Theory of Distortion II

Mixing of Two (or More) Signals - Intermodulation Distortion

A. Output Response from a Purely Linear System

If *two* pure tones of *different* frequencies, f_1 and f_2 are simultaneously applied as input stimuli to a system, the overall input stimulus is a linear combination of the two individual input stimuli, $S_{1i}(t)$ and $S_{2i}(t)$:

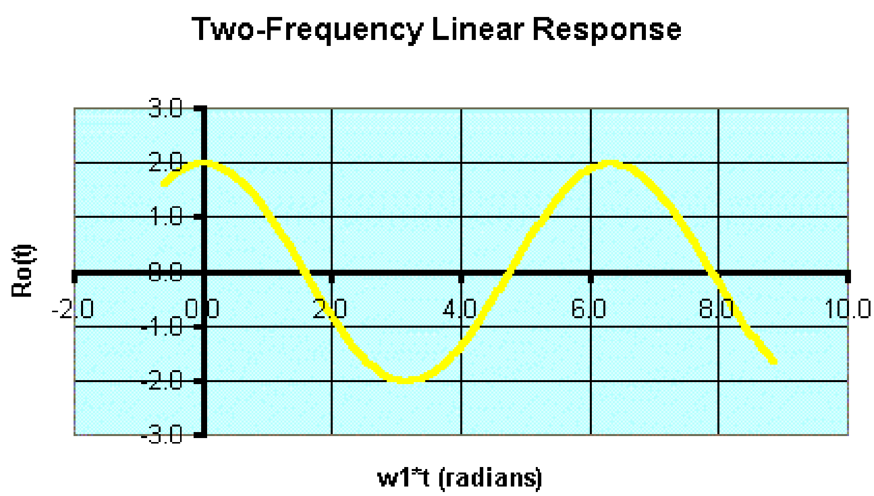
$$S_i(t) = S_{1i}(t) + S_{2i}(t) = A_{1i} \cos(\omega_1 t) + A_{2i} \cos(\omega_2 t)$$

where A_{1i} is the amplitude of input stimulus # 1 and A_{2i} is the amplitude of input stimulus # 2, and $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$. The output response, $R_o(t)$ for a system with a purely linear response, for two pure tones simultaneously applied to the input of such a system is given by:

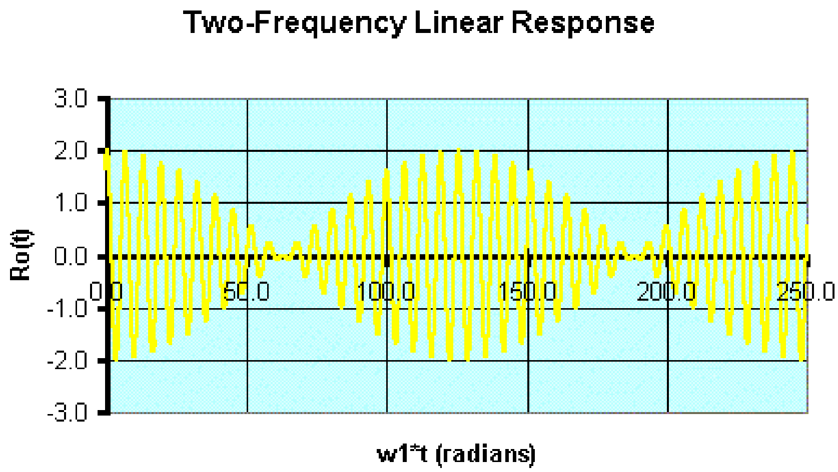
$$\begin{aligned} R_o(t) &= K\, S_i(t) = K\,(S_{1i}(t) + S_{2i}(t)) = K\, S_{1i}(t) + K\, S_{2i}(t) \\ &= K\,(A_{1i} \cos(\omega_1 t) + A_{2i} \cos(\omega_2 t)) = K\, A_{1i} \cos(\omega_1 t) + K\, A_{2i} \cos(\omega_2 t) \\ &= R_{1o}(t) + R_{2o}(t) \end{aligned}$$

Thus, the overall output response of a linear system to two pure tones simultaneously applied to the input of such a system is simply the linear sum of the individual responses to their associated pure input tones - no harmonics, or other frequencies of any kind are generated.

The figure below shows the two-frequency linear response waveform, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 990\ Hz$, and $A_{1i} = A_{2i} = K = 1.0$. Because the two frequencies are close together, this linear response waveform appears very similar to that for a single frequency linear response.



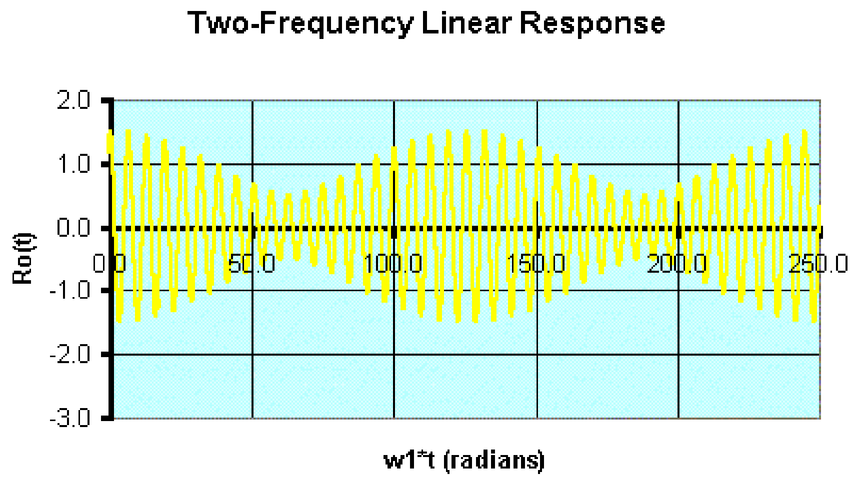
However, if we watch the two-frequency overall output response waveform for a long time, we can observe an interesting effect of the difference frequency, $\Delta f = |f_1 - f_2|$ on this waveform, if the two input frequencies *and* the amplitudes of the input signals are relatively close to each other. The following plot shows the linear output response waveform, for parameter values of $f_1 = 1000$ *Hz*, $f_2 = 950$ *Hz*, $A_{1i} = A_{2i} = K = 1.0$, for an elapsed time of $\Delta t = 0.040$ seconds, or *40 (38)* cycles of the *1000 (950) Hz* signal, respectively. This time interval also corresponds to *2* cycles of the *beat* frequency, i.e. the frequency difference, $f_B = \Delta f = |f_1 - f_2| = 50$ *Hz*, since the period of oscillation, τ is related to the frequency by $\tau = 1/f$.



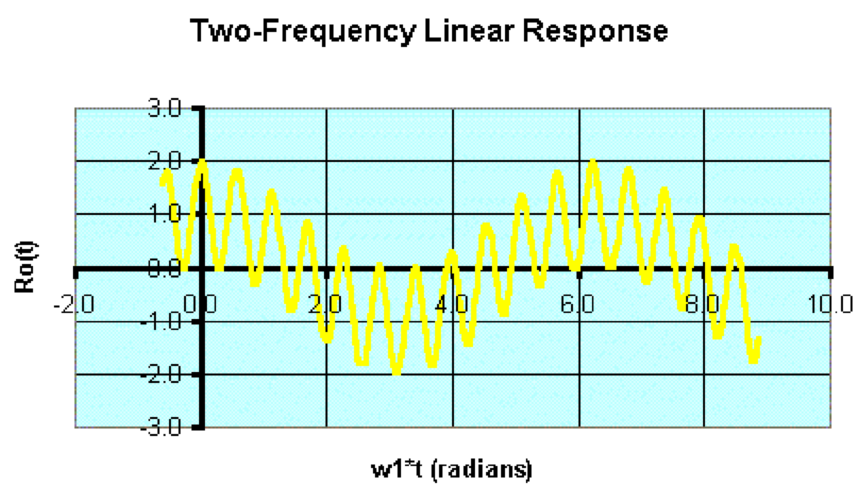
The two-frequency linear output response waveform associated with two input signals whose frequencies and amplitudes are close to each other exhibit the phenomenon of *beats* – the output response is largest in magnitude when both input signals, with frequency f_1 and f_2, respectively, are instantaneously at their maximum values, *and* exactly in phase with each other. As time progresses from this point, because of the difference in frequencies of the two input wave forms, the two signals get progressively further apart in phase from each other, and at some point in time, the two input signals are instantaneously at their maximum values, but exactly out-of-phase with each other, thus canceling (i.e. destructively interfering) with each other. At this point in time, the two-frequency output response is at its least value, magnitude-wise. As time progresses further, the two input signals will get back in phase with each other, where the two-frequency overall output response waveform will be maximum in magnitude again (actually, the input signal with the lower frequency will now be 2π radians behind in phase, with respect to the higher frequency input signal). And so on.

The beat frequency effect is maximally apparent when the beat frequency, $f_B = |f_1 - f_2|$ is small in comparison to either of the two frequencies, f_1 and/or f_2. In other words, the two frequencies, f_1 and f_2 need to be close to each other, in order to observe beats in the overall output response waveform. As the beat frequency becomes comparable to the lower of the two frequencies, say f_1 (implying that $f_2 \sim 2\, f_2$), the beat frequency effect dies out, vanishing completely when f_B becomes larger than the lower frequency (f_1).

In *addition*, the beat frequency effect can be maximal only when the amplitudes of the two input stimuli are equal, i.e. $A_{1i} = A_{2i}$. Even if the two frequencies are close to each other, but the amplitudes of the two input signals are unequal, then complete cancellation, as a beat frequency phenomenon cannot occur, as shown figure below, for parameter values of $f_1 = 1000$ *Hz*, $f_2 = 950$ *Hz*, $A_{1i} = 1.0$, $A_{2i} = 0.5$ and $K = 1.0$, for an elapsed time of $\Delta t = 0.040$ seconds, or *40 (38)* cycles of the *1000 (950) Hz* signal, respectively.



In the figure below, we also show the two-frequency linear output response waveform, for the case for two frequencies that are very different from each other, $f_1 = 1000$ *Hz* and $f_2 = 11.111$ *KHz*, but for $A_{1i} = A_{2i} = K = 1.0$, and for an elapsed time of $\Delta t = 0.0015$ seconds, or *1.5 (~16.7)* cycles of the *1000 Hz (11.111 KHz)* signal, respectively. This time interval corresponds to ~ *15.2* cycles of the beat frequency, $f_B = \Delta f = |f_1 - f_2| =$ *10.111 KHz.*



Even if the amplitudes are equal, but the frequencies of the two input signals differ greatly from each other, e.g. $f_2 >> f_1$, the beat-frequency effect on the two-frequency overall output response waveform is non-existent – the higher frequency input signal completes an entire cycle, while the lower frequency input signal hardly changes its value at all during this same time interval.

The above figure demonstrates the principle of *amplitude modulation*, as used in AM radio transmission and reception. An AM radio station broadcasts at a "carrier" frequency in the AM radio band ($540\ KHz < f_2 < 1600\ KHz$), with a bandwidth, $\Delta f = 10\ KHz$ wide, centered on its nominal frequency, f_o. Thus, up to $5\ KHz$ of the audio (or "voice") signal, in the audio frequency range ($\sim 50\ Hz < f_1 < 20\ KHz$) is *mixed* with the carrier signal, which results in *modulating* the amplitude of the carrier signal. In an AM radio receiver, the antenna picks up all radio signals, passes them through a tunable, narrow-band "pre-filter", an amplifier then amplifies the signal output from this filter. The tuner control of the AM radio receiver sets the frequency of a local oscillator to the carrier frequency of the AM radio station. The amplified radio signal and local oscillator signal are then mixed together, *demodulating* the carrier portion of the radio signal, resulting in an output signal which is the originally broadcast audio/voice signal from the AM radio station!

It is straightforward to show that the overall output response of a linear system, $R_o(t)$ to an arbitrarily large number of pure input tones, $S_{1i}(t)$, $S_{2i}(t)$, $S_{3i}(t)$, ... etc. simultaneously applied to the input of such a system will also simply be the linear sum of the individual responses to their associated pure input tones.

For n such pure-tone input stimuli, each with frequency, f_k and amplitude, A_{ki} for the k^{th} input stimulus, $S_{ki}(t) = A_{ki} \cos(\omega_k t)$ the overall input stimulus is:

$$S_i(t) = \sum_{k=1}^{k=n} S_{ki}(t) = \sum_{k=1}^{k=n} A_{ki} \cos(\omega_k t)$$

The overall output response, $R_o(t)$ for a linear system is:

$$R_o(t) = K \sum_{k=1}^{k=n} A_{ki} \cos(\omega_k t) = \sum_{k=1}^{k=n} R_{ko}(t)$$

where $R_{ko}(t) = K\, A_{ki} \cos(\omega_k t)$ is the individual output response associated with the k^{th} input stimulus, for this linear response system.

B. Output Response from a Quadratically Non-Linear System

If two pure tones of different frequencies, f_1 and f_2 are simultaneously applied as input stimuli to a system which has a small, quadratic nonlinear response,

$$R_o(S_i) = K\,(S_i + \varepsilon S_i^2) = K\,S_i\,(1 + \varepsilon S_i) \qquad (\,|\varepsilon S_i| << 1)$$

then if

$$S_i(t) = S_{1i}(t) + S_{2i}(t) = A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)$$

the overall output response is:

$$R_o(t) = R_o(S_i(t)) = K\,(A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)) + \varepsilon K\,[A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)]^2$$

or:

$$\begin{aligned} R_o(t) &= K\,(A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)) \\ &+ \varepsilon K\,[A_{1i}^2\cos^2(\omega_1 t)\,) + A_{2i}^2\cos^2(\omega_2 t) + 2A_{1i}A_{2i}\cos(\omega_1 t)\cos(\omega_2 t)] \end{aligned}$$

Again, using the trigonometric identity:

$$\cos^2\theta = \cos\theta * \cos\theta = ½\,(\cos 0 + \cos 2\theta) = ½\,(1 + \cos 2\theta)$$

and the generalized relation:

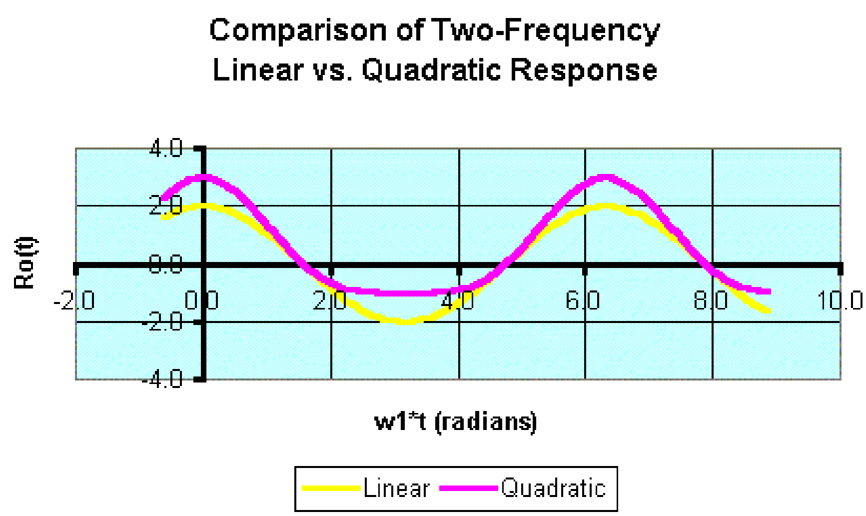
$$\cos\theta_1 * \cos\theta_2 = ½\,[\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)]$$

we have:

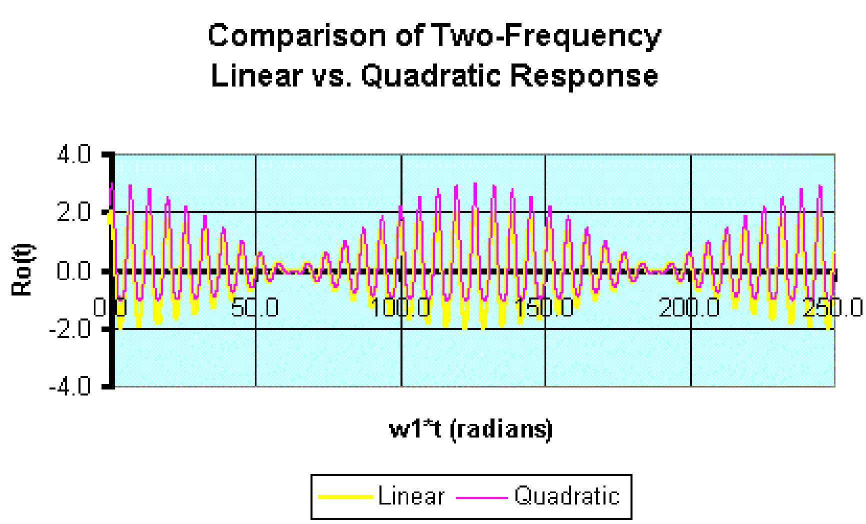
$$\begin{aligned} R_o(t) &= ½\,\varepsilon K A_{1i}^2 + ½\,\varepsilon K A_{2i}^2 + K\,(A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)) \\ &+ ½\,\varepsilon K A_{1i}^2\cos(2\omega_1 t) + ½\,\varepsilon K A_{2i}^2\cos(2\omega_2 t) \\ &+ \varepsilon K A_{1i}A_{2i}\,[\cos((\omega_1 - \omega_2)\,t) + \cos((\omega_1 + \omega_2)\,t)] \end{aligned}$$

The overall output response of this quadratically non-linear system is dominated by the two fundamental frequency contributions, $K A_{1i}\cos(\omega_1 t)$ and $K A_{2i}\cos(\omega_2 t)$ that are also associated with the linear output response system. In addition, we see that we have two d.c. level (zero-frequency) contributions, $½\,\varepsilon K A_{1i}^2$ and $½\,\varepsilon K A_{2i}^2$, as well as two second-harmonic contributions, $½\,\varepsilon K A_{1i}^2\cos(2\omega_1 t)$ and $½\,\varepsilon K A_{2i}^2\cos^2(2\omega_2 t)$ that are associated with the individual quadratic non-linear responses to the pure-tone inputs, applied to this system one at a time. However, there are also two new response terms, $\varepsilon K A_{1i}A_{2i}\cos((\omega_1 - \omega_2)\,t)$ and $\varepsilon K A_{1i}A_{2i}\cos((\omega_1 + \omega_2)\,t)$ which contribute harmonics at the (absolute value of the) *difference* between the two frequencies, $|f_1 - f_2|$ (since the cosine function is an *even* function of its argument), and the *sum* of the two frequencies, $(f_1 + f_2)$, respectively. These latter two output response harmonics arise from the <u>non-linear</u> *mixing* of the two input signals. This effect is known as *intermodulation distortion*.

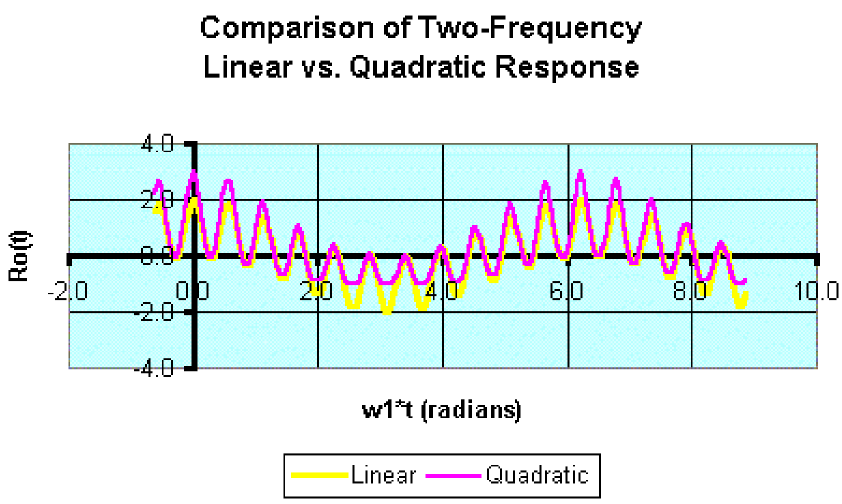
In the figure below, we show a comparison of the linear vs. quadratic non-linear output response waveforms, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 990\ Hz$, $A_{1i} = A_{2i} = K = 1.0$, and a relatively large value of the non-linearity parameter, $\varepsilon = +0.25$, so as to exaggerate the effect of the non-linearity term, to make it easily visible on the graph.



The following plot shows a comparison of the linear vs. quadratic non-linear output response waveforms, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 950\ Hz$, $A_{1i} = A_{2i} = K = 1.0$, and a value of $\varepsilon = +0.25$, for an elapsed time of $\Delta t = 0.040$ seconds, or *40 (38)* cycles of the *1000 (950) Hz* signal, respectively. This time interval also corresponds to *2* cycles of the *beat* frequency, i.e. the frequency difference, $f_B = \Delta f = |f_1 - f_2| = 50\ Hz$, since the period of oscillation, τ is related to the frequency by $\tau = 1/f$.



We also show a comparison of the linear vs. quadratic non-linear output response waveforms in the figure below, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 11.111\ KHz$, $A_{1i} = A_{2i} = K = 1.0$, and $\varepsilon = +0.25$.



In the above (and following) discussion(s) associated with mixing two pure tones of different frequencies, f_1 and f_2, we have tacitly assumed both signals to be in phase with each other at our defined zero of time, $t = 0$. The input signals were $S_{1i}(t) = A_{1i} \cos(\omega_1 t)$ and $S_{2i}(t) = A_{2i} \cos(\omega_2 t)$. We could have instead carried out these discussion(s) by chosing one of the input signals to be $\pm 90^o$ (or $\pm 180^o$) out of phase with respect to the other at $t = 0$, for example, $S_{2i}(t) = \pm A_{2i} \sin(\omega_2 t)$ (or $S_{2i}(t) = -A_{2i} \cos(\omega_2 t)$), respectively. We could also have left the phases for both signals, δ_1 and δ_2 completely arbitrary, i.e.

$$S_{1i}(t) = A_{1i} \cos(\omega_1 t + \delta_1) \quad \text{and} \quad S_{2i}(t) = A_{2i} \cos(\omega_2 t + \delta_2)$$

No loss of generality, nor any difference in the corresponding physics is encountered, with or without the explicit inclusion of such phases - they are merely equivalent to a redefinition of the choice of the zero of time for the problem. We leave it as an exercise for the rabidly enthusiastic reader to work through the above (and following) examples for two input signals that are not in phase with each other at $t = 0$.

It is also straightforward to extend the above discussions(s) associated with simultaneous mixing of two pure-tone input signals, to the case(s) for simultaneous mixing of three (or more) pure-tone input signals, for a system with a quadratic, non-linear response, $R_o(S_i) = K(S_i + \varepsilon S_i^2)$.

For n pure-tone input stimuli, each with frequency, f_k and amplitude, A_{ki} for the k^{th} input stimulus, $S_{ki}(t) = A_{ki} \cos(\omega_k t)$ the overall input stimulus is:

$$S_i(t) = \sum_{k=1}^{k=n} S_{ki}(t) = \sum_{k=1}^{k=n} A_{ki} \cos(\omega_k t)$$

The overall output response, $R_o(t)$ for a quadratically non-linear system for n pure-tone input stimuli is:

$$R_o(t) = K\sum_{k=1}^{k=n} A_{ki}\cos(\omega_k t) + \tfrac{1}{2}\varepsilon K\sum_{k=1}^{k=n} A_{ki}^2 + \tfrac{1}{2}\varepsilon K\sum_{k=1}^{k=n} A_{ki}^2\cos(2\omega_k t) + \varepsilon K\sum_{k=1}^{k=n}\sum_{\substack{j=1,\\ j\neq k,\\ j>k}}^{j=n} A_{ki}A_{ji}\left(\cos[(\omega_k - \omega_j)t] + \cos[(\omega_k + \omega_j)t]\right)$$

$$R_o(t) = K\left[\sum_{k=1}^{k=n} A_{ki}\cos(\omega_k t) + \tfrac{1}{2}\varepsilon\left(\sum_{k=1}^{k=n} A_{ki}^2\left(1+\cos(2\omega_k t)\right) + \sum_{k=1}^{k=n}\sum_{\substack{j=1,\\ j\neq k}}^{j=n} A_{ki}A_{ji}\left(\cos[(\omega_k - \omega_j)t] + \cos[(\omega_k + \omega_j)t]\right)\right)\right]$$

This expression for the overall output response, $R_o(t)$ for a quadratically non-linear system for n pure-tone input stimuli can be written most compactly as:

$$R_o(t) = K\left[\sum_{k=1}^{k=n} A_k\cos(\omega_k t) + \tfrac{1}{2}\varepsilon\sum_{k=1}^{k=n}\sum_{j=1}^{j=n} A_{ki}A_{ji}\left(\cos[(\omega_k - \omega_j)t] + \cos[(\omega_k + \omega_j)t]\right)\right]$$

C. Output Response from a Cubically Non-Linear System

If two pure tones of different frequencies, f_1 and f_2 are simultaneously applied as input stimuli to a system which has a small, cubic nonlinear response,

$$R_o(S_i) = K\,(S_i + \varepsilon S_i^3) = K\,S_i\,(1+\varepsilon S_i^2) \qquad (|\varepsilon S_i^2| << 1)$$

then if

$$S_i(t) = S_{1i}(t) + S_{2i}(t) = A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)$$

the overall output response is:

$$R_o(t) = R_o(S_i(t)) = K\,(A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)) + \varepsilon K\,[A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)]^3$$

or:

$$R_o(t) = K\,(A_{1i}\cos(\omega_1 t) + A_{2i}\cos(\omega_2 t)) + \varepsilon K\,[A_{1i}^3\cos^3(\omega_1 t)\,) + A_{2i}^3\cos^3(\omega_2 t)] \\ + \varepsilon K\,[3A_{1i}^2A_{2i}\cos^2(\omega_1 t)\cos(\omega_2 t) + 3A_{1i}A_{2i}^2\cos(\omega_1 t)\cos^2(\omega_2 t)]$$

Again, using the trigonometric identities:

$$\cos^2\theta = ½\,(\cos 0 + \cos 2\theta) = ½\,(1 + \cos 2\theta) \quad \text{and} \quad \cos^3\theta = {}^3\!/_4\cos\theta + {}^1\!/_4\cos 3\theta$$

and the generalized relation:

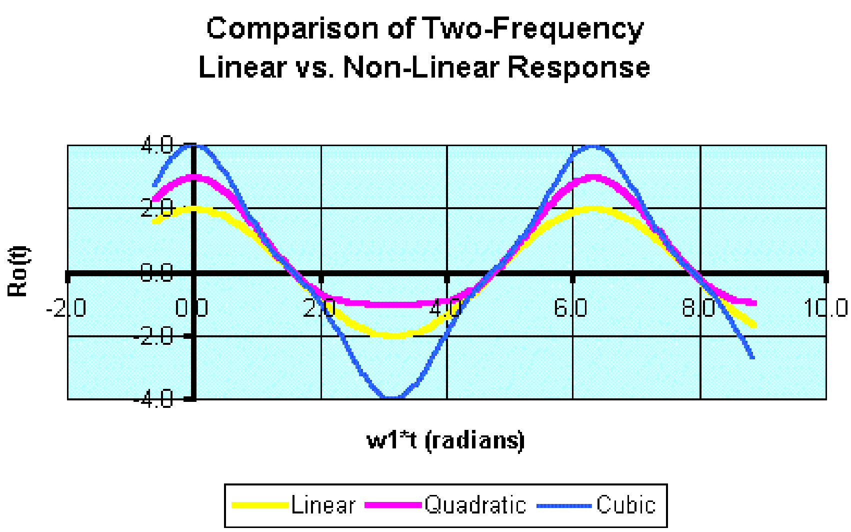
$$\cos\theta_1 * \cos\theta_2 = ½\,[\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)]$$

we obtain, after some algebra, combining similar terms:

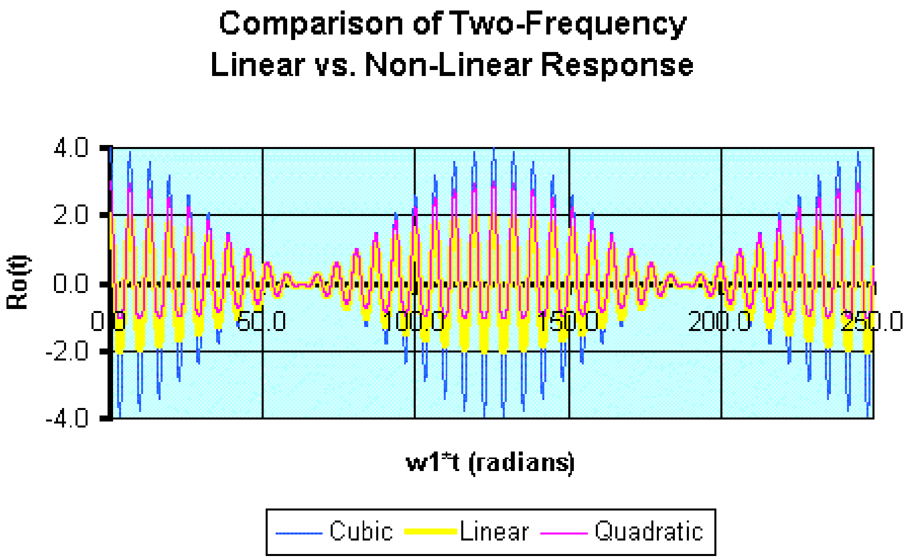
$$
\begin{aligned}
R_o(t) = &\ KA_{1i}\,[1 + {}^3/_2\,\varepsilon\,(\tfrac{1}{2} A_{1i}^2 + A_{2i}^2)]\cos(\omega_1 t) \\
&+ KA_{2i}\,[1 + {}^3/_2\,\varepsilon\,(\tfrac{1}{2} A_{2i}^2 + A_{1i}^2)]\cos(\omega_2 t) \\
&+ {}^1/_4\,\varepsilon K A_{1i}^3 \cos(3\omega_1 t) + {}^1/_4\,\varepsilon K A_{2i}^3 \cos(3\omega_2 t) \\
&+ {}^3/_4\,\varepsilon K A_{1i}^2 A_{2i}\,[\cos((2\omega_1 - \omega_2)t) + \cos((2\omega_1 + \omega_2)t)] \\
&+ {}^3/_4\,\varepsilon K A_{1i} A_{2i}^2\,[\cos((\omega_1 - 2\omega_2)t) + \cos((\omega_1 + 2\omega_2)t)]
\end{aligned}
$$

The overall output response of this cubically non-linear system is dominated by the two fundamental frequency contributions, $K A_{1i}\,[1 + {}^3/_2\,\varepsilon\,(\tfrac{1}{2} A_{1i}^2 + A_{2i}^2)]\cos(\omega_1 t)$ and $K A_{2i}\,[1 + {}^3/_2\,\varepsilon\,(\tfrac{1}{2} A_{2i}^2 + A_{1i}^2)]\cos(\omega_2 t)$. Note that *each* of these have terms associated with the linear portion of the response for each separate frequency, as well as cubic non-linear contributions associated with <u>*both*</u> frequencies! In addition, we see that we have two third-harmonic contributions, ${}^1/_4\,\varepsilon K A_{1i}^3 \cos(3\omega_1 t)$ and ${}^1/_4\,\varepsilon K A_{2i}^3 \cos^2(3\omega_2 t)$ that are associated with the individual quadratic non-linear responses to the pure-tone inputs, applied to this system one at a time. There are *also* four new response terms, ${}^3/_4\,\varepsilon K A_{1i}^2 A_{2i} \cos((2\omega_1 - \omega_2)t)$ and ${}^3/_4\,\varepsilon K A_{1i} A_{2i}^2 \cos((\omega_1 - 2\omega_2)t)$ which contribute harmonics at the (absolute value of the) *difference* between the 2nd harmonic of one input signal and the fundamental of the other, e.g. the frequency difference, $|2f_1 - f_2|$, and the terms ${}^3/_4\,\varepsilon K A_{1i}^2 A_{2i} \cos((2\omega_1 + \omega_2)t)$ and ${}^3/_4\,\varepsilon K A_{1i} A_{2i}^2 \cos((\omega_1 + 2\omega_2)t)$ which contribute harmonics at the *sum* of the 2nd harmonic of one input signal and the fundamental of the other, e.g., $(2f_1 + f_2)$. These latter four output response harmonics arise from the <u>non-linear</u> *mixing* of the two input signals, and correspond to *intermodulation distortion* of the output response waveform.

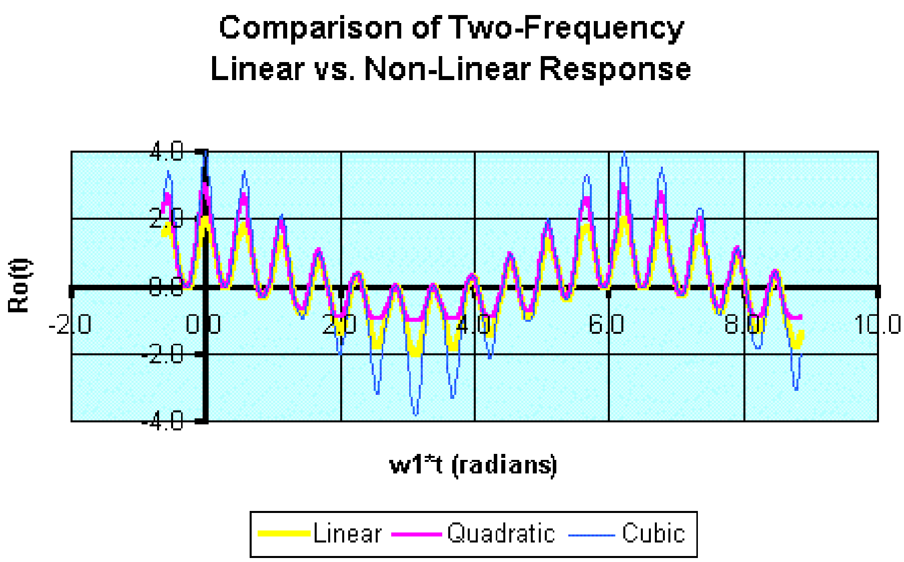
In the figure below, we show a comparison of the linear vs. quadratic and cubic non-linear output response waveforms, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000$ Hz, $f_2 = 990$ Hz, $A_{1i} = A_{2i} = K = 1.0$, and a relatively large value of the non-linearity parameter, $\varepsilon = +0.25$, so as to exaggerate the effect of the non-linearity term, to make it easily visible on the graph.



The following plot shows a comparison of the linear vs. quadratic and cubic non-linear output response waveforms, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 950\ Hz$, $A_{1i} = A_{2i} = K = 1.0$, and a value of $\varepsilon = +\ 0.25$, for an elapsed time of $\Delta t = 0.040$ seconds, or *40 (38)* cycles of the *1000 (950) Hz* signal, respectively. This time interval also corresponds to *2* cycles of the *beat* frequency, i.e. the frequency difference, $f_B = \Delta f = |f_1 - f_2| = 50\ Hz$, since the period of oscillation, τ is related to the frequency by $\tau = 1/f$.

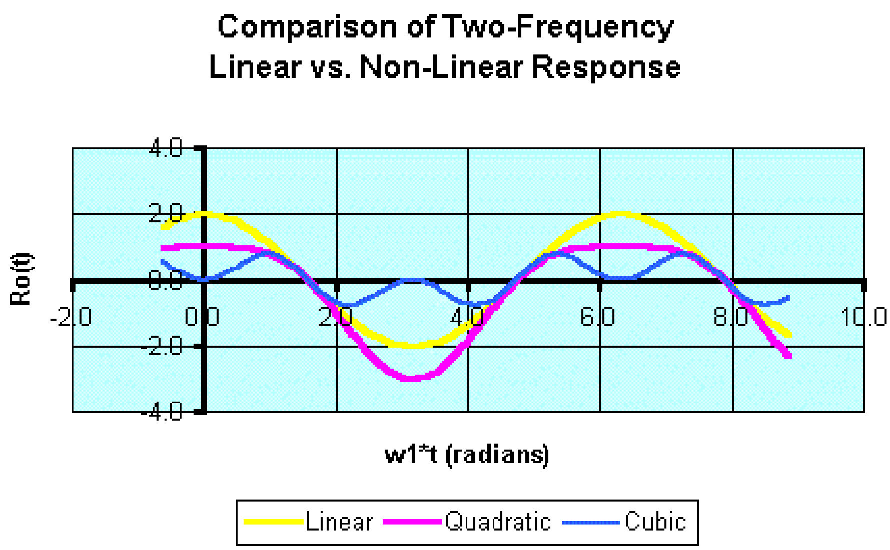


We also show a comparison of the linear vs. quadratic and cubic non-linear output response waveforms in the figure below, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 11.111\ KHz$, $A_{1i} = A_{2i} = K = 1.0$, and $\varepsilon = +\ 0.25$.

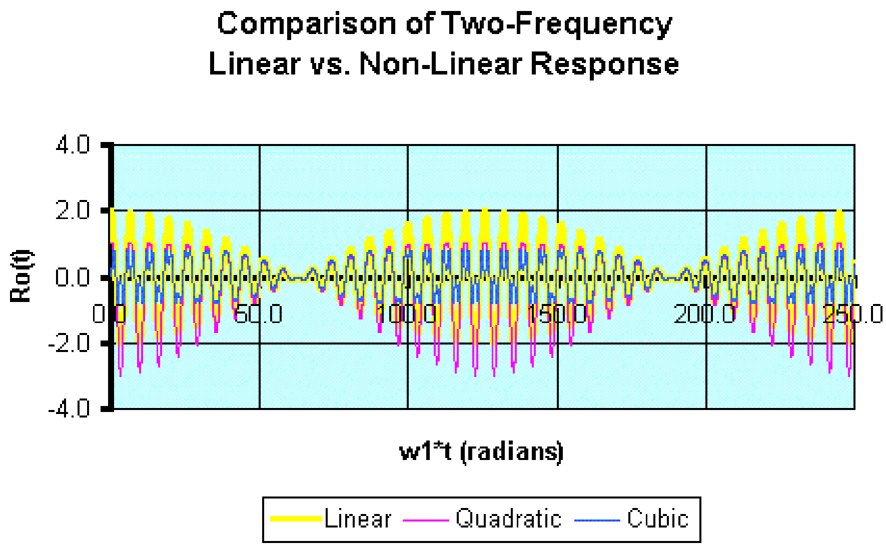


As we saw for the single-frequency quadratic and cubic non-linear response waveforms, if the sign of the non-linearity parameter is changed from e.g. $\varepsilon = +0.25$ to $\varepsilon = -0.25$, then for the two-frequency quadratic non-linear reponse, the phase of the second harmonic, relative to the fundamental is shifted by 180^o, however for the two-frequency cubic non-linear response, the amplitudes of both the fundamental and third harmonic components of the output response are affected by this sign change.

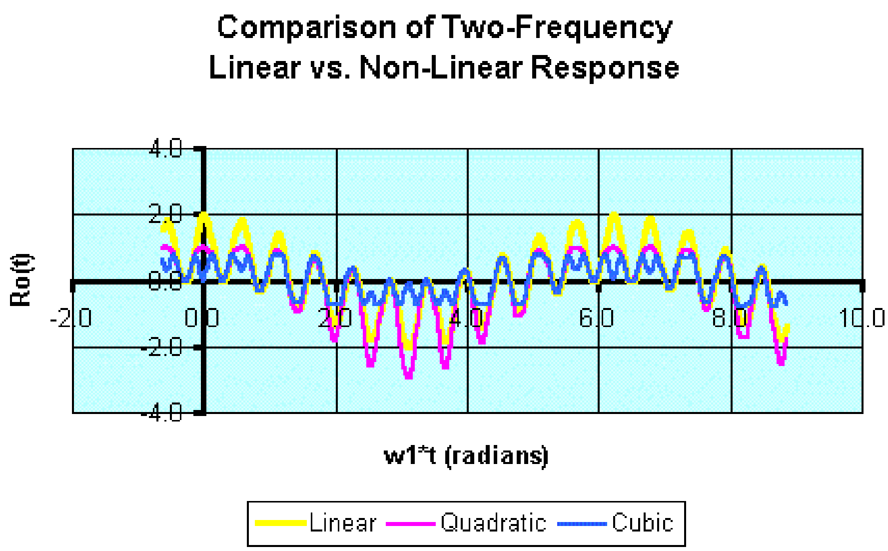
In the figure below, we show a comparison of the linear vs. quadratic and cubic non-linear output response waveforms, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 990\ Hz$, $A_{1i} = A_{2i} = K = 1.0$, and a relatively large value of the non-linearity parameter, $\varepsilon = -\ 0.25$, so as to exaggerate the effect of the non-linearity term, to make it easily visible on the graph. As can be seen by comparing this figure with its corresponding one above, the cubic non-linear output response here has been significantly altered by this sign change of the ε parameter.



The following plot shows a comparison of the linear vs. quadratic and cubic non-linear output response waveforms, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 950\ Hz$, $A_{1i} = A_{2i} = K = 1.0$, but for a value of $\varepsilon = -0.25$, for an elapsed time of $\Delta t = 0.040$ seconds, or *40 (38)* cycles of the *1000 (950) Hz* signal, respectively. As can be seen by comparing this figure with its corresponding one above, the cubic non-linear output response here has also been significantly altered by this sign change of the ε parameter.



We also show a comparison of the linear vs. quadratic and cubic non-linear output response waveforms in the figure below, for an elapsed time of $\Delta t = 0.0015$ seconds, for parameter values of $f_1 = 1000\ Hz$, $f_2 = 11.111\ KHz$, $A_{1i} = A_{2i} = K = 1.0$, and $\varepsilon = -0.25$. Again, the cubic non-linear output response here has also been significantly altered by this sign change of the ε parameter



Again, it is also straightforward (but now more tedious) to extend the above discussions associated with simultaneous mixing of two pure-tone input signals, to the case(s) for simultaneous mixing of three (or more) pure-tone input signals, for a system with a cubic, non-linear response, $R_o(S_i) = K\,(S_i + \varepsilon S_i^3)$.

For n pure-tone input stimuli, each with frequency, f_k and amplitude, A_{ki} for the k^{th} input stimulus, $S_{ki}(t) = A_{ki}\cos(\omega_k t)$ the overall input stimulus is:

$$S_i(t) = \sum_{k=1}^{k=n} S_{ki}(t) = \sum_{k=1}^{k=n} A_{ki}\cos(\omega_k t)$$

The overall output response, $R_o(t)$ for a cubically non-linear system for n pure-tone input stimuli, after much algebra, combining similar terms, is:

$$\begin{aligned} R_o(t) &= K\sum_{k=1}^{k=n} A_{ki}\left[1+\tfrac{3}{2}\varepsilon\left(\sum_{j=1}^{j=n} A_{ji}^2 - \tfrac{1}{2}A_{ki}^2\right)\right]\cos(\omega_k t) + \tfrac{1}{4}\varepsilon K\sum_{k=1}^{k=n} A_{ki}^3\cos(3\omega_k t) \\ &+ \tfrac{3}{2}\varepsilon K\sum_{k=1}^{k=n}\sum_{\substack{j=1,\\ j\neq k}}^{j=n} A_{ki}^2 A_{ji}\left(\cos[(2\omega_k - \omega_j)t] + \cos[(2\omega_k + \omega_j)t]\right) \\ &+ \tfrac{3}{2}\varepsilon K\sum_{l=1}^{l=n}\sum_{\substack{k=1,\\ k\neq l}}^{k=n}\sum_{\substack{j=1,\\ j\neq k,\\ j\neq l}}^{j=n} A_{li}A_{ki}A_{ji}\begin{pmatrix}\cos[(\omega_l - \omega_k + \omega_j)t] + \cos[(\omega_l + \omega_k - \omega_j)t] \\ +\cos[(\omega_l - \omega_k - \omega_j)t] + \cos[(\omega_l + \omega_k + \omega_j)t]\end{pmatrix} \end{aligned}$$

The overall output response of this cubically non-linear system is dominated by the n fundamental frequency contributions, associated with the first summation over terms proportional to $\cos(\omega_k t)$. Note that *each* of these have terms associated with the linear portion of the response for each separate frequency, as well as cubic non-linear contributions associated with <u>*all*</u> n input signal frequencies! In addition, in the second summation, we see that we have n third-harmonic contributions, associated with the summation over terms proportional to $\cos(3\omega_1 t)$, that are the individual quadratic non-linear responses to the pure-tone inputs, applied to this system one at a time. In the double summation, there are $2n$ response terms that are proportional to $\cos[(2\omega_k - \omega_j)t]$ which contribute harmonics at the (absolute value of the) *difference* between the 2nd harmonic of one input signal and the fundamental of the other, e.g. the frequency difference, $|2f_1 - f_2|$, and $2n$ response terms, that are proportional to $\cos[(2\omega_k + \omega_j)t]$ which contribute harmonics at the *sum* of the 2nd harmonic of one input signal and the fundamental of the other, e.g., $(2f_1 + f_2)$. Lastly, the triple summation, which exists <u>*only*</u> for $n \geq 3$ input signals, contains $(n-2)^2$ terms that are proportional to three types of triple fundamental frequency differences and a sum of three fundamental frequencies, with terms proportional to $\cos[(\omega_l - \omega_k + \omega_j)t]$, $\cos[(\omega_l + \omega_k - \omega_j)t]$, $\cos[(\omega_l - \omega_k - \omega_j)t]$, and $\cos[(\omega_l + \omega_k + \omega_j)t]$. Here again, the terms in the output response harmonics associated with frequency differences and sums of frequencies are output response harmonics that arise from the <u>non-linear</u> *mixing* of the n input signals, and correspond to *intermodulation distortion* of the output response waveform.

Output Response of a System with Higher-Order Non-Linearities

By following the above methodology, one can also show that non-linear output responses, $R_o(S_i)$ associated with systems that have purely *quartic* ($\varepsilon K S_i^4$), *quintic* ($\varepsilon K S_i^5$), and/or higher-order terms (e.g $\varepsilon K S_i^6$, etc.) will also produce higher harmonics – 4th, 5th, 6th, etc. harmonics, respectively, of the fundamental frequencies, f_k associated with two or more pure-tone input stimuli of the system, as well as harmonic components associated with sums and differences of frequencies, and even higher-order effects. One can also work through cases for systems e.g. with non-linear exponential-type responses, approximating the non-linear exponential response of such systems by the Taylor series expansion to the desired order. While these cases are more complicated and lengthy to carry out in detail, with some determination, stamina and care, the rabidly enthusiastic reader can work through them and discover many interesting phenomena associated with each such system!

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