Fourier Analysis I:

Determination of the Harmonic Content of a Periodic Waveform

The harmonic content of a *periodic* waveform - one which repeats itself in time or in space, can be obtained using the mathematical formalism known as *Fourier analysis* (also known as *harmonic analysis*), named after the French mathematician, Joseph Fourier (1768-1830). The periodic waveform(s) analyzed using this method could be e.g. either a poly-phonic input stimulus to a given system, and/or the linear or non-linear output response waveform associated with that system. Another example of the use of Fourier analysis is to determine the harmonic distortion content and/or the intermodulation distortion content associated with the non-linear response of a system, to which a pure-tone input stimulus is applied.

Mathematically, any arbitrary function, *f(x)* that is *finite*, *single-valued* and *piece-wise continuous* over the interval $x_1 \le x \le x_2$, can be exactly represented by a power series (with suitably-chosen values of the constant coefficients, a_n), due to the fact that the powers of x, x^n form a complete set of basis vectors for the function "*space*" associated with the interval $x_1 \le x \le x_2$:

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots.. = \sum_{n=0}^{n=\infty} a_n x^n$$

In this abstract, infinite-dimensional mathematical space, each of the x^n, as basis vectors, are analogous to the *x*, *y* and *z* axes in real, 3-dimensional space. Except that the complete set of basis vectors, x^n aren't all mutually perpendicular (i.e. *orthogonal*) to each other, like the the *x*, *y* and *z* axes are to each other, in our real, 3-dimensional space. However, *certain linear combinations* of the complete set of x^n are orthogonal to each other. Thus, these certain linear combinations of the x^n in this abstract, infinite-dimensional mathematical space *do* behave exactly analogously to the *x*, *y* and *z* axes in our real, 3-dimensional space. Also, just as one can carry out an infinitude of possible rotations in our real, 3-dimensional space, to obtain a entirely new sets of *x*, *y* and *z* axes in our real, 3-dimensional space, obtaining new *x′*, *y′* and *z′* axes (which are linear combinations of the original *x*, *y* and *z* axes), one can also carry out analogous rotations in the abstract, infinite-dimensional mathematical space, to obtain new complete sets of othogonal basis vectors there, too.

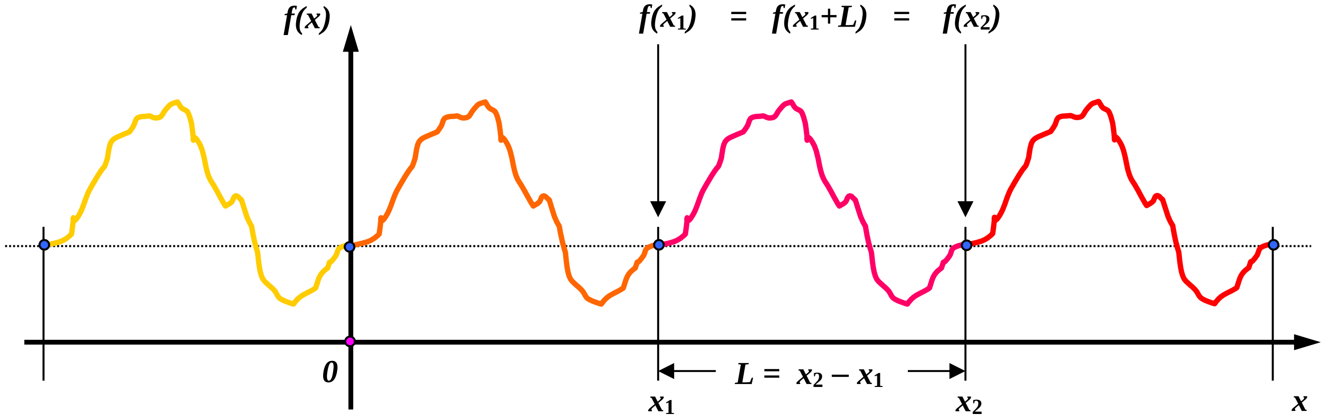
Now, the *sine* and *cosine* functions, *sin (x)* and *cos (x)* have Taylor series expansions in powers of *x* - i.e. the *sin (x)* and *cos (x)* functions are certain specific linear combinations of the x^n:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

and:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!}$$

For an as-above-defined well-behaved, but arbitrary function, $f(x)$, defined in the spatial interval $x_1 \leq x \leq x_2$ (with $x_2 = x_1 + L$), if $f(x)$ is *periodic* - i.e. it *repeats* with a spatial period, L, such that $f(x+L) = f(x)$, as shown in the figure below:



Then the periodic function, $f(x)$ in the space-domain, can be *precisely* replicated by the following *Fourier series* expansion:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\tfrac{2\pi nx}{L}\right) + \sum_{n=1}^{n=\infty} b_n \sin\left(\tfrac{2\pi nx}{L}\right)$$

The constant coefficient, a_0 is needed, as it represents a d.c. offset (i.e. constant) term. The constant coefficients a_n and b_n are the (harmonic) amplitudes associated with the *cosine* and *sine* functions, for the n^{th} term (n = 1, 2, 3, ...) in each of the above sums, respectively.

Note also that the spatial period, L physically corresponds to the (spatial) *wavelength*, λ, i.e. $L = \lambda$. The *wavenumber*, $k \equiv 2\pi/\lambda$. Thus, we can rewrite the above Fourier series expansion as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(nkx) + \sum_{n=1}^{n=\infty} b_n \sin(nkx)$$

It needs to be stated here that the wavelength, λ and hence the wavenumber, k are associated with the *lowest*, or *fundamental* frequency, f (i.e. when n = 1 in the above summations) since $f\lambda = v$, where v is the speed of propagation of the wave. The harmonics of the fundamental are then integer multiples of the fundamental frequency, i.e. $f_n = nf$, and thus the wavelengths and wavenumbers associated with the n^{th} harmonic are $\lambda_n = \lambda/n$ and $k_n = nk$, respectively, for n = 1, 2, 3, 4, 5, ... etc.

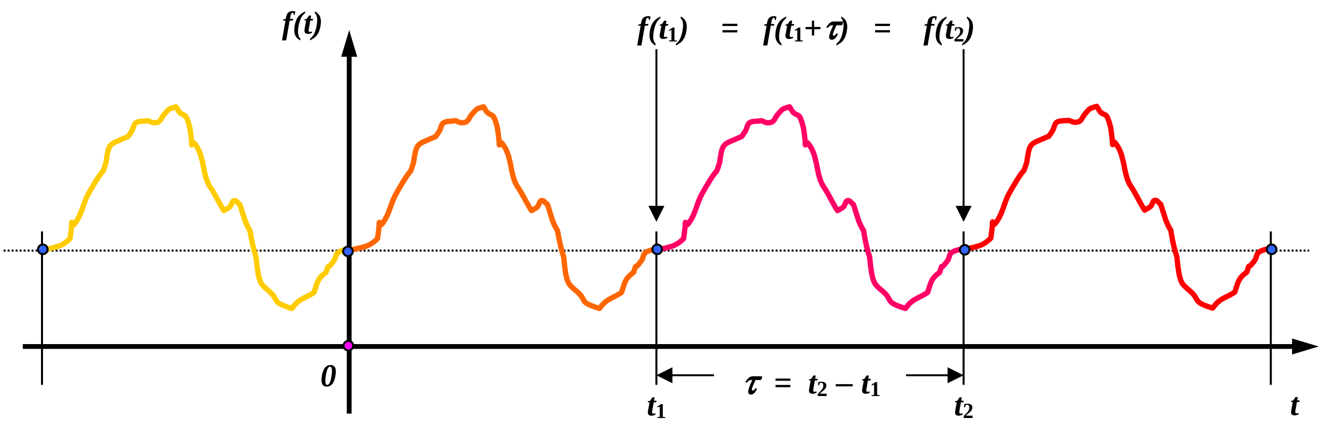
Note that we can also write the Fourier series expansion of *f(x)* in the time-domain, simply by changing the variable $x \to t$ and changing the spatial period, L to the temporal (i.e. time) period, τ, i.e. $L \to \tau$. Then since the frequency, $f = 1/\tau$, and $\omega = 2\pi f$, also with the relation $\omega / k = v$, we have:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(\tfrac{2\pi n t}{\tau}\right) + \sum_{n=1}^{n=\infty} b_n \sin\left(\tfrac{2\pi n t}{\tau}\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(2\pi n f t\right) + \sum_{n=1}^{n=\infty} b_n \sin\left(2\pi n f t\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\left(n\omega t\right) + \sum_{n=1}^{n=\infty} b_n \sin\left(n\omega t\right)$$

In the time-domain, the corresponding figure for the periodic temporal function, *f(t)* is:



Note further that since the *sine* and *cosine* functions, *sin (x)* and *cos (x)*, respectively, are linear combinations of powers of *x*,(i.e. their Taylor series expansions), then together with 1, they encompass *__all__* powers of *x*. Since the x^n form a complete set of basis vectors for the function "*space*" associated with the interval $x_1 \le x \le x_2$, then 1, and the Taylor series expansions for *sin(x)* and *cos(x)* also form a complete set of basis vectors for the function "*space*" associated with the interval $x_1 \le x \le x_2$. This is the reason that any mathematically well-behaved, periodic function, *f(x)* can be precisely replicated by an appropriate linear combination of 1, *sin(nkx)* and *cos(nkx)* - i.e. a Fourier series expansion, as defined above.

Now it turns out that, as basis vectors in the mathematical space associated with the interval $x_1 \leq x \leq x_2$, the *sin (nkx)* and *cos (nkx)* functions, and 1 *are* orthogonal (i.e. mutually perpendicular) to each other. In real, 3-dimensional space, the orthogonality of two vectors, $\boldsymbol{A} = A_x \boldsymbol{x} + A_y \boldsymbol{y} + A_z \boldsymbol{z}$ and $\boldsymbol{B} = B_x \boldsymbol{x} + B_y \boldsymbol{y} + B_z \boldsymbol{z}$, where ($A_x$, A_y, A_z) are the (x, y, z)-components of the vector $\boldsymbol{A}$, and (B_x, B_y, B_z) are the (x, y, z)-components of the vector $\boldsymbol{B}$, and $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are *unit* vectors (i.e. vectors with unit length) pointing along the x, y and z axes, respectively, is defined by the so-called *dot*, or *inner product* of the two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{A} \bullet \boldsymbol{B} \equiv A_x B_x + A_y B_y + A_z B_z$$

The two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal (i.e. perpendicular to each other) if their dot product, $\boldsymbol{A} \bullet \boldsymbol{B} = 0$. For example, if the vector, $\boldsymbol{A}$ is oriented entirely along the x-direction, then $\boldsymbol{A} = A_x \boldsymbol{x} + 0\,\boldsymbol{y} + 0\,\boldsymbol{z}$, or equivalently, $\boldsymbol{A} = (A_x, 0, 0)$. If the vector, $\boldsymbol{B}$ is oriented, e.g. only along the y-direction, then $\boldsymbol{B} = 0\,\boldsymbol{x} + B_y \boldsymbol{y} + 0\,\boldsymbol{z}$, or equivalently, $\boldsymbol{B} = (B_x, B_y, B_z)$. Then here, the dot product $\boldsymbol{A} \bullet \boldsymbol{B} = A_x * 0 + 0 * B_y + 0 * 0 = 0$. The length (i.e. magnitude) of a vector, $\boldsymbol{A}$ is defined as $|\,\boldsymbol{A}\,| \equiv (A_x^2 + A_y^2 + A_z^2)^{½}$. Thus, the dot, or inner product, $\boldsymbol{A} \bullet \boldsymbol{B}$ has physical units of $(length)^2$.

In this abstract, infinite-dimensional mathematical function space associated with the interval $x_1 \leq x \leq x_2$, the analog of the dot, or inner product between two mathematically well-behaved, but arbitrary "vectors" in this space - the functions, *f(x)* and *g(x)* is defined as:

$$\langle f(x), g(x) \rangle \equiv \int_{x=x_1}^{x=x_2} f(x) * g(x)\,dx$$

If this integral is zero, then the two functions, *f(x)* and *g(x)* are orthogonal to each other.

Since *f(x)* and *g(x)*, as well-behaved functions over the interval, $x_1 \leq x \leq x_2$ can each be represented as separate Fourier series expansions, then the above inner product becomes:

$$\langle f(x), g(x) \rangle = \int_{x=x_1}^{x=x_2} \left(\frac{a_o}{2} + \sum_{n=1}^{n=\infty} a_n \cos(nkx) + \sum_{n=1}^{n=\infty} b_n \sin(nkx) \right) * \left(\frac{c_o}{2} + \sum_{m=1}^{m=\infty} c_m \cos(mkx) + \sum_{m=1}^{m=\infty} d_m \sin(mkx) \right) dx$$

If we expand this expression out, term-by term, then there will be an infinite number of integrals on the right hand side. If the two arbitrary functions, *f(x)* and *g(x)* are to be orthogonal to each other, then *each* of these integrals *must* vanish, separately from each other. Thus, the inner product term:

$$\int_{x=x_1}^{x=x_2} \frac{a_o}{2} * \frac{c_o}{2}\,dx = \frac{a_o}{2} * \frac{c_o}{2} \int_{x=x_1}^{x=x_2} dx = \frac{a_o c_o}{4}\left[x_2 - x_1\right] = \frac{a_o c_o}{4} L = 0$$

which, in general can vanish *only* if either of the coefficients, a_o or c_o (or both) are zero, for an arbitrary interval, $x_1 \leq x \leq x_2$. Since the constant ($n = m = 0$) terms in the Fourier series, e.g. $a_o = a_o * 1$, then obviously the inner product of the basis vector, 1 with itself, i.e. < 1, 1 > *cannot* vanish, since (any) basis vector cannot be orthogonal to itself!

Similarly, *each* of the following inner products must vanish, for <u>*all*</u> values of *n* and *m*:

$$\int_{x=x_1}^{x=x_2} \frac{a_o}{2} * c_m \cos(mkx)dx = \frac{a_o c_m}{2}\int_{x=x_1}^{x=x_2} \cos(mkx)dx = +\frac{a_o c_m}{2}\left[\frac{\sin(mkx_2)}{mk} - \frac{\sin(mkx_1)}{mk}\right] = 0$$

$$\int_{x=x_1}^{x=x_2} \frac{a_o}{2} * d_m \sin(mkx)dx = \frac{a_o d_m}{2}\int_{x=x_1}^{x=x_2} \sin(mkx)dx = -\frac{a_o d_m}{2}\left[\frac{\cos(mkx_2)}{mk} - \frac{\cos(mkx_1)}{mk}\right] = 0$$

$$\int_{x=x_1}^{x=x_2} \frac{c_o}{2} * a_n \cos(nkx)dx = \frac{c_o a_n}{2}\int_{x=x_1}^{x=x_2} \cos(nkx)dx = +\frac{c_o a_n}{2}\left[\frac{\sin(nkx_2)}{nk} - \frac{\sin(nkx_1)}{nk}\right] = 0$$

$$\int_{x=x_1}^{x=x_2} \frac{d_o}{2} * b_n \sin(nkx)dx = \frac{d_o b_n}{2}\int_{x=x_1}^{x=x_2} \sin(nkx)dx = -\frac{d_o b_n}{2}\left[\frac{\cos(nkx_2)}{nk} - \frac{\cos(nkx_1)}{nk}\right] = 0$$

Each of these terms <u>*does*</u> vanish, because the functions *f(x)* and *g(x)* are periodic - i.e. they repeat themselves for $x_2 = x_1 + L$. Since the wavenumber, $k = 2\pi/L$, then for arbitrary values of *n*, *m* (= 1,2,3,...), then, e.g.:

$$\sin(mkx_2) = \sin(2\pi m x_2/L) = \sin(2\pi m(x_1+L)/L) = \sin(2\pi m x_1/L + 2\pi m) = \sin(2\pi m x_1/L)$$

$$\cos(mkx_2) = \cos(2\pi m x_2/L) = \cos(2\pi m(x_1+L)/L) = \cos(2\pi m x_1/L + 2\pi m) = \cos(2\pi m x_1/L)$$

These results explicitly demonstrate that, since the constant ($n = m = 0$) terms in the Fourier series, e.g. $a_o = a_o * 1$, that the *sin* (*mkx*) and *cos* (*mkx*) functions (with $m > 0$), as basis vectors, are orthogonal to 1 on the interval, $x_1 \le x \le x_2$.

Similarly, *each* of the following inner products must all vanish, for <u>*all*</u> values of *n* and *m*:

$$\int_{x=x_1}^{x=x_2} a_n c_m \cos(nkx)\cos(mkx)dx = +a_n c_m\left\{\left[\frac{\sin(n-m)kx_2}{2(n-m)k} + \frac{\sin(n+m)kx_2}{2(n+m)k}\right] - \left[\frac{\sin(n-m)kx_1}{2(n-m)k} + \frac{\sin(n+m)kx_1}{2(n+m)k}\right]\right\}$$

$$\int_{x=x_1}^{x=x_2} b_n c_m \sin(nkx)\cos(mkx)dx = -b_n c_m\left\{\left[\frac{\cos(n-m)kx_2}{2(n-m)k} - \frac{\cos(n+m)kx_2}{2(n+m)k}\right] - \left[\frac{\cos(n-m)kx_1}{2(n-m)k} - \frac{\cos(n+m)kx_1}{2(n+m)k}\right]\right\}$$

$$\int_{x=x_1}^{x=x_2} b_n d_m \sin(nkx)\sin(mkx)dx = +b_n d_m\left\{\left[\frac{\sin(n-m)kx_2}{2(n-m)k} - \frac{\sin(n+m)kx_2}{2(n+m)k}\right] - \left[\frac{\sin(n-m)kx_1}{2(n-m)k} - \frac{\sin(n+m)kx_1}{2(n+m)k}\right]\right\}$$

For the cases where $n \neq m$, each of the above three types of integrals *does* vanish, because the *sin* (*mkx*) and *cos* (*mkx*) functions are periodic on the interval, $x_1 \le x \le x_2$. These results explicitly demonstrate that for $n \neq m$, that the *cos* (*nkx*) and *cos* (*mkx*) functions, as basis vectors, are orthogonal to each other; the *sin* (*nkx*) and *cos* (*mkx*) functions are also orthogonal to each other; and the *sin* (*nkx*) and *sin* (*mkx*) functions are also orthogonal to each other on the interval, $x_1 \le x \le x_2$.

For the cases where $n = m$, these integrals become:

$$\int_{x=x_1}^{x=x_2} a_n c_n \cos^2(nkx)dx = a_n c_n \left\{\left[\frac{x_2}{2} + \frac{\sin(2nkx_2)}{4nk}\right] - \left[\frac{x_1}{2} + \frac{\sin(2nkx_1)}{4nk}\right]\right\} = a_n c_n \left\{\frac{x_2 - x_1}{2}\right\} = a_n c_n \frac{L}{2}$$

$$\int_{x=x_1}^{x=x_2} b_n c_n \sin(nkx)\cos(nkx)dx = b_n c_n \left\{\left[\frac{\sin^2(nkx_2)}{2nk}\right] - \left[\frac{\sin^2(nkx_1)}{2nk}\right]\right\} = 0$$

$$\int_{x=x_1}^{x=x_2} b_n d_n \sin^2(nkx)dx = b_n d_n \left\{\left[\frac{x_2}{2} - \frac{\sin(2nkx_2)}{4nk}\right] - \left[\frac{x_1}{2} - \frac{\sin(2nkx_1)}{4nk}\right]\right\} = b_n d_n \left\{\frac{x_2 - x_1}{2}\right\} = b_n d_n \frac{L}{2}$$

The first and third of these type of integrals, the inner product of *cos* (*nkx*) with itself and the inner product of *sin* (*nkx*) with itself, respectively, vanish only when e.g. either of the coefficients, a_n or c_n (or both) are zero, and either of the coefficients, b_n or d_n (or both) are zero, respectively, for an arbitrary interval, $x_1 \le x \le x_2$. The second of these type of integrals vanishes, because the *sin* (*mkx*) and *cos* (*mkx*) functions are periodic on the interval, $x_1 \le x \le x_2$, thus explicitly demonstrating that for $n = m$, the *sin* (*nkx*) and *cos* (*nkx*) functions, as basis vectors, are orthogonal to each on the interval, $x_1 \le x \le x_2$.

Thus, we have proved that the basis vectors 1, the *sin (nkx)* and *cos (nkx)* functions in this abstract, infinite-dimensional mathematical function space *are* orthogonal (i.e. mutually perpendicular) to each other over the interval $x_1 \le x \le x_2$.

We have also shown that, on the interval $x_1 \le x \le x_2$, that two arbitrary, but mathematically well-behaved, periodic functions, *f(x)* and *g(x)*, each expressed as a Fourier series, cannot be orthogonal to each other unless certain of their respective Fourier coefficients, (a_n and/or b_n) and (c_n and/or d_n) vanish in such a way to enable the inner product, $< f(x), g(x) >$ to vanish - this result is described by the so-called *generalized Parseval identity* - the inner product of the functions *f(x)* with *g(x)*:

$$\langle f(x), g(x)\rangle = \int_{x=x_1}^{x=x_2} f(x) * g(x)dx = \frac{L}{2}\left[\frac{a_o c_o}{2} + \sum_{n=1}^{n=\infty}\left(a_n c_n + b_n d_n\right)\right]$$

The inner product of the function, *f(x)* with *itself* is known as *Parseval's identity*:

$$\langle f(x), f(x)\rangle = \int_{x=x_1}^{x=x_2} f(x) * f(x)dx = \frac{L}{2}\left[\frac{a_o^2}{2} + \sum_{n=1}^{n=\infty}\left(a_n^2 + b_n^2\right)\right]$$

These identities are named in honor of the French mathematician, Marc Antoine Parseval des Chenes (1755-1836), who derived them. Physically, Parseval's identity, $<f(x), f(x)> = ...$ in the space-domain (time-domain) is proportional to the total average *linear* energy density, $< u_{tot} >$ (power, $< P_{tot} >$) in the waveform over one cycle, respectively. The average linear energy density (power) associated with the n^{th} harmonic, $< u_n >$ ($< P_n >$), respectively, can therefore be obtained from this relation!

If the periodic function, *f(x)* is known on the interval $x_1 \le x \le x_2$, then we can *use* the orthogonality properties of the basis vectors, 1, the *sin (nkx)* and *cos (nkx)* functions to determine each of the Fourier coefficients, a_n and b_n in the Fourier series! By taking the inner product of *f(x)* with *each* of the basis vectors, because of the orthogonality properties of the basis vectors, the inner product of the function, *f(x)* with a given basis vector "projects" out *that* component of the "vector" *f(x)* in this infinite-dimensional function space lying along, or parallel to that basis vector, i.e.:

$$\langle f(x),1\rangle = \int_{x=x_1}^{x=x_2} f(x) * 1dx = \int_{x=x_1}^{x=x_2} \frac{a_o}{2} * 1dx = \frac{a_o}{2}\left[x_2 - x_1\right] = \frac{a_o}{2} L$$

Thus, the d.c. (i.e. $n = 0$) term in the Fourier series expansion can be determined from:

$$a_o = \frac{2}{L}\langle f(x),1\rangle = \frac{2}{L}\int_{x=x_1}^{x=x_2} f(x)dx$$

Similarly, the inner product of the function, *f(x)* with the *cos (nkx)* and *sin (nkx)* basis vectors projects out the a_n and b_n coefficients, respectively, of the Fourier series expansion of *f(x)*, i.e.:

$$\langle f(x),\cos(nkx)\rangle = \int_{x=x_1}^{x=x_2} a_n \cos^2(nkx)dx = a_n\left[\frac{x_2}{2} - \frac{x_1}{2}\right] = a_n \frac{L}{2}$$

$$\langle f(x),\sin(nkx)\rangle = \int_{x=x_1}^{x=x_2} b_n \sin^2(nkx)dx = b_n\left[\frac{x_2}{2} - \frac{x_1}{2}\right] = b_n \frac{L}{2}$$

Thus, the Fourier coefficients, a_n and b_n can be determined from:

$$a_n = \frac{2}{L}\langle f(x),\cos(nkx)\rangle = \frac{2}{L}\int_{x=x_1}^{x=x_2} f(x)\cos(nkx)dx$$

$$b_n = \frac{2}{L}\langle f(x),\sin(nkx)\rangle = \frac{2}{L}\int_{x=x_1}^{x=x_2} f(x)\sin(nkx)dx$$

By a simple change of variables, we can write the Fourier series expansion of a "generic" periodic function, $f(\theta)$, where θ (in units of radians) is a "generic" variable, e.g. defined as $\theta = kx$ (for work in the space-domain), or $\theta = \omega t$ (for work in the time-domain). Then the "generic" variable, $\theta_n = nkx = n\theta$, or $\theta_n = n\omega t = n\theta$. Thus:
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$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\theta_n + \sum_{n=1}^{n=\infty} b_n \sin\theta_n$$

Since $\theta = kx$ or $\theta = \omega t$, then in the space-domain, since $f(x)$ is a periodic function, i.e. $f(x_2) = f(x_1)$ with $x_2 = x_1 + L$, or, in the time-domain, since $f(t)$ is a periodic function, i.e. $f(t_2) = f(t_1)$ with $t_2 = t_1 + \tau$, then generically-speaking, $f(\theta)$ is also periodic function, i.e. $f(\theta_2) = f(\theta_1)$ with $\theta_2 = \theta_1 + 2\pi$. Thus, $x_2 - x_1 = \Delta x = L$, $t_2 - t_1 = \Delta t = \tau$, and we also have $\theta_2 - \theta_1 = \Delta\theta = 2\pi$, since e.g. $\theta_2 - \theta_1 = k(x_2 - x_1) = 2\pi/\lambda * (x_2 - x_1) = 2\pi/L * (x_2 - x_1) = 2\pi(L/L) = 2\pi$, since $L = \lambda$ ($= 2\pi/k$), the wavelength of the fundamental, whose frequency is $f = \omega/2\pi$, and period $\tau = 1/f$.

The inner products, used to determine the Fourier coefficients, can also be written "generically" as:

$$a_o = \frac{1}{\pi}\langle f(x),1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta$$

$$a_n = \frac{1}{\pi}\langle f(\theta),\cos(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta$$

$$b_n = \frac{1}{\pi}\langle f(\theta),\sin(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta$$

We can also write the "generic" Fourier series expansion of the periodic function, $f(\theta)$ in *complex form*, using the relations:

$$exp(+i\theta_n) = e^{+i\theta n} = \cos\theta_n + i\sin\theta_n \quad \text{and} \quad exp(-i\theta_n) = e^{-i\theta n} = \cos\theta_n - i\sin\theta_n$$

Where i is defined as $i \equiv \sqrt{(-1)}$, thus $i * i = -1$, and $i * -i = +1$. (One can *prove* these relations e.g. by using the Taylor series expansions for both sides of each equation.) Conversely, one can also show that:

$$\cos\theta_n = ½(e^{+i\theta n} + e^{-i\theta n}) \quad \text{and} \quad i\sin\theta_n = ½(e^{+i\theta n} - e^{-i\theta n})$$

Then:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n\cos(\theta_n) + \sum_{n=1}^{n=\infty} b_n\sin(\theta_n) = \frac{a_o}{2} + \sum_{n=1}^{n=\infty} a_n\frac{(e^{i\theta_n} + e^{-i\theta_n})}{2} + \sum_{n=1}^{n=\infty} b_n\frac{(e^{i\theta_n} - e^{-i\theta_n})}{2i}$$

This expression for the periodic function, $f(\theta)$ can be written as a single sum, if we define *complex* Fourier coefficients, c_n that are linear combinations of the a_n and b_n Fourier coefficients:

$$c_o \equiv a_o, \quad c_n \equiv ½(a_n - i\,b_n) \quad \text{and} \quad c_{-n} \equiv ½(a_n + i\,b_n) \;\; (= c_n^*)$$

Then the "generic" periodic function, $f(\theta)$ can be written compactly as:

$$f(\theta) = \frac{c_o}{2} + \sum_{n=1}^{n=+\infty} c_n e^{+i\theta_n} + \sum_{n=-\infty}^{n=-1} c_{-n}e^{-i\theta_n} = \sum_{n=-\infty}^{n=+\infty} c_n e^{+i\theta_n} = \sum_{n=-\infty}^{n=+\infty} c_{-n}^* e^{-i\theta_n}$$

Note that the last two sums on the right hand side extends (in integer steps) from $n = -\infty$ to $n = +\infty$.

The complex Fourier coefficients, c_n can be determined by taking the inner product(s) of the periodic function, $f(\theta)$ with each of these new, complex basis vectors, $exp(+i\theta_n)$:

$$c_n = \frac{1}{\pi}\left\langle f(\theta), e^{+i\theta_n}\right\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)^* e^{+i\theta_n}\, d\theta$$

We can also write the Fourier series expansion of the periodic function, $f(\theta)$ in yet another way, thereby gaining some additional physical insight as to the meaning of the harmonic terms in the series. Consider the n^{th} harmonic term in the Fourier series:

$$a_n \cos\theta_n + b_n \sin\theta_n = ½\, a_n\,(e^{+i\theta n} + e^{-i\theta n}) - ½\, ib_n\,(e^{+i\theta n} - e^{-i\theta n})$$
$$= ½\,(a_n - ib_n)\, e^{+i\theta n} + ½\,(a_n + ib_n)\, e^{-i\theta n}$$

then, again defining $c_n \equiv ½\,(a_n - i\, b_n)$ and thus $c_n^* \equiv ½\,(a_n + i\, b_n)$, the *magnitude* of c_n (a *real* number) is defined as:

$$|c_n| \equiv (c_n\, c_n^*)^{½} = ½\,[(a_n - i\, b_n)(a_n + i\, b_n)]^{½} = ½\,(a_n^2 + b_n^2)^{½}$$

However, we can define a *new* complex variable, r_n such that $r_n \equiv ½\, c_n = (a_n - i\, b_n)$ and $r_n^* \equiv ½\, c_n^* = (a_n + i\, b_n)$. The *magnitude* of r_n (a *real* number) is thus defined as $|r_n| \equiv (r_n\, r_n^*)^{½} = (a_n^2 + b_n^2)^{½} = ½\,|c_n|$. Thus, we can define a *phase angle*, δ_n (in units of radians) such that:

$$\cos\delta_n \equiv a_n / |r_n| \quad \text{and} \quad \sin\delta_n \equiv b_n / |r_n|$$

or equivalently:

$$a_n \equiv |r_n| \cos\delta_n \quad \text{and} \quad b_n \equiv |r_n| \sin\delta_n$$

thus:

$$\tan\delta_n = \sin\delta_n / \cos\delta_n = (b_n / |r_n|\,) / (a_n / |r_n|\,) = b_n / a_n, \quad \text{and thus } \delta_n = \tan^{-1}(b_n / a_n).$$

Then the n^{th} harmonic term in the Fourier series becomes:

$$a_n \cos\theta_n + b_n \sin\theta_n = ½\,(a_n - ib_n)\, e^{+i\theta n} + ½\,(a_n + ib_n)\, e^{-i\theta n}$$
$$= ½\,|r_n|\,(\cos\delta_n - i\sin\delta_n)\, e^{+i\theta n} + ½\,|r_n|\,(\cos\delta_n + i\sin\delta_n)\, e^{-i\theta n}$$
$$= ½\,|r_n|\, e^{-i\delta n} e^{+i\theta n} + ½\,|r_n|\, e^{+i\delta n} e^{-i\theta n} = ½\,|r_n|\,[e^{+i(\theta n - \delta n)} + e^{-i(\theta n - \delta n)}]$$
$$= |r_n| \cos(\theta_n - \delta_n)$$

Thus, the "generic" Fourier series expansion for the periodic function, $f(\theta)$ may also be equivalently written as (defining $|r_0| \equiv a_0$):

$$f(\theta) = \frac{|r_0|}{2} + \sum_{n=1}^{n=\infty} |r_n| \cos(\theta_n - \delta_n)$$

Physically, then, it can be seen that the "generic" periodic function, $f(\theta)$ consists of a superposition (i.e. a linear combination) of waveforms, consisting of a d.c. offset (i.e. time-averaged, or frequency-independent/constant) term, $|r_0|/2$, a fundamental harmonic, $\cos\theta_1$ with amplitude, $|r_1|$ and phase angle, δ_1, with additional contributions from all higher (i.e. $n > 1$) harmonics, $\cos\theta_n$, each with amplitude, $|r_n|$ and phase angle, δ_n.

Defining a *new* phase angle, $\delta_n' \equiv \pi/2 - \delta_n$, or $\delta_n \equiv \pi/2 - \delta_n'$, it can be easily shown that $a_n \equiv |r_n| \sin \delta_n'$ and $b_n \equiv |r_n| \cos \delta_n'$, thus $\tan \delta_n' = \sin \delta_n' / \cos \delta_n' = a_n / b_n$, and thus $\delta_n' = \tan^{-1}(a_n / b_n)$, and therefore we may also equivalently write the "generic" Fourier series expansion for the periodic function, $f(\theta)$ as:

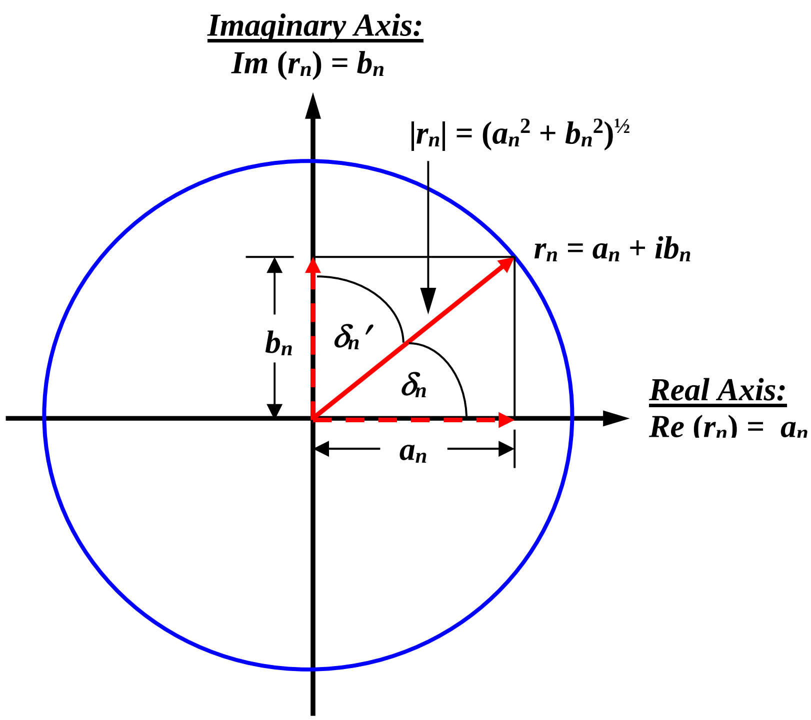
$$f(\theta) = \frac{|r_0|}{2} + \sum_{n=1}^{n=\infty} |r_n| \sin(\theta_n + \delta_n')$$

While these latter two mathematical forms of the Fourier series expansion for a "generic" periodic function, $f(\theta)$ are perhaps more physically intuitive, operationally, they are more difficult to work with, because of the fact that one of the parameters (δ_n or δ_n') for the n^{th} harmonic appears ***inside*** the argument of the *cosine* (or *sine*) function for that harmonic. This makes for difficulties, e.g. in computing inner products, for determining *both* of the parameters, $|r_n|$ and δ_n (or δ_n') for each harmonic.

Operationally-speaking, it is easier to use the above-defined inner products that enable one to determine the Fourier coefficients, a_0, a_n and b_n for each harmonic. After having determined these, ***then*** one can compute the magnitude of the complex amplitude, $|r_n|$ and phase angle, δ_n (or δ_n') for each harmonic, using the relations:

$$|r_n| = (a_n^2 + b_n^2)^{1/2} \quad \text{and} \quad \delta_n = \tan^{-1}(b_n / a_n) \quad (\text{or: } \delta_n' = \tan^{-1}(a_n / b_n))$$

We can obtain an even better physical understanding of these relations if we draw what is happening in the *complex plane*, as shown in the figure below:



We see that the Fourier coefficients, a_n and b_n are the *real* ("in-phase") and *imaginary* ("90° out-of-phase") components of the n^{th} *complex* harmonic amplitude, r_n, respectively. The Fourier coefficients, $a_n = |r_n| \cos \delta_n = |r_n| \sin \delta_n'$ and $b_n = |r_n| \sin \delta_n = |r_n| \cos \delta_n'$, where the *magnitude* of the the n^{th} complex harmonic amplitude is $|r_n| = (a_n^2 + b_n^2)^{½}$.

We also see that δ_n and δ_n' are *complementary* phase angles associated with the n^{th} harmonic, since they are related to each other by $\delta_n' = \pi/2 - \delta_n = 90^o - \delta_n$. Note also that the phase angles, δ_n (δ_n') are referenced to the real (imaginary) axes of the complex plane, respectively. By convention, usually we are most interested in the phase angle, δ_n.

Exercises:

1. Work your way through the mathematical details of changing over from the representation(s) of the Fourier series in the space-domain, to those in the time-domain.

2. Work your way through the mathematical details of obtaining the Fourier coefficients, a_0, a_n and b_n from their inner products, in the time-domain.

3. Prove, using the Taylor series expansions for e^x, *sin* (x) and *cos* (x) that $e^{+i\theta_n} = \cos\theta_n + i \sin\theta_n$ and $e^{-i\theta_n} = \cos\theta_n - i \sin\theta_n$, where $i \equiv \sqrt{(-1)}$, thus $i * i = -1$, and $i * -i = +1$.

4. Work your way through the mathematical details of obtaining the *complex* Fourier series expansion(s) with the c_n & c_{-n} Fourier coefficients, from that with the a_0, a_n and b_n Fourier coefficients.

5. Work your way through the mathematical details of deriving

$$f(\theta) = \frac{|r_0|}{2} + \sum_{n=1}^{n=\infty} |r_n| \cos(\theta_n - \delta_n)$$

from the Fourier series expansion with the the a_0, a_n and b_n Fourier coefficients.

References for Fourier Analysis and Further Reading:

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3. Mathematical Methods of Physics, 2^{nd} Edition, Jon Matthews and R.L. Walker, W.A. Benjamin, Inc., 1964.

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