations:** Transmission of light (at normal incidence) through a refractive medium, such as glass: Output intensity,  $I_o$  related to input intensity,  $I_i$  via transmission coefficient,  $T$ :  $I_o = T I_i$  (slope,  $K = T$ ).

In each of the above three examples, the linearity of each relationship is in fact only approximately true, even though each relation may be very linear – there are in fact *deviations* from linearity, or *distortions*, in each of the above relationships, when the range of the independent variable becomes extremely large. For example, Ohm's Law does not hold exactly for extremely large (e.g. kilo-volt) potential differences across a resistor - the current increases *faster* than linear in this regime. Hooke's Law for springs is in fact invalid for extremely small forces due to friction in the spring; Hooke's Law is also invalid for very large forces, when the material of the spring is stretched beyond its elastic limit. The transmission of light, e.g. through a piece of glass depends on the wavelength (frequency) of the light - *i.e.* the transmission coefficient,  $T$  is wavelength / frequency dependent - because the index of refraction of the glass is wavelength / frequency dependent. When the intensity of incident light (for a given, or fixed wavelength / frequency of light) becomes *extremely* large (e.g. using a very powerful

laser), the transmission coefficient,  $T$  can also change in a non-linear manner, depending on the details of the nature of the glass!

In fact, in just about *every* physical system one can think of, such systems may be thought as being linear to a certain degree, and in fact may be highly linear. However, if one scrutinizes the system very carefully, one almost always finds that such systems do indeed have some degree of non-linearity, most often for very large values of input stimulus parameters - the independent variable. Some systems may be much more linear than others. Some systems may in fact be very non-linear in their response, or may in fact be linear only for extremely small values of input stimuli, or linear only for extremely small *changes* (i.e. extremely small variations/deviations) from a non-zero, equilibrium value of input stimulus. We will see that a necessary requirement for distortion to occur is that a given system *must* have some form of non-linear response.

In high-end audio (and audio recording) applications, it is highly desirable for all components in the audio system - recording transducers (i.e. microphones), the recording devices and recording media (record, tape, CD), the play-back pre-amplifier, power amplifier and sound transducers (i.e. loudspeakers) to be as linear as possible, over the full audio spectrum, so as not to “color” the sound in any manner – thus providing as faithfully as possible, an accurate, unbiased reproduction of the music as it actually sounded when it was originally recorded.

However, even in high-end audio systems, certain non-linearities of response do indeed exist in each of the components of the system. Some kinds of non-linearities, and their resultant impact on the final sound are undesirable, especially if they are large - they produce “overt” coloration effects on the overall sound output from the audio system which the human ear deems as undesirable, or unpleasing to hear. On the other hand, other types of non-linearities in these systems may be such that they are in fact very pleasing to the ear - coloring the sound in a much more subtle, but characteristic way, such that these non-linearities becomes a highly desirable attribute, uniquely associated with that piece of equipment! While superficially this may at first seem quite odd, it must be kept in mind that the human ear is itself a non-linear audio “device” - the ear adds its own non-linearities to the sound input to it. Thus, it is in fact not surprising that by adding small non-linearities somewhere in the audio reproduction chain, the *perceived* sound may be overall enhanced, relative to that from a purely linear-response system!

Such is indeed the case in vacuum tube amplifiers – the non-linear behavior of vacuum tubes in the preamplifier and power amplifier sections of the amp gives rise to an overall “warmth” to the sound output from such a system, in comparison to that output from a solid-state/transistorized system. Part of the reason vacuum tube amplifiers have a “warmth” to them is due to their overload/transient response characteristics, which is “soft” (i.e. the signal output from the tubes compresses) relative to the “hard” clipping associated with solid-state/transistorized amplifiers, when they reach their limits. However, another reason for the “warmth” of vacuum tube amplifiers is due to their inherently non-linear output response behavior to an input signal.

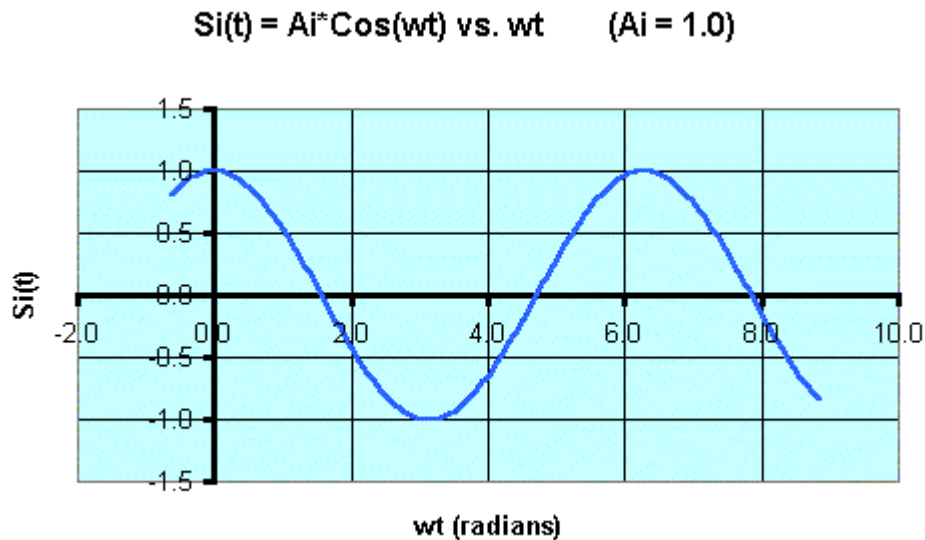
As we have seen above, a perfectly linear system has an output response

$$R_o(S_i) = K S_i$$

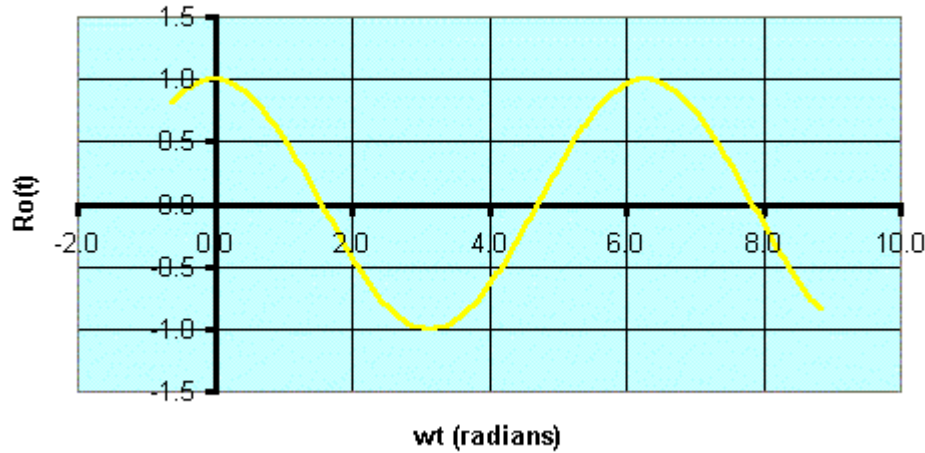
Suppose the input stimulus is a pure tone, i.e. a signal of a single frequency,  $f$  (having units of cycles per second, or Hertz (= Hz)). We define the “angular” frequency as  $\omega \equiv 2\pi f$  (having units of radians per second). Then the input stimulus,  $S_i$  becomes a time-dependent function, i.e.  $S_i(t) = A_i \cos(2\pi f t) = A_i \cos(\omega t)$ , where  $A_i$  is the amplitude of the input stimulus and  $t$  is the time, in seconds, relative to some reference time,  $t = 0$ . Then the output response,  $R_o(S_i)$  also becomes explicitly dependent on time, i.e.

$$R_o(t) = R_o(S_i(t)) = K S_i(t) = K A_i \cos(\omega t) = A_o \cos(\omega t)$$

where  $A_o = K A_i$  is the amplitude of the output response. It can be seen that the output response for a pure tone input stimulus is also a pure tone, for a perfectly linear system. The input stimulus and linear output response waveforms are shown in the figures below, for an elapsed time of  $\Delta t = 0.0015$  seconds, for parameter values of  $f = 1000$  Hz,  $A_i = 1.0$ , and  $A_o = K A_i = 1.0$ . Note that the product,  $\omega t$  has units of radians; thus  $2\pi$  radians =  $360^\circ$ .



$$R_o(t) = A_o \cdot \cos(\omega t) \text{ vs. } \omega t \quad (A_o = K \cdot A_i = 1.0)$$



The output response,  $R_o(S_i(t))$  of this system to a pure-tone input stimulus,  $S_i(t) = A_i \cos(\omega t)$  may be such that it is *not precisely in phase* with the input stimulus. In general, a system may have some *reactance* to the input stimulus, which simply means that there exists a *non-zero phase relation* between the output response, *referenced* to the driving input stimulus. For a system with a linear response,  $R_o(S_i(t)) = K S_i(t)$ , a non-zero phase relation between output response and input stimulus means that the constant of proportionality,  $K$ , is not purely *real*, but is said to be *complex* - i.e. in general,  $K$  has both a *real* component,  $K_r$  and a so-called *imaginary* component,  $K_i$ :

$$K = K_r + iK_i$$

The real component,  $Re(K) = K_r$  is in phase with the driving input stimulus, if  $K_r$  is positive. If  $K_r$  is negative, then  $K_r$  is  $180^\circ$  out of phase with the input stimulus. The imaginary component,  $Im(K) = K_i$  is  $+90^\circ$  ahead in phase of the driving input stimulus if  $K_i$  is positive. If  $K_i$  is negative, this indicates that the imaginary component,  $K_i$  is  $-90^\circ$  behind in phase of the driving input stimulus. The  $i$  in the above formula is what enables us to mathematically describe this phase relation between output response and driving input stimulus; it is defined as  $i \equiv \sqrt{-1}$ . Thus  $i * i = -1$ , and  $i * -i = +1$ .

The actual phase angle relation between the output response and the driving input stimulus is then given by:

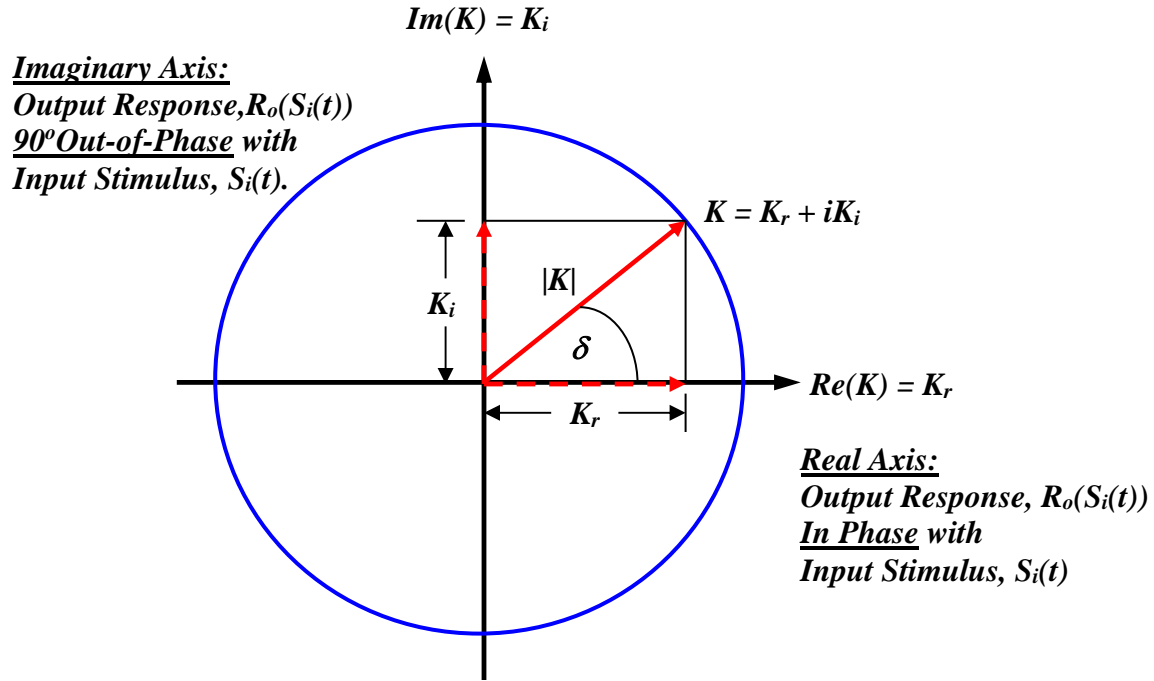
$$K_r = |K| \cos \delta \quad \text{and} \quad K_i = |K| \sin \delta$$

where the *magnitude* of  $K$ , which is a purely real quantity, is given by:

$$|K| \equiv (K K^*)^{1/2} = (K_r^2 + K_i^2)^{1/2} = [(|K| \cos \delta)^2 + (|K| \sin \delta)^2]^{1/2} = |K| [\cos^2 \delta + \sin^2 \delta]^{1/2} = |K|$$

Where  $\delta$  is the *phase angle* between the (reactive) output response,  $R_o(S_i(t))$  and the driving input stimulus,  $S_i(t)$ . The *complex conjugate* of  $K$  is defined as  $K^* \equiv K = K_r - iK_i$ .

We show these complex relations graphically for  $K$ , which can be thought of as a two-dimensional vector lying in the complex plane, as shown in the figure below:



If the in-phase (i.e. real),  $K_r$  and out-of-phase (i.e. imaginary),  $K_i$  components of the (complex)  $K$  are known (e.g. via measurement), then the phase angle,  $\delta$  can be computed from the ratio:

$$\tan \delta = K_i / K_r$$

This relation can also be understood geometrically, from the above figure. If the phase angle,  $\delta$  is *positive* (i.e.  $Im(K)$  is *positive, above* the real axis) then the output response,  $R_o(S_i(t))$  is said to lead the input stimulus,  $S_i(t)$  by the phase angle,  $\delta$ . If the phase angle,  $\delta$  is *negative* (i.e.  $Im(K)$  is *negative, below* the real axis), then the output response,  $R_o(S_i(t))$  is said to lag the input stimulus,  $S_i(t)$  by the phase angle,  $\delta$ .

Alternatively, we can instead use so-called complex notation to mathematically equivalently describe a reactive output response (i.e. a non-zero phase relation between output response and driving input stimulus). Noting that:

$$\exp(+i\theta) = e^{+i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad \exp(-i\theta) = e^{-i\theta} = \cos \theta - i \sin \theta$$

one can show:

$$\cos \theta = \frac{1}{2} (e^{+i\theta} + e^{-i\theta}) \quad \text{and} \quad i \sin \theta = \frac{1}{2} (e^{+i\theta} - e^{-i\theta})$$

Then:

$$K = K_r + iK_i = |K| (\cos \delta + i \sin \delta) = |K| e^{i\delta}$$

Then a generalized, possibly reactive, but linear output response,  $R_o(S_i(t))$  to a driving, pure-tone input stimulus,  $S_i(t)$  can be written as:

$$R_o(S_i(t)) = K S_i(t) = |K| e^{i\delta} S_i(t)$$

Note that the constant of proportionality,  $K$  and the phase angle,  $\delta$  may in fact be frequency dependent, i.e.  $K = K(\omega)$  and  $\delta = \delta(\omega)$ . Thus, most generally, the output response can also have frequency dependence:

$$R_o(S_i(t), \omega) = K(\omega) S_i(t) = |K(\omega)| e^{i\delta(\omega)} S_i(t)$$

However, note that a *given*, (i.e. fixed) frequency,  $K(\omega)$  and  $\delta(\omega)$  behave as constants.

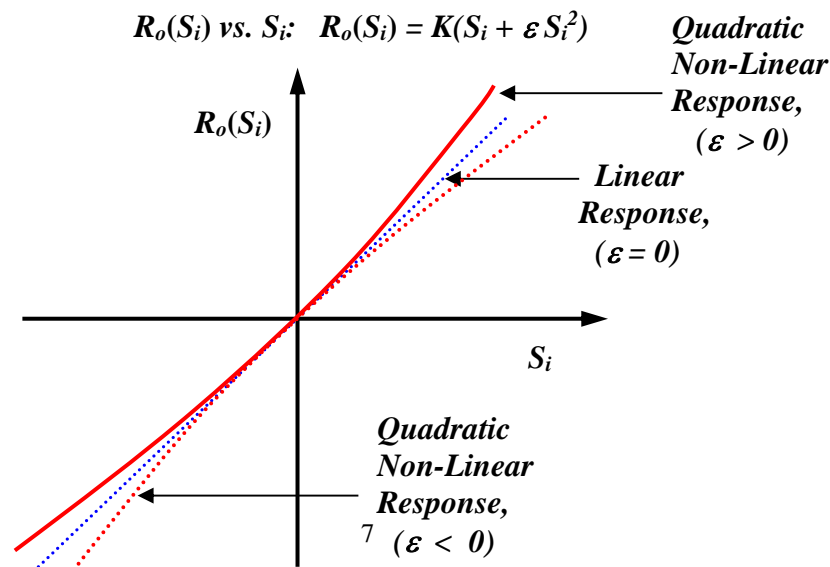
In the above, and following discussions on various systems with either linear or non-linear output responses to e.g. pure-tone input stimuli, we will discuss these systems as having a purely real, (i.e. in-phase) response. However, from the above, it can be seen that it is a straight-forward extension to more generally allow a reactive output response to an input driving stimulus.

### Output Response of a System with a Quadratic Non-Linearity

Now let us consider a generic system with a slightly non-linear, quadratic response:

$$R_o(S_i) = K(S_i + \varepsilon S_i^2) = K S_i (1 + \varepsilon S_i) \quad (|\varepsilon S_i| \ll 1)$$

The non-linearity parameter,  $\varepsilon$ , which has dimensions (i.e. units) of  $1/S_i$ , is assumed to be small, i.e.  $|\varepsilon S_i| \ll 1$  (the non-linearity parameter,  $\varepsilon$  can be either positive or negative, but here  $\varepsilon$  must such that it is very much less in *magnitude* than one). Thus, in addition to the *linear* response term,  $K S_i$  there is also now a small, non-linear *quadratic* response term,  $\varepsilon K S_i^2$ . Note that the quadratic response term,  $\varepsilon K S_i^2$  is the *lowest-order possible deviation* from a purely linear response,  $R_o(S_i) = K S_i$ . The overall output response,  $R_o(S_i)$  as a function of the input stimulus,  $S_i$  for this (non-linear) system is shown in the figure below:



Let us again assume that the input stimulus is a pure tone, i.e. a signal of a single frequency,  $f$ . Then again, the input stimulus,  $S_i(t) = A_i \cos(\omega t)$ . Then the output response,  $R_o(S_i)$  again also becomes explicitly dependent on time, i.e.

$$\begin{aligned} R_o(t) &= R_o(S_i(t)) = K(S_i(t) + \varepsilon S_i^2(t)) = K S_i(t) + \varepsilon K S_i^2(t) \\ &= K A_i \cos(\omega t) + \varepsilon K A_i^2 \cos^2(\omega t) \end{aligned}$$

Now, using the trigonometric identity:

$$\cos^2 \theta = \cos \theta * \cos \theta = \frac{1}{2} (\cos 0 + \cos 2\theta) = \frac{1}{2} (1 + \cos 2\theta)$$

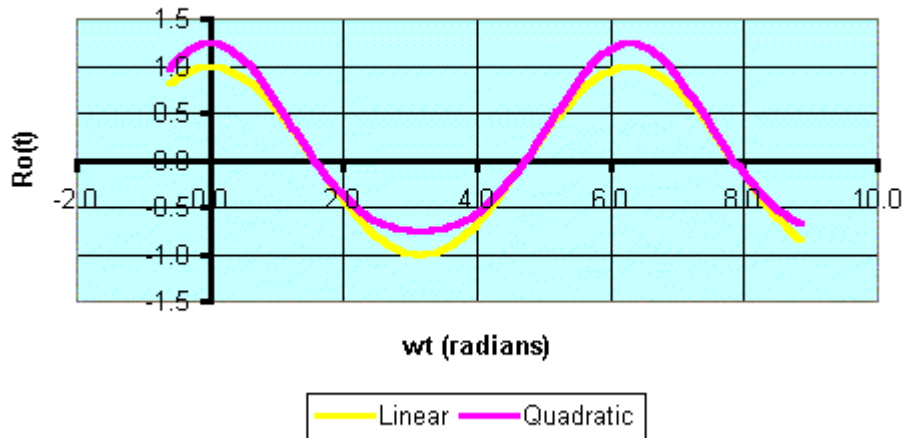
we have:

$$R_o(t) = \frac{1}{2} \varepsilon K A_i^2 + K A_i \cos(\omega t) + \frac{1}{2} \varepsilon K A_i^2 \cos(2\omega t)$$

Thus, for this kind of quadratic non-linear response to a pure input tone of frequency  $f$ , the output response not only has a component at the fundamental frequency,  $f$  (also known as the *first harmonic*), but with amplitude,  $K A_i$  that was present at the input (with amplitude  $A_i$ ), but the output response also has a small *second harmonic* component, with amplitude  $\frac{1}{2} \varepsilon K A_i^2$ , due to the existence of the  $\cos(2\omega t)$  (i.e.  $2f$ ) term! In addition, the output response also has a *shift* in its average, or d.c. value, due to the constant term, of amplitude  $\frac{1}{2} \varepsilon K A_i^2$ . The process of producing a shift in the average value of the output response from its linear-response value is known as *rectification*.

We show a comparison of the linear vs. quadratic non-linear output response waveforms in the figure below, for an elapsed time of  $\Delta t = 0.0015$  seconds, for parameter values of  $f = 1000$  Hz,  $A_i = K = 1.0$ , and a relatively large value of the non-linearity parameter,  $\varepsilon = +0.25$ , so as to exaggerate the effect of the non-linearity term, to make it easily visible on the graph.

### Comparison of Linear vs. Non-Linear Responses

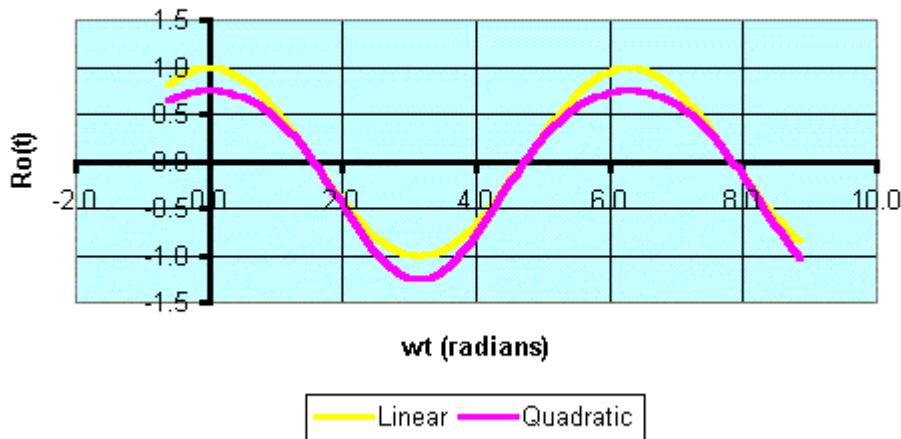




The output response waveform for this system is no longer a pure cosine function. It is more sharply peaked at the top and flatter at the bottom than the pure cosine input waveform (note that the  $\cos^2(\omega t)$  term is always positive, adding both when  $\cos(\omega t)$  is positive and also when  $\cos(\omega t)$  is negative). Thus, the output waveform is *distorted* from the pure-tone input waveform, due to the non-linear response. Such a distorted output waveform, for a quadratic non-linearity has, in addition to the pure tone fundamental, a second harmonic component and also a d.c. offset/zero-frequency component!

The following plot shows the same comparison, except for *reversing* the sign on the nonlinearity parameter, i.e.  $\epsilon = -0.25$ . Comparing this plot with the one immediately above, for which  $\epsilon = +0.25$ , one observes that flipping the sign of the non-linearity parameter simply results in *shifting the phase* of the second harmonic component by  $180^\circ$  relative to the fundamental. Note that the human ear is *not* sensitive to the *relative phases* of one musical tone to another.

**Comparison of Linear vs. Non-Linear Responses**



The *ratio* of amplitudes for the second harmonic (i.e.  $2f$ ) component to the fundamental (i.e.  $f$ ) component of the output response waveform,  $R_o(S_i(t))$  is:

$$\text{Amplitude of } 2^{\text{nd}} \text{ harmonic} / \text{Amplitude of fundamental} = (|\frac{1}{2} \epsilon K A_i^2|) / |K A_i| = \frac{1}{2} |\epsilon A_i|$$

Thus, the  $2^{\text{nd}}$  harmonic *fraction*, relative to the fundamental component of the output response waveform,  $R_o(S_i(t))$  increases *linearly* with the pure-tone input amplitude,  $A_i$  of the input response stimulus,  $S_i(t)$ . This amplitude ratio, or fraction is often referred to as the harmonic distortion content, and usually expressed in per cent (%).

However, if this output response “signal” is e.g. output through a loudspeaker, converting it to sound, the human ear perceives the *loudness*,  $L$  of this sound (units of *deci-Bels*, abbreviated as *dB*) which is *logarithmically* proportional to the *intensity*,  $I$ ) of

the sound wave (units of  $Watts/m^2$ ), which in turn is linearly proportional power,  $P$  of the sound wave (units of  $Watts$ ), which in turn is proportional to the *square* of the output response amplitude,  $R_o(S_i(t))$ , i.e.

$$\text{Loudness, } L \equiv 10 \log_{10}(I / I_o) \quad (\text{units} = \text{deci-Bels, dB})$$

$$\text{Intensity, } I \text{ (Watts/m}^2) \propto \text{Power, } P \text{ (Watts)} \propto \{\text{Output Response, } R_o(S_i(t))\}^2$$

The threshold of human hearing - i.e. the faintest possible sound that is detectable as such by the (average) human ear is defined as *Loudness*,  $L \equiv 0 \text{ dB}$ , which corresponds to a sound intensity,  $I_o$  associated with the threshold of human hearing of  $I_o = 10^{-12} \text{ Watts/m}^2$ .

If the loudness of the fundamental tone is  $L_{\text{fund}} = 60 \text{ dB}$  ( $100 \text{ dB}$ ), this corresponds to an intensity associated with the fundamental tone of  $I_{\text{fund}} = 10^{-6}$  ( $10^{-2}$ )  $Watts/m^2$ , respectively. Then if the ratio of amplitudes for the second harmonic component to the fundamental component of the output response waveform,  $R_o(S_i(t))$  is e.g.

$\frac{1}{2} |\varepsilon A_i| = \frac{1}{2} * 0.25 = 0.125 = 12.5\%$ , then  $I_{2\text{nd}} / I_{\text{fund}} = (\frac{1}{2} \varepsilon A_i)^2$ , and the terms:

$$\log_{10}(I_{2\text{nd}} / I_{\text{fund}}) = \log_{10}(\frac{1}{2} \varepsilon A_i)^2 = 2 \log_{10}(\frac{1}{2} \varepsilon A_i) = 2 \log_{10}(0.125) = -1.806$$

and

$$\log_{10}(I_{\text{fund}} / I_o) = 6 \text{ (10)} \quad \text{for } I_{\text{fund}} = 10^{-6} \text{ (} 10^{-2} \text{) } \text{Watts/m}^2, \text{ respectively.}$$

Thus, the human ear will perceive the loudness,  $L_{2\text{nd}}$  of the  $2^{\text{nd}}$  harmonic component of the output response, relative to perceived loudness,  $L_{\text{fund}}$  of the fundamental component of the output response, as heard e.g. through a loudspeaker as:

$$\begin{aligned} L_{2\text{nd}} / L_{\text{fund}} &= 10 \log_{10}(I_{2\text{nd}} / I_o) / 10 \log_{10}(I_{\text{fund}} / I_o) = \log_{10}(I_{2\text{nd}} / I_o) / \log_{10}(I_{\text{fund}} / I_o) \\ &= \log_{10}[(I_{2\text{nd}} / I_{\text{fund}}) * (I_{\text{fund}} / I_o)] / \log_{10}(I_{\text{fund}} / I_o) \\ &= [\log_{10}(I_{2\text{nd}} / I_{\text{fund}}) + \log_{10}(I_{\text{fund}} / I_o)] / \log_{10}(I_{\text{fund}} / I_o) \\ &= \{\log_{10}(I_{2\text{nd}} / I_{\text{fund}}) / \log_{10}(I_{\text{fund}} / I_o)\} + 1 \\ &= 1 + \{\log_{10}(I_{2\text{nd}} / I_{\text{fund}}) / \log_{10}(I_{\text{fund}} / I_o)\} \\ &= 1 - \{1.806 / 6\} \quad (= 1 - \{1.806 / 10\}) \\ &= 69.9\% \quad (= 81.9\%) \end{aligned}$$

for  $I_{\text{fund}} = 10^{-6}$  ( $10^{-2}$ )  $Watts/m^2$ , respectively. This is the (fractional) amount of second harmonic distortion, as heard by the human ear for this system. Note that the ratio,  $L_{2\text{nd}} / L_{\text{fund}}$  increases (logarithmically) with increasing input amplitude,  $A_i$  - it is *not* a constant! In other words, for a loudness of the fundamental tone of  $L_{\text{fund}} = 60 \text{ dB}$  ( $100 \text{ dB}$ ), the loudness of the second harmonic, for  $\frac{1}{2} |\varepsilon A_i| = 12.5\%$  is:

$$\begin{aligned} L_{2\text{nd}} &= 10 \log_{10}(I_{2\text{nd}} / I_o) = 10 \log_{10}[(I_{2\text{nd}} / I_{\text{fund}}) * (I_{\text{fund}} / I_o)] \\ &= 10 \log_{10}(I_{2\text{nd}} / I_{\text{fund}}) + 10 \log_{10}(I_{\text{fund}} / I_o) \\ &= 20 \log_{10}(0.125) + 60 \text{ dB (100 dB)} \\ &= -18.06 \text{ dB} + 60 \text{ dB (100 dB)} \\ &= 41.94 \text{ dB (81.94 dB)}, \text{ respectively.} \end{aligned}$$

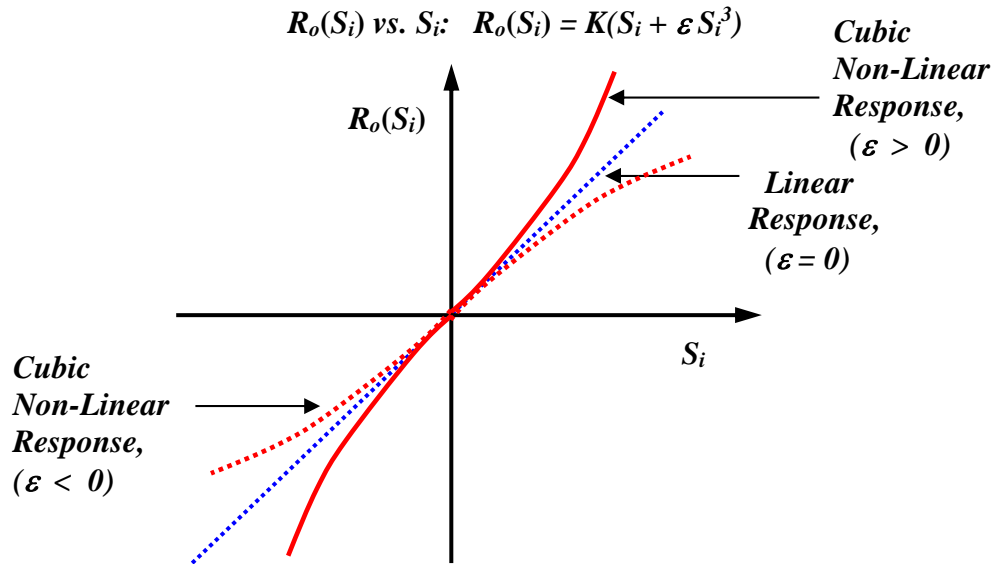
Thus, a value of the ratio of amplitudes for the second harmonic component to the fundamental component of the output response waveform,  $R_o(S_i(t))$ , of  $\frac{1}{2} |\varepsilon A_i| = 12.5\%$  is perceived by the human ear as *extremely* “rich” in second harmonic content. The human ear is capable of detecting quite small harmonic overtone components, where  $\frac{1}{2} |\varepsilon A_i| \sim 0.5\%$  (or less) because these (still) correspond to rather large values of the ratio  $L_{2nd}/L_{fund} \sim 25\%$  ( $\sim 55\%$ ), for  $L_{fund} = 60\text{ dB}$  ( $100\text{ dB}$ ), respectively!

**Output Response of a System with a Cubic Non-Linearity**

As another example, we can consider a generic system with a (purely) *cubic* non-linear response:

$$R_o(S_i) = K (S_i + \varepsilon S_i^3) = K S_i (1 + \varepsilon S_i^2) \quad (|\varepsilon S_i^2| \ll 1)$$

where again, the non-linearity parameter,  $\varepsilon$ , which (here) has units of  $1/S_i^2$ , is assumed to be small, i.e.  $|\varepsilon S_i^2| \ll 1$ . In addition to the *linear* response term,  $K S_i$  there is also now a small, *cubic* non-linear response term,  $\varepsilon K S_i^3$ . The cubic non-linear response term,  $\varepsilon K S_i^3$  is the *next-to-lowest* order possible deviation from a purely linear response,  $R_o(S_i) = K S_i$ , after the quadratic non-linear response term,  $\varepsilon K S_i^2$ . The overall output response,  $R_o(S_i)$  as a function of the input stimulus,  $S_i$  for the cubic non-linear response of system is shown in the figure below:



Assuming again an input stimulus that is a pure tone, i.e. a signal of a single frequency,  $f$  the input stimulus,  $S_i(t) = A_i \cos(\omega t)$ . Then the output response,  $R_o(S_i)$  again also becomes explicitly dependent on time, i.e.

$$\begin{aligned} R_o(t) &= R_o(S_i(t)) = K(S_i(t) + \varepsilon S_i^3(t)) = K S_i(t) + \varepsilon K S_i^3(t) \\ &= K A_i \cos(\omega t) + \varepsilon K A_i^3 \cos^3(\omega t) \end{aligned}$$

Now, we can write the  $\cos^3\theta$  term as:

$$\cos^3\theta = \cos\theta * \cos^2\theta = \cos\theta * \frac{1}{2}(1 + \cos 2\theta) = \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta * \cos 2\theta$$

thus:

$$R_o(t) = K A_i (1 + \frac{1}{2}\varepsilon A_i^2) \cos(\omega t) + \frac{1}{2}\varepsilon K A_i^3 \cos(\omega t) * \cos(2\omega t)$$

Noting that the cosine function is an *even* function, i.e. that  $\cos(-\beta) = +\cos\beta$ , we can write the  $\cos\theta * \cos 2\theta$  term as:

$$\begin{aligned} \cos\theta * \cos 2\theta &= \frac{1}{2} [\cos(\theta - 2\theta) + \cos(\theta + 2\theta)] = \frac{1}{2} [\cos(2\theta - \theta) + \cos(2\theta + \theta)] \\ &= \frac{1}{2} [\cos\theta + \cos 3\theta] \end{aligned}$$

Physically, the  $\cos(\omega t) * \cos(2\omega t) = \frac{1}{2} [\cos(\omega t) + \cos(3\omega t)]$  term can be thought of as the *modulation* of a wave of fundamental frequency,  $f$  by another wave having twice the frequency,  $2f$  resulting in an output wave which is a linear combination of two waves, one with a frequency which is the *sum* of the two original frequencies ( $f + 2f = 3f$ ), the other with a frequency which is the *difference* of the two frequencies,  $(2f - f) = f$ !

Putting this altogether, the  $\cos^3\theta$  term is

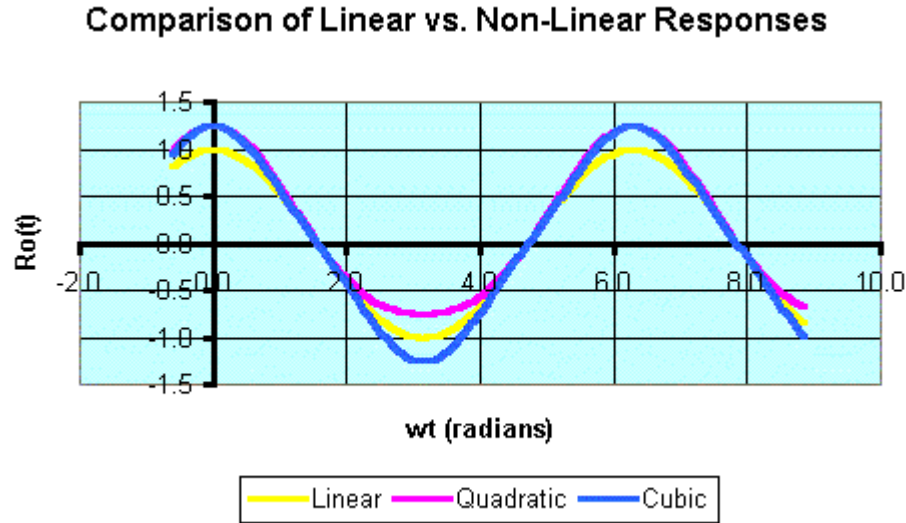
$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$$

Using this result, the output response,  $R_o(t) = K A_i \cos(\omega t) + \varepsilon K A_i^3 \cos^3(\omega t)$  becomes:

$$R_o(t) = K A_i (1 + \frac{3}{4}\varepsilon A_i^2) \cos(\omega t) + \frac{1}{4}\varepsilon K A_i^3 \cos(3\omega t)$$

Thus, for this kind of cubic non-linear response to a pure input tone of frequency  $f$ , the output response has a component at the fundamental frequency, that was present at the input, but the output response also has some of the *third* harmonic, due to the existence of the  $\cos(3\omega t)$  term! Note here, that there is no shift in its average, or d.c. value, due to the absence of a constant term. Thus, we can now also understand, for the previous example of a quadratic non-linear response, why the 2<sup>nd</sup> harmonic and d.c. terms arose – they are the *sum* ( $f + f = 2f$ ) and *difference* ( $f - f = 0f$ ) frequencies associated with the  $\cos(\omega t) * \cos(\omega t) = \frac{1}{2} [\cos(2\omega t) + \cos(0\omega t)]$  term!

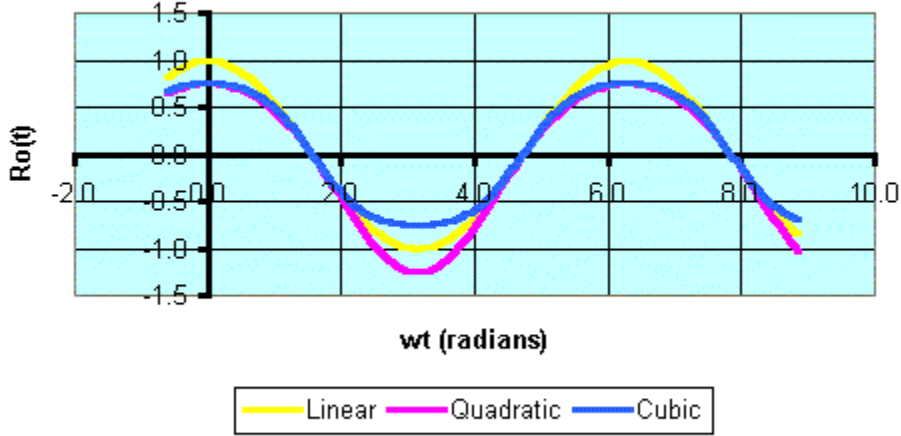
We show a comparison of the linear vs. quadratic and cubic non-linear output response waveforms in the figure below, for an elapsed time of  $\Delta t = 0.0015$  seconds, for parameter values of  $f = 1000 \text{ Hz}$ ,  $A_i = K = 1.0$ , and a relatively large value of the non-linearity parameter,  $\varepsilon = +0.25$ , so as to exaggerate the effect of the non-linearity term, to make it easily visible on the graph.



As we saw for the case of a quadratic non-linearity, the cubic non-linearity output response waveform is no longer a pure cosine function. It is more sharply peaked at *both* the top and bottom than the pure cosine input waveform (note that the  $\cos^3(\omega t)$  term has the *same* sign as the  $\cos(\omega t)$  term, adding when  $\cos(\omega t)$  is positive and subtracting when  $\cos(\omega t)$  is negative). Thus, this output response waveform is also distorted from the input waveform, due to the cubic non-linear response. Such a distorted output waveform, for a cubic non-linearity, has, in addition to the pure tone of the fundamental, a third harmonic (i.e.  $3f$ ) component.

The following plot shows the same comparison, except for reversing the sign on the nonlinearity parameter, i.e.  $\varepsilon = -0.25$ .

### Comparison of Linear vs. Non-Linear Responses



Comparing this plot with the one immediately above, for which  $\varepsilon = +0.25$ , one observes, just as we saw for the quadratic nonlinearity case, that flipping the sign of the cubic non-linearity parameter results in *shifting the phase* of the third harmonic component by  $180^\circ$  relative to the fundamental. But here, for  $\varepsilon = -0.25$ , the cubic non-linear response term also has the consequence of *flattening both* of the + and - peaks of the waveform, rather than *sharpening both* them, as for the cubic non-linear case with  $\varepsilon = +0.25$ ! Note that the cubic non-linear output response,  $R_o(t)$  for the term associated with the fundamental tone, has an amplitude of  $K A_i (1 + \frac{3}{4}\varepsilon A_i^2)$ . Thus, the output response amplitude associated with the *fundamental* component depends on the sign of the non-linearity parameter,  $\varepsilon$  for this system! Thus, even though the human ear is not sensitive to the relative phase of one musical tone to another, this cubic non-linear response waveform, for  $\varepsilon = -0.25$  will not sound the same as that for a cubic non-linear response waveform, for  $\varepsilon = +0.25$ , because the amplitudes of the fundamental components are *not* the same in both cases!

This can also be seen from the *ratio* of amplitudes for the third harmonic (i.e.  $3f$ ) component to the fundamental (i.e.  $f$ ) component of the output response waveform,  $R_o(S_i(t))$ , which is:

$$\begin{aligned} \text{Amplitude of } 3^{\text{rd}} \text{ harmonic} / \text{Amplitude of fundamental} &= \left| \frac{1}{4}\varepsilon K A_i^3 \right| / \left| K A_i (1 + \frac{3}{4}\varepsilon A_i^2) \right| \\ &= \frac{1}{4} |\varepsilon A_i^2| / |1 + \frac{3}{4}\varepsilon A_i^2| = |\varepsilon A_i^2| / |4 + 3\varepsilon A_i^2| \end{aligned}$$

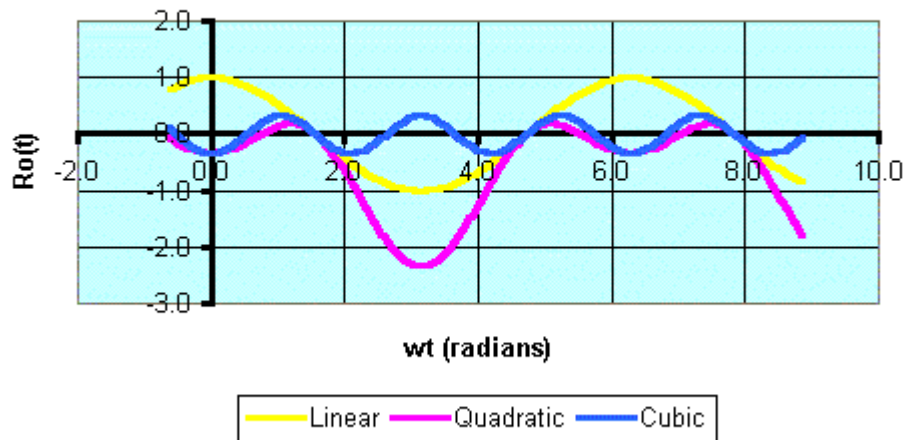
For  $\varepsilon > 0$ , the  $3^{\text{rd}}$  harmonic *fraction*, relative to the fundamental component of the cubic non-linear output response waveform,  $R_o(S_i(t))$  increases  $\sim$  *linearly* with small values of the pure-tone input amplitude,  $A_i$  of the input response stimulus,  $S_i(t)$ . Note

however, that as  $A_i$  becomes extremely large, this ratio *asymptotically* reaches a value of 33.3% - this ratio *cannot* exceed this value, for any value of  $\varepsilon > 0$  and/or for any value of input amplitude  $A_i$ .

However, for the case where  $\varepsilon < 0$ , when the quantity  $^{3/4}\varepsilon A_i^2 = -1$ , i.e. when  $\varepsilon A_i^2 = -^{4/3}$ , the output response amplitude associated with the fundamental,  $K A_i (1 + ^{3/4}\varepsilon A_i^2)$  *vanishes*, and this ratio becomes *infinite* - i.e. *only* the third harmonic is heard by the human ear, when  $\varepsilon A_i^2 = -^{4/3}$  for this system! This is an example of totally destructive interference (i.e. cancellation), at the *amplitude* level. When  $\varepsilon A_i^2 = -^{4/3}$ , a *zero* in the output response amplitude for the fundamental occurs. Thus, for such a cubically non-linear system, if one inputs a large amplitude stimulus at frequency,  $f$  such that  $\varepsilon A_i^2 = -^{4/3}$  then the signal output from the system will be *entirely* at  $3f$ ! Such a system is known as a *frequency tripler*.

In the figure below, we show a comparison of the linear vs. quadratic and cubic non-linear output response waveforms, for an elapsed time of  $\Delta t = 0.0015$  seconds, for parameter values of  $f = 1000$  Hz,  $A_i = K = 1.0$ , and a value of the non-linearity parameter,  $\varepsilon = -^{4/3}$ , i.e. for the case when  $\varepsilon A_i^2 = -^{4/3}$ , when the cubic non-linear output response amplitude associated with the *fundamental*,  $K A_i (1 + ^{3/4}\varepsilon A_i^2)$  vanishes.

### Comparison of Linear vs. Non-Linear Responses



Note that a system with a value of the non-linearity response parameter,  $\varepsilon = -^{4/3}$ , whether it be for a quadratic or cubic non-linear response term, is such that it is *not* a small deviation from a linear response - it is in fact extremely large! Note also, that for even *more* negative values of the non-linearity parameter than  $\varepsilon = -^{4/3}$ , i.e.  $\varepsilon < -^{4/3}$ , then the cubic non-linear output response amplitude associated with the fundamental,  $K A_i (1 + ^{3/4}\varepsilon A_i^2)$  does *not* vanish - it “reappears”, and continues to grow in magnitude as  $\varepsilon$  becomes increasingly more negative in value, from its value of  $\varepsilon = -^{4/3}$ .

**Output Response of a System with Higher-Order Non-Linearities**

By following the above methodology, one can also show that non-linear output responses,  $R_o(S_i)$  associated with systems that have purely *quartic* ( $\epsilon K S_i^4$ ), *quintic* ( $\epsilon K S_i^5$ ), and/or higher-order terms (e.g.  $\epsilon K S_i^6$ , etc.) will indeed produce higher harmonics – 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup>, etc. harmonics, respectively, of the fundamental frequency,  $f$  associated with a pure-tone input stimulus of the system! While these cases are more complicated and lengthy to carry out in detail, the rabidly enthusiastic reader can work through them and discover many interesting phenomena associated with each one!

**Output Response of a System with an Exponentially-Growing Non-Linearity**

Now let us consider a more physically realistic system, one with an *exponentially growing* non-linear output response,  $R_o(S_i)$  which we model as follows:

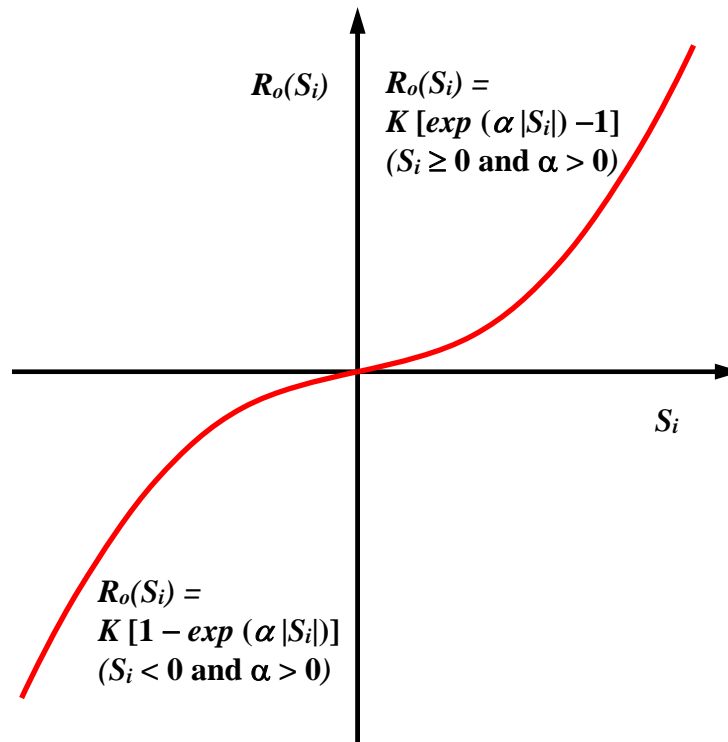
$$R_o(S_i) = K [\exp(\alpha |S_i|) - 1] \quad (\text{for } S_i \geq 0, \text{ and } \alpha > 0)$$

and

$$R_o(S_i) = K [1 - \exp(\alpha |S_i|)] \quad (\text{for } S_i < 0, \text{ and } \alpha > 0)$$

The parameter,  $K$  is (as before) a (positive) constant; the parameter,  $\alpha$  is also a (positive) constant. This type of non-linear response is shown in the figure below:

***Example of an Exponential Non-linear Response:  
Current through a Resistor at Very High Voltages***





A physical example of a system modelled approximately as an exponentially-growing non-linear response is the electrical current flowing through a resistor when extreme voltages ( $|S_i| \gg 1$ ) are applied across it. The resistance of a resistor under such conditions actually depends on the voltage across it - i.e. the resistance becomes voltage dependent (i.e.  $R = R(V)$ ), for very large voltages across the resistor, deviating from the linear  $I = V/R$  relation of Ohm's Law! Another example of such a system is the electrical current flowing through a back-to-back pair of e.g. silicon and/or germanium diodes.

Now, the Taylor series expansion of the exponential function,  $\exp(x) = e^x$  is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{n=\infty} \frac{x^n}{n!}$$

Where the *factorial* function,  $n! \equiv n(n-1)(n-2)(n-3) \dots 3*2*1$ . Thus,  $0! = 1$ ,  $1! = 1$ ,  $2! = 2$ ,  $3! = 6$ ,  $4! = 24$ ,  $5! = 120$ , ... etc. Note then, that for  $|x| \ll 1$ , that the higher-order terms in the Taylor series expansion for  $\exp(x)$  beyond the linear term are quite small. They become increasingly important as  $x$  increases.

The exponentially-growing non-linear response function,  $R_o(S_i)$  as given above, for positive  $S_i$  (i.e.  $S_i \geq 0$ ) can be expanded in a Taylor series:

$$\begin{aligned} R_o(S_i) &= K [\exp(\alpha |S_i|) - 1] = K [(\alpha |S_i|) + (\alpha |S_i|)^2/2! + (\alpha |S_i|)^3/3! + (\alpha |S_i|)^4/4! + \dots] \\ &= \alpha K |S_i| [1 + (\alpha |S_i|)/2! + (\alpha |S_i|)^2/3! + (\alpha |S_i|)^3/4! + (\alpha |S_i|)^4/5! + \dots] \end{aligned}$$

The Taylor series expansion for the exponentially-growing non-linear response function,  $R_o(S_i)$  for negative  $S_i$  (i.e.  $S_i < 0$ ) is:

$$\begin{aligned} R_o(S_i) &= K [1 - \exp(\alpha |S_i|)] = -K [(\alpha |S_i|) + (\alpha |S_i|)^2/2! + (\alpha |S_i|)^3/3! + (\alpha |S_i|)^4/4! + \dots] \\ &= -\alpha K |S_i| [1 + (\alpha |S_i|)/2! + (\alpha |S_i|)^2/3! + (\alpha |S_i|)^3/4! + (\alpha |S_i|)^4/5! + \dots] \end{aligned}$$

Then the Taylor series expansion for the exponentially-growing non-linear response function,  $R_o(S_i)$  valid for any value of  $S_i$  is given by:

$$R_o(S_i) = \alpha K S_i [1 + (\alpha |S_i|)/2! + (\alpha |S_i|)^2/3! + (\alpha |S_i|)^3/4! + (\alpha |S_i|)^4/5! + \dots]$$

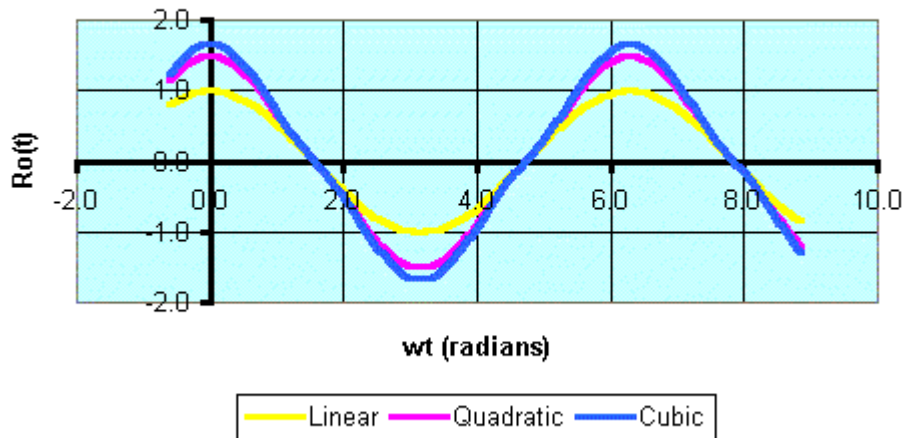
Thus, we see from this Taylor series expansion, that for (very) small values of input stimulus,  $|S_i|$ , the output response is in fact quite linear - the contribution(s) from the higher-order terms are negligibly small. However, for increasingly larger values of  $S_i$ , each of the successive quadratic, cubic, quartic, quintic, etc. higher-order terms becomes increasingly important.

If a pure tone signal,  $S_i(t) = A_i \cos(\omega t)$  is input to this system, then due to the exponentially-growing non-linear nature of the response of this system, the output response,  $R_o(S_i)$  will be dominated by the fundamental tone at frequency,  $f$ , however there will also be (decreasingly important) contributions from higher-order harmonics of the fundamental, at frequencies  $2f, 3f, 4f, 5f, 6f, \dots$  etc. corresponding to each of the higher-order terms in the Taylor series expansion of the exponentially-growing non-linear response function,  $R_o(S_i(t))$ :

$$R_o(t) = R_o(S_i(t)) = \alpha K A_i \cos(\omega t) * [1 + (\alpha A_i |\cos(\omega t)|)/2! + (\alpha A_i |\cos(\omega t)|)^2/3! + (\alpha A_i |\cos(\omega t)|)^3/4! + (\alpha A_i |\cos(\omega t)|)^4/5! + \dots]$$

In the figure below, we show a comparison of the output response waveforms for this exponentially-growing non-linear system, truncating the above Taylor series expansion of the exponential response to linear, linear + quadratic, linear + quadratic + cubic terms, for an elapsed time of  $\Delta t = 0.0015$  seconds, for parameter values of  $f = 1000$  Hz,  $A_i = K = 1.0$ , and a (deliberately chosen, very large) value of the exponential parameter,  $\alpha = 1.0$  (so that differences between the output response waveforms are easily visible).

**Comparison of Linear vs. Non-Linear Responses**



**Output Response of a System with an Exponentially-Decaying Non-Linearity**

We consider here another physically realistic system, one with an *exponentially decaying* non-linear output response,  $R_o(S_i)$  which we model as follows:

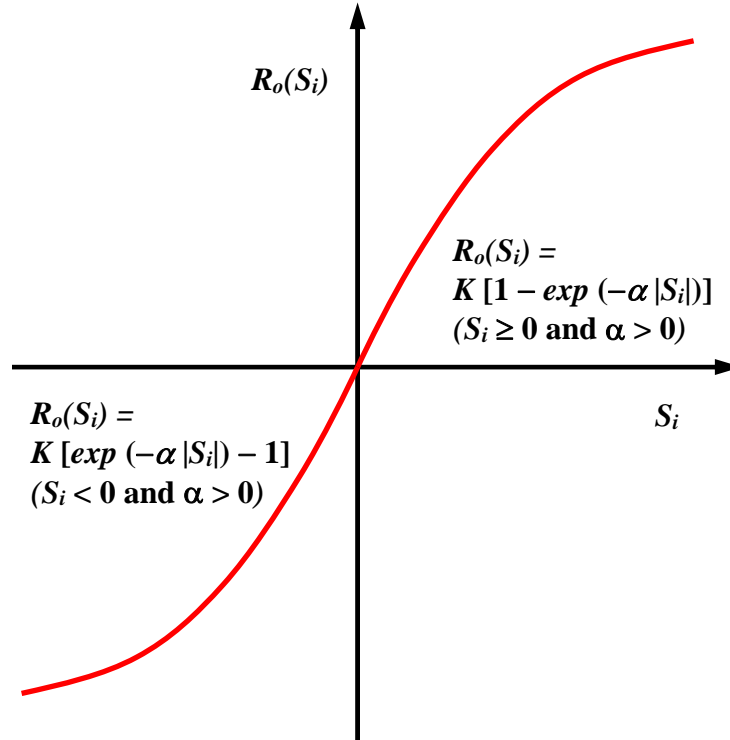
$$R_o(S_i) = K [1 - \exp(-\alpha |S_i|)] \quad (\text{for } S_i \geq 0, \text{ and } \alpha > 0)$$

and

$$R_o(S_i) = K [\exp(-\alpha |S_i|) - 1] \quad (\text{for } S_i < 0, \text{ and } \alpha > 0)$$

The parameter,  $K$  is (as before) a (positive) constant; the parameter,  $\alpha$  is also a (positive) constant. This type of non-linear response is shown in the figure below:

**Example of an Exponential Non-linear Response:  
Overdriven Class-AB Push-Pull Tube Amplifier**



A physical example of a system modelled approximately as exponentially-decaying non-linear response is the electrical current associated with the power tubes flowing through the output transformer of an overdriven class AB, push-pull tube amplifier. Another example of such a system is the electrical current output from an op-amp IC driving e.g. an output load resistance which is too low. Another example is the voltage output from an op-amp IC which has a pair of back-to-back diodes in the feedback loop of the op-amp.

Now, the Taylor series expansion of the exponential function,  $\exp(-x) = e^{-x}$  is:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots = \sum_{n=0}^{n=\infty} \frac{(-1)^n x^n}{n!}$$

The exponentially-decaying non-linear response function,  $R_o(S_i)$  as given above, for positive  $S_i$  (i.e.  $S_i \geq 0$ ) can be expanded in a Taylor series:

$$\begin{aligned} R_o(S_i) &= K [1 - \exp(-\alpha |S_i|)] = K [(\alpha |S_i|) - (\alpha |S_i|)^2/2! + (\alpha |S_i|)^3/3! - (\alpha |S_i|)^4/4! + \dots] \\ &= \alpha K |S_i| [1 - (\alpha |S_i|)/2! + (\alpha |S_i|)^2/3! - (\alpha |S_i|)^3/4! + (\alpha |S_i|)^4/5! + \dots] \end{aligned}$$

The Taylor series expansion for the exponentially-decaying non-linear response function,  $R_o(S_i)$  for negative  $S_i$  (i.e.  $S_i < 0$ ) is:

$$\begin{aligned} R_o(S_i) &= K [\exp(-\alpha |S_i| - 1)] = -K [(\alpha |S_i|) - (\alpha |S_i|)^2/2! + (\alpha |S_i|)^3/3! - (\alpha |S_i|)^4/4! + \dots] \\ &= -\alpha K |S_i| [1 - (\alpha |S_i|)/2! + (\alpha |S_i|)^2/3! - (\alpha |S_i|)^3/4! + (\alpha |S_i|)^4/5! + \dots] \end{aligned}$$

Then the Taylor series expansion for the exponentially-decaying non-linear response function,  $R_o(S_i)$  valid for any value of  $S_i$  is given by:

$$R_o(S_i) = \alpha K S_i [1 - (\alpha |S_i|)/2! + (\alpha |S_i|)^2/3! - (\alpha |S_i|)^3/4! + (\alpha |S_i|)^4/5! + \dots]$$

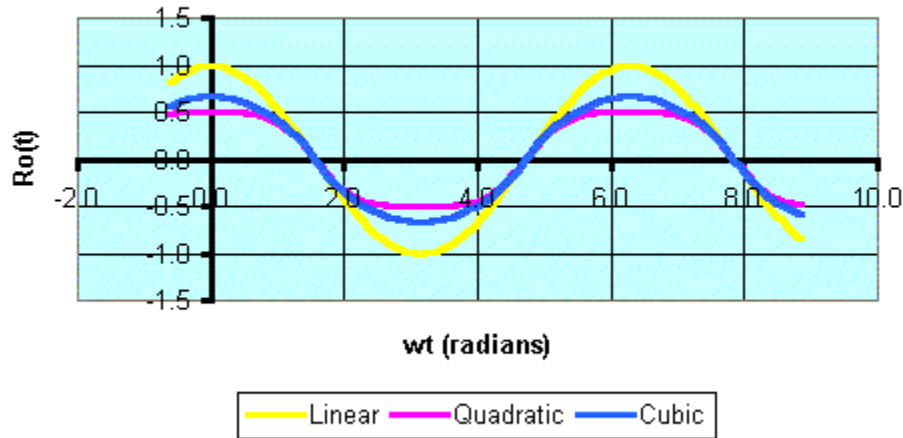
Thus, we (again) see from this Taylor series expansion, that for (very) small values of input stimulus,  $|S_i|$ , the output response is in fact quite linear - the contribution(s) from the higher-order terms are negligibly small. However, for increasingly larger values of  $S_i$ , each of the successive quadratic, cubic, quartic, quintic, etc. higher-order terms becomes increasingly important.

If a pure tone signal,  $S_i(t) = A_i \cos(\omega t)$  is input to this system, then due to the exponentially-decaying non-linear nature of the response of this system, the output response,  $R_o(S_i)$  will be dominated by the fundamental tone at frequency,  $f$ , however there will also be (decreasingly important) contributions from higher-order harmonics of the fundamental, at frequencies  $2f, 3f, 4f, 5f, 6f, \dots$  etc. corresponding to each of the higher-order terms in the Taylor series expansion of the exponentially-decaying non-linear response function,  $R_o(S_i(t))$ :

$$\begin{aligned} R_o(t) = R_o(S_i(t)) &= \alpha K A_i \cos(\omega t) * [1 - (\alpha A_i |\cos(\omega t)|)/2! + (\alpha A_i |\cos(\omega t)|)^2/3! \\ &\quad - (\alpha A_i |\cos(\omega t)|)^3/4! + (\alpha A_i |\cos(\omega t)|)^4/5! + \dots] \end{aligned}$$

In the figure below, we show a comparison of the output response waveforms for this exponentially-decaying non-linear system, truncating the above Taylor series expansion of the exponential response to linear, linear + quadratic, linear + quadratic + cubic terms, for an elapsed time of  $\Delta t = 0.0015$  seconds, for parameter values of  $f = 1000$  Hz,  $A_i = K = 1.0$ , and a (deliberately chosen, very large) value of the exponential parameter,  $\alpha = 1.0$  (so that differences between the output response waveforms are easily visible).

### Comparison of Linear vs. Non-Linear Responses



### Output Response of a System with a Non-Linearity of Arbitrary Functional Form

Any *analytic* function, such as  $\sin(x)$ ,  $\cos(x)$ ,  $\tan(x)$ ,  $\sin^{-1}(x)$ ,  $\cos^{-1}(x)$ ,  $\tan^{-1}(x)$ ,  $\exp(x)$ ,  $\log(x)$ , .... etc., as well as *arbitrary linear combinations* of these analytic functions, can be represented by a Taylor series expansion (with certain restriction(s) on the allowed range(s) of  $x$ , depending on the function). Any good mathematical handbook will give the expressions for the Taylor series expansions of these, and many other analytic functions. Thus, if the non-linear response function,  $R_o(S_i)$  of a system can be modelled using analytic functions, these functions can be expanded in a Taylor series, e.g. to enable the investigation and study of the harmonic content of the non-linear response function,  $R_o(S_i(t))$  that arises as a consequence of deviations from a purely linear response, e.g. to a pure-tone input stimulus.

### Generalized Theory of Distortion

The most general mathematical theory of distortion allows a nearly arbitrary functional form of the non-linear output response  $R_o(S_i)$  – the only restrictions are that the output response,  $R_o(S_i)$  must be *finite*, and a *single-valued* function of the input stimulus,  $S_i$  - i.e. that for any given value of the input stimulus,  $S_i$ , the output response,  $R_o(S_i)$  cannot *simultaneously* have two (or more) values (i.e. take on multiple values). The output response,  $R_o(S_i)$  must also be *piece-wise continuous* - i.e. it can have discrete “jumps” up or down for certain values of the input stimulus,  $S_i$ , but the output response,  $R_o(S_i)$  cannot have “gaps” (i.e. *no* value) for certain values, or a continuous range of values of  $S_i$ . Thus, these restrictions on the behavior of  $R_o(S_i)$  simply mean that the output response,  $R_o(S_i)$  must depend in a physically-sensible, realistic/every-day world-type manner on the input stimulus,  $S_i$ .

For all electronic signal-processing and/or audio signal applications, the mathematical functions we can envision *are* mathematically well-behaved, since the arrow of time points always in the time-increasing direction.

If the output response function,  $R_o(S_i)$  satisfies these mathematical restrictions, then we can perfectly describe such a function as a linear combination of ever-increasing powers of  $S_i$ , with suitably-chosen constant coefficients for each term:

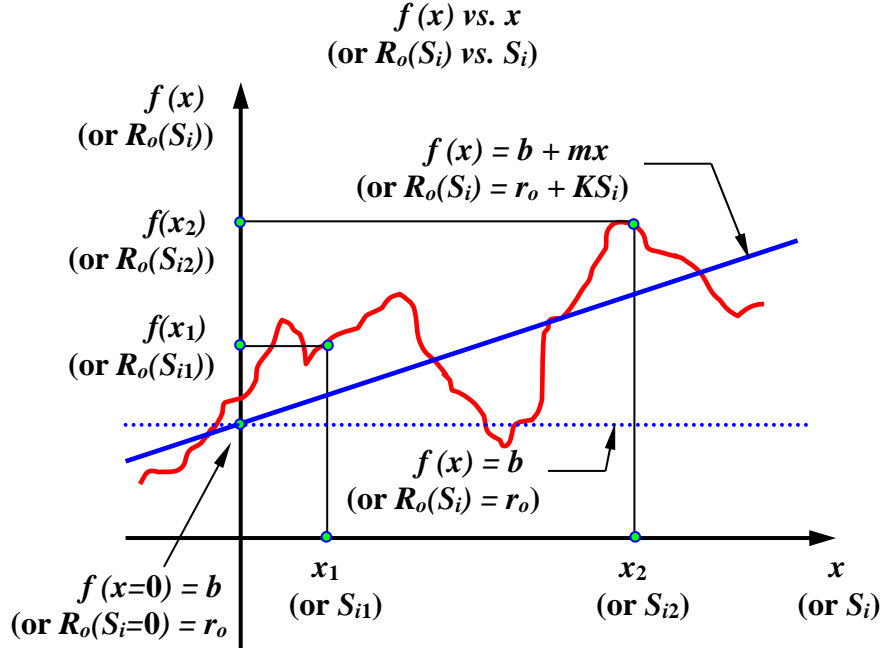
$$R_o(S_i) = C_0S_i^0 + C_1S_i^1 + C_2S_i^2 + C_3S_i^3 + C_4S_i^4 + C_5S_i^5 + C_6S_i^6 + C_7S_i^7 + \dots$$

This is simply a power series expansion of  $R_o(S_i)$ , which we can write compactly as:

$$R_o(S_i) = \sum_{n=0}^{n=\infty} C_n S_i^n$$

This result follows from the fact that for any mathematically well-behaved function,  $f(x)$  which is *finite*, *single-valued* and *piece-wise continuous* over the  $x$ -range of interest (for example, the interval,  $x_1 \leq x \leq x_2$ ), then that function,  $f(x)$ , an example of which is shown in the figure below, is perfectly described by a linear combination of powers of  $x$ , with suitably chosen constant coefficients,  $a_n$ , i.e. a power series expansion, over the interval  $x = x_1$  to  $x = x_2$ :

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + \dots = \sum_{n=0}^{n=\infty} a_n x^n$$



In the above figure, for the example function,  $f(x)$  we have drawn (red curve), we also show, graphically, the  $f(x)$  that results from using only the first term in the power series expansion ( $n = 0$ ), where  $f(x) = b = a_0$  (the d.c. value of  $f(x)$ , dotted blue horizontal line), and using only the first two terms in the power series expansion ( $n = 0$  &  $n = 1$ ), where  $f(x) = b + mx = a_0 + a_1x$  (the straight line/linear relation for  $f(x)$ , blue line). Thus, it becomes evident that higher-order terms beyond the linear term in the power series expansion of  $f(x)$  are what is needed for replicating all of the wiggles in the red curve, where  $f(x)$  deviates from the straight-line relation. In general, the more wiggles there are in  $f(x)$ , the more the higher-order terms,  $x^n$  in the power series expansion of  $f(x)$  will contribute - i.e. the coefficients,  $a_n$  in the power series expansion associated with the higher-order terms,  $x^n$  will be non-zero, and relatively large, in comparison e.g. to a  $f(x)$  which has relatively little, or no wiggles at all.

Mathematically, the reason that *any* as-above-defined well-behaved function,  $f(x)$  over the interval  $x_1 \leq x \leq x_2$ , can be exactly represented by a power-series is due to the fact that the powers of  $x$ , (i.e. the  $x^n$ ) form what is called a complete set of basis vectors in the infinite-dimensional function “space” associated with the interval  $x_1 \leq x \leq x_2$ . Proving this in a rigorous manner, mathematically is quite tedious and involved, and is beyond the scope (and need) of the discussion here. The interested reader is referred to any decent mathematics book that discusses the topic of *vector spaces* in detail.

Operationally, nowadays, one can use computer-based mathematics packages, such as MathCad, Mathematica, Minuit, etc., inputting your function,  $f(x)$ , either as an analytic expression, or, as more often is the case,  $f(x)$  is defined (or known) only as a set of data points on the interval,  $x_1 \leq x \leq x_2$ . One then *approximates* the function,  $f(x)$  by a polynomial of degree,  $n$ ,  $P_n(x)$  which is finite, but can be quite large, e.g. an eighth-order polynomial:

$$P_8(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8$$

The computer program then carries out e.g. a *least-squares fit* of the finite- $n$  polynomial expression representing the function,  $f(x)$  on this interval, minimizing a  $\chi^2$  function on how well the finite- $n$  polynomial expression *matches* the function,  $f(x)$  over the interval  $x_1 \leq x \leq x_2$ , e.g.  $\chi^2 \equiv \int_{x_1}^{x_2} [f(x) - P_n(x)]^2 dx$  for an analytic representation of  $f(x)$ , or e.g.  $\chi^2 \equiv \sum_{i=1}^k [f(x_i) - P_n(x_i)]^2$  for  $f(x)$  represented by a set of ( $i = 1, 2, 3, \dots, k$ ) data points, systematically varying the values of each of the coefficients,  $a_n$  until the “best” (i.e. lowest) minimum  $\chi^2$  is obtained, and then outputting the values of these coefficients,  $a_n$  as determined from the least-squares fitting process. Depending on the detailed nature of the function,  $f(x)$ , in general, a higher-order polynomial,  $P_n(x)$  is required for a good  $\chi^2$  fit result if the function,  $f(x)$  has a lot of wiggles in it, over the interval  $x_1 \leq x \leq x_2$ . If the function,  $f(x)$  is less-wiggly, then usually a lower-order polynomial,  $P_n(x)$  gives an excellent fit  $\chi^2$ .

Thus, the above formalism is applicable for obtaining e.g. an accurate polynomial expression for the response function,  $R_o(S_i)$  associated with a given physical system, to which an input stimulus,  $S_i$  is applied. If an accurate analytic form of the response function,  $R_o(S_i)$  is already known, e.g. from either first principles, if it is a particularly simple response function, or e.g. a least-squares fit to some other analytic expression, such as an exponential relation, then using the above formalism is obviously not necessary. However, in many circumstances an accurate, quantitative mathematical expression for the response function,  $R_o(S_i)$  is not apriori known. It may be extremely complicated. The above formalism is a method for obtaining such an expression.

Even having obtained such an expression for the response function,  $R_o(S_i)$ , this is only the first step, since the graph of the response function,  $R_o(S_i)$  vs. input stimulus,  $S_i$  is (in and of itself) not very useful. We want to *use* this response function,  $R_o(S_i)$  for doing other things - e.g. carrying out detailed computer simulations of this physical system, or e.g. determining the harmonic content of the output response,  $R_o(S_i(t))$  for a pure-tone stimulus, e.g.  $S_i(t) = A_i \cos(\omega t)$  that is input to the physical system, or the response function,  $R_o(S_i(t))$  associated with a more complex input stimulus waveform,  $S_i(t)$ , etc.



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