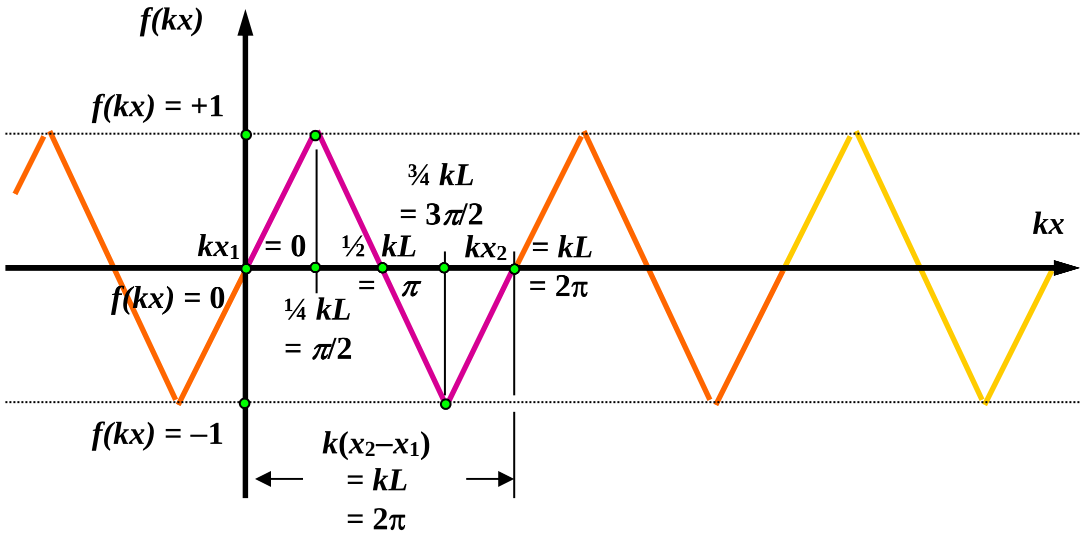
Fourier Analysis III:

More Examples of the Use of Fourier Analysis

D. Fourier Analysis of a Periodic, Symmetrical Triangle Wave

We now consider a *spatially*-periodic, symmetrical, bipolar triangle wave of *unit* amplitude, as shown in the figure below:



Mathematically, this *odd*-symmetry waveform, on the "generic" interval $0 \le \theta < 2\pi$ (i.e. one cycle of this waveform) is described as:

$$f(\theta) = f(kx) = +(2/\pi)\theta \quad \text{for} \quad 0 \le \theta < \pi/2$$

and:

$$f(\theta) = f(kx) = -(2/\pi)\theta + 2 \quad \text{for} \quad \pi/2 \le \theta < 3\pi/2$$

and:

$$f(\theta) = f(kx) = +(2/\pi)\theta - 4 \quad \text{for} \quad 3\pi/2 \le \theta < 2\pi$$

Where we used the straight line equation, $y = mx + b$ to determine the slopes, m and the intercepts, b associated with each of the three line segments in the above waveform on this θ-interval.

We determine the Fourier coefficients, a_0, a_n and b_n from the inner products:

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\langle f(\theta),1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/2} f(\theta)d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} f(\theta)d\theta + \int_{\theta=3\pi/2}^{\theta=2\pi} f(\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/2} (\tfrac{2}{\pi})\theta d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2}\left((\tfrac{-2}{\pi})\theta + 2\right)d\theta + \int_{\theta=3\pi/2}^{\theta=2\pi}\left((\tfrac{2}{\pi})\theta - 4\right)d\theta\right] \\
&= \frac{1}{\pi}\left[(\tfrac{2}{\pi})\tfrac{1}{2}\theta^2 \,\Big|_0^{\pi/2} + [(\tfrac{-2}{\pi})\tfrac{1}{2}\theta^2 + 2\theta]\,\Big|_{\pi/2}^{3\pi/2} + [(\tfrac{2}{\pi})\tfrac{1}{2}\theta^2 - 4\theta]\,\Big|_{3\pi/2}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\tfrac{1}{\pi}\left[\tfrac{1}{4}\pi^2 - \tfrac{9}{4}\pi^2 + \tfrac{1}{4}\pi^2 + 4\pi^2 - \tfrac{9}{4}\pi^2\right] + \left[3\pi - \pi - 8\pi + 6\pi\right]\right] = 0
\end{aligned}
$$

Since this waveform is <u>*bipolar*</u>, it has no d.c. offset, thus $a_0 = 0$.

The Fourier coefficients, a_n and b_n for $n > 0$ are:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\langle f(\theta),\cos(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/2} f(\theta)\cos(n\theta)d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} f(\theta)\cos(n\theta)d\theta + \int_{\theta=3\pi/2}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/2} (\tfrac{2}{\pi})\theta\cos(n\theta)d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2}\left((\tfrac{-2}{\pi})\theta + 2\right)\cos(n\theta)d\theta + \int_{\theta=3\pi/2}^{\theta=2\pi}\left((\tfrac{2}{\pi})\theta - 4\right)\cos(n\theta)d\theta\right]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\langle f(\theta),\sin(\theta_n)\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/2} f(\theta)\sin(n\theta)d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2} f(\theta)\sin(n\theta)d\theta + \int_{\theta=3\pi/2}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/2} (\tfrac{2}{\pi})\theta\sin(n\theta)d\theta + \int_{\theta=\pi/2}^{\theta=3\pi/2}\left((\tfrac{-2}{\pi})\theta + 2\right)\sin(n\theta)d\theta + \int_{\theta=3\pi/2}^{\theta=2\pi}\left((\tfrac{2}{\pi})\theta - 4\right)\sin(n\theta)d\theta\right]
\end{aligned}
$$

Now the *indefinite* integrals:

$$\int \cos(n\theta)d\theta = +\frac{\sin(n\theta)}{n} \qquad \int \sin(n\theta)d\theta = -\frac{\cos(n\theta)}{n}$$

$$\int \theta\cos(n\theta)d\theta = \frac{\cos(n\theta)}{n^2} + \frac{\theta\sin(n\theta)}{n} \qquad \int \theta\sin(n\theta)d\theta = \frac{\sin(n\theta)}{n^2} - \frac{\theta\cos(n\theta)}{n}$$

Using these relations in the above formulae for determining the Fourier coefficients, a_n and b_n we obtain, after much algebra and using the fact that $sin(3n\pi/2) = -\,sin(n\pi/2)$, that:

$$a_n = 0 \text{ for } all\ n > 0$$

and:

$$b_n = 2*(2/n\pi)^2\, sin(n\pi/2)$$

The *even* Fourier coefficients, $b_n = 0$ for n = 2, 4, 6, 8, ... etc.

The *odd* Fourier coefficients, $b_n = +2*(2/n\pi)^2$ for n = 1, 5, 9, 13, ... etc.

The *odd* Fourier coefficients, $b_n = -2*(2/n\pi)^2$ for n = 3, 7, 11, 15, ... etc.

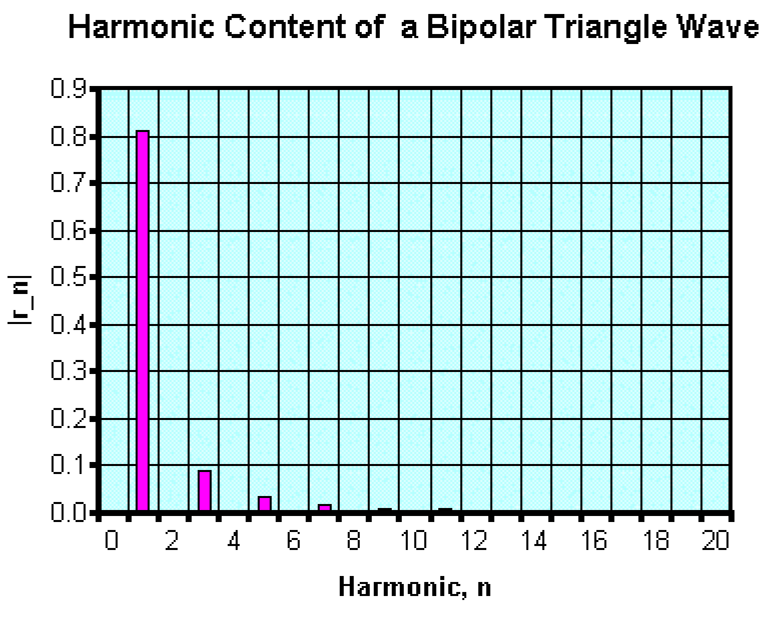
Thus, the Fourier series for the symmetrical, bipolar triangle wave of *unit* amplitude, as shown in the above figure is given by:

$$f(\theta)\big|_{\substack{triangle\\-wave}} = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\theta_n + \sum_{n=1}^{n=\infty} b_n \sin\theta_n = 2\sum_{\substack{n=1\\odd-n}}^{n=\infty} (-1)^{(n-1)/2}\left(\tfrac{2}{n\pi}\right)^2 \sin(n\theta)$$

Using the replacement: $n_{odd} = 2\,m - 1$, m = 1, 2, 3, 4, ... in the above summation, we can alternatively write the Fourier series expansion for this triangle wave as:

$$f(\theta)\big|_{\substack{triangle\\-wave}} = 2\sum_{m=1}^{m=\infty} (-1)^{m-1}\left(\tfrac{2}{(2m-1)\pi}\right)^2 \sin[(2m-1)\theta] = \tfrac{8}{\pi^2}\left\{\sin\theta - \tfrac{1}{9}\sin 3\theta + \tfrac{1}{25}\sin 5\theta - \tfrac{1}{49}\sin 7\theta + ...\right\}$$

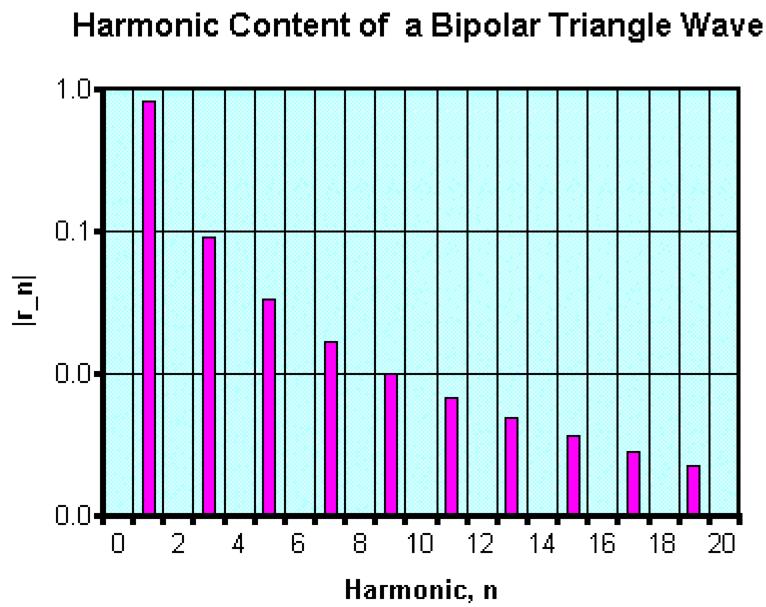
Note that the *magnitudes* of the non-zero amplitudes of the harmonics, $|r_n| = |b_n| = 8/(n\pi)^2$, as shown in the figure(s) below for the first 20 harmonics.



The non-zero amplitudes of the harmonics, $|r_n|$ associated with the bipolar triangle wave decrease much faster with increasing harmonic #, *n* than e.g. those associated with the bipolar, 50% duty-cycle square wave. The harmonic amplitudes, $|r_n|$ associated with the bipolar triangle wave vary with *n* as $|r_n| \sim 1/n^2$, whereas the harmonic amplitudes, $|r_n|$ for the bipolar, 50% duty cycle square wave vary as $|r_n| \sim 1/n$.

As can be seen from the above figure, in addition to the fundamental, at frequency, *f*, only the *odd* harmonics, at frequencies 3*f*, 5*f*, 7*f*, 9*f*, ... etc. contribute to creating this waveform.

For comparison purposes, we also show the harmonic amplitudes, $|r_n|$ associated with the bipolar triangle wave on a *semi-log* plot, in the following figure:



The human ear hears a triangle-wave audio signal as being "bright", relative to e.g. a pure-tone (sine-wave) audio signal at the same frequency, but less "bright" than a square wave. The triangle wave, like the square wave audio signal also sounds a bit "harsh" to the human ear, because of the presence of all of the *odd* harmonics, at 3*f*, 5*f*, 7*f*, 9*f*, ... etc. But again, the triangle wave is not as harsh-sounding as the square wave is to the human ear, because its higher harmonics are not as strong as those associated with the square wave.

If the loudness of the fundamental ($n = 1$) is $L_1 = 60\ dB\ (100\ dB)$ for a triangle wave, this corresponds to an intensity associated with the fundamental tone of $I_1 = 10^{-6}\ (10^{-2})\ Watts/m^2$, respectively. If the *ratio* of the amplitude for the n^{th} harmonic to the amplitude of the fundamental associated with the triangle wave is $|r_n| / |r_1| = 1/n^2$, for *odd* $n = 3, 5, 7, 9, \ldots$ etc. Then the ratio of intensity for the n^{th} harmonic to the intensity for the fundamental associated with the triangle wave is $I_n / I_1 = (1/n)^4$, and the terms, e.g for $n = 3$ are:

$$log_{10}(I_n / I_1) = log_{10}(1/n)^4 = 4\ log_{10}(1/n) = 4\ log_{10}(0.3333) = -1.9085$$

and

$$log_{10}(I_1 / I_o) = 6\ (10) \quad \text{for} \quad I_1 = 10^{-6}\ (10^{-2})\ Watts/m^2, \quad \text{respectively.}$$

Thus, the human ear will perceive the loudness, L_n of the n^{th} harmonic, relative to perceived loudness, L_1 of the fundamental of the triangle wave, as heard e.g. through a loudspeaker as:

$$L_n / L_1 = 1 + \{log_{10}(I_n / I_1) \,/\, log_{10}(I_1 / I_o)\}$$

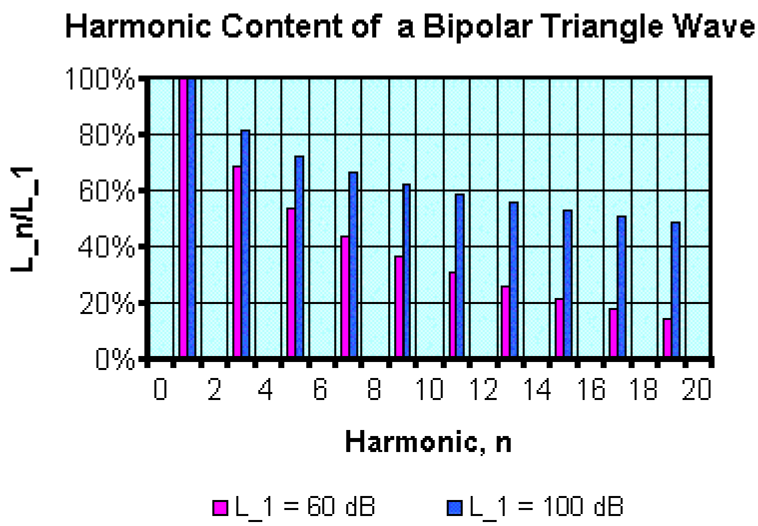
Then for the 3rd harmonic:

$$L_3 / L_1 = 1 - \{1.9085 / 6\} \quad (= 1 - \{1.9085 / 10\})$$
$$= 68.2\% \qquad (= 80.9\%)$$

for $I_1 = 10^{-6}\ (10^{-2})\ Watts/m^2$, respectively. This is the (fractional) amount of third harmonic, as heard by the human ear for a triangle wave. This is quite large, but again, not as large as that for the square wave! Again, note that the ratio, L_n / L_1 increases (logarithmically) with increasing amplitude of the square wave! For a loudness of the fundamental tone of $L_1 = 60\ dB\ (100\ dB)$, the loudness of the third harmonic, for $|r_3| / |r_1| = 1/3 = 33.3\%$ is:

$$L_3 = 10\ log_{10}(I_3 / I_1) + 10\ log_{10}(I_1 / I_o)$$
$$= 40\ log_{10}(0.3333) + 60\ dB\ (100\ dB)$$
$$= -19.08\ dB + 60\ dB\ (100\ dB)$$
$$= 40.92\ dB\ (80.92\ dB), \text{ respectively.}$$

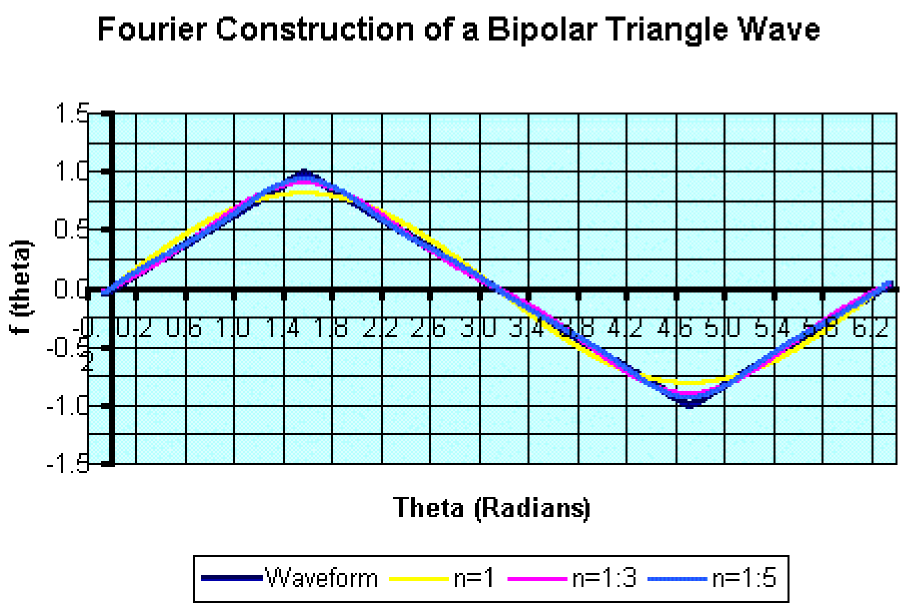
The following figure shows the loudness ratios, L_n / L_1 for the first twenty harmonics (i.e. $n < 20$) associated with the bipolar triangle wave, for loudness values of the fundamental of $L_1 = 60\ dB$ (~ quiet) and for $L_1 = 100\ dB$ (~ quite loud). This is what the human ear perceives as the loudness of the harmonics relative to that of the fundamental. Note that the decrease in the loudness ratio, L_n / L_1 with increasing harmonic #, n is quite slow.



The following two figures show the "Fourier construction" of a periodic, bipolar, unit-amplitude triangle wave. The waveforms in these figures were generated using truncated, finite-term version(s) of the Fourier series expansion for this waveform:

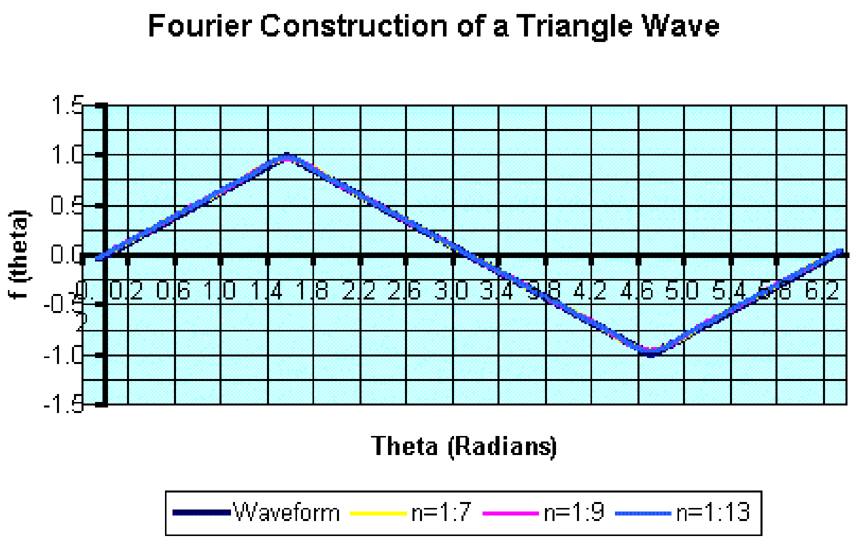
$$f(\theta)\big|_{\substack{triangle\\-wave}} = 2\sum_{m=1}^{m=\infty}(-1)^{m-1}\left(\tfrac{2}{(2m-1)\pi}\right)^2 \sin[(2m-1)\theta] = \tfrac{8}{\pi^2}\left\{\sin\theta - \tfrac{1}{9}\sin 3\theta + \tfrac{1}{25}\sin 5\theta - \tfrac{1}{49}\sin 7\theta + ...\right\}$$

The first figure shows the bipolar triangle wave (labelled as "Waveform") overlaid with three other waveforms: that associated with just the fundamental ("*n* = 1"), then the waveform associated with fundamental + 3rd harmonic ("*n* = 1:3"), then the waveform associated with fundamental + 3rd + 5th harmonic ("*n* = 1:5").



Note that the fundamental, a *sine* wave, is already a quite good approximation to the triangle wave (visually-speaking, but not auditorially so!). Just adding the first two harmonics to the fundamental brings this waveform into quite good visual agreement with the triangle wave, except at the sharp peak(s) of the triangle wave.

The second figure shows the bipolar triangle wave (labelled as "Waveform") overlaid with three other waveforms: that associated with the fundamental through the 7^{th} harmonic ("n = 1:7"), then the waveform associated with fundamental through the 9^{th} harmonic ("n = 1:9"), then the waveform associated with fundamental through the 13^{th} harmonic ("n = 1:13").



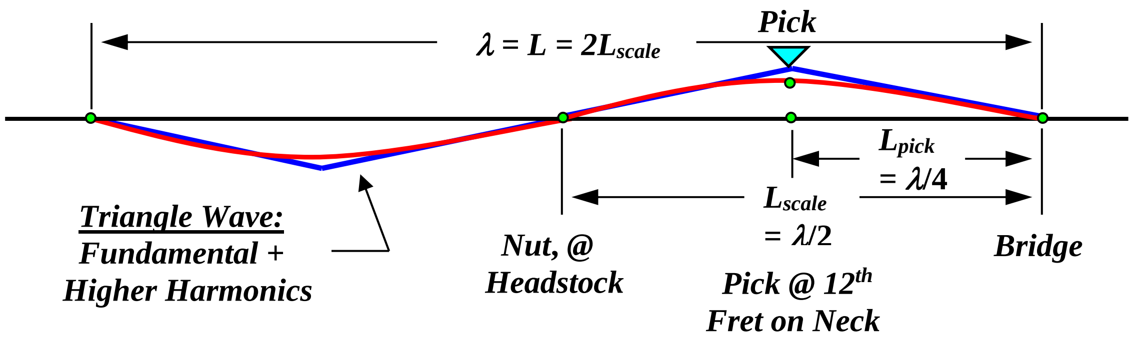
Thus, adding on higher harmonics to the lower-order harmonics associated with the triangle wave makes for only small visual changes in the overall waveform - primarily, just the peak(s) sharpen as the higher harmonics are added.

The bipolar triangle wave has physical relevance in stringed instruments, such as the guitar or violin, when the strings are *plucked* at the *mid-point* of the string, along its length, e.g. using one's fingernail, or a guitar pick (aka *plectrum*).

The *scale length*, L_{scale} of a guitar is the *physical* length of the string(s) from the *bridge* to the *nut* at the headstock on the neck of the guitar. When one of the open (i.e. unfretted) strings vibrates, the fundamental mode of vibration of frequency, f with a *wavelength*, λ equal to *twice* the scale length of the guitar, i.e. $\lambda = 2L_{scale}$. In other words, the scale length of a guitar is *half* the wavelength of the fundamental, i.e. $L_{scale} = \lambda/2$. In our discussion of Fourier analysis, the wavelength, λ of the fundamental is *equal* to the space-domain length parameter, L, i.e. $\lambda = L$. Thus, the scale length, $L_{scale} = \lambda/2 = L/2$.

As shown in the figure below, the fundamental has a *node* (i.e. points of zero transverse displacement) at both ends of its wavelength, *and* at its midpoint. All harmonic waves on a guitar *must* have nodes at the bridge and nut, since these do not vibrate (to a first approximation). These *boundary conditions* mandate *sine* wave-type solutions!

Vibration of the Fundamental of a Guitar String



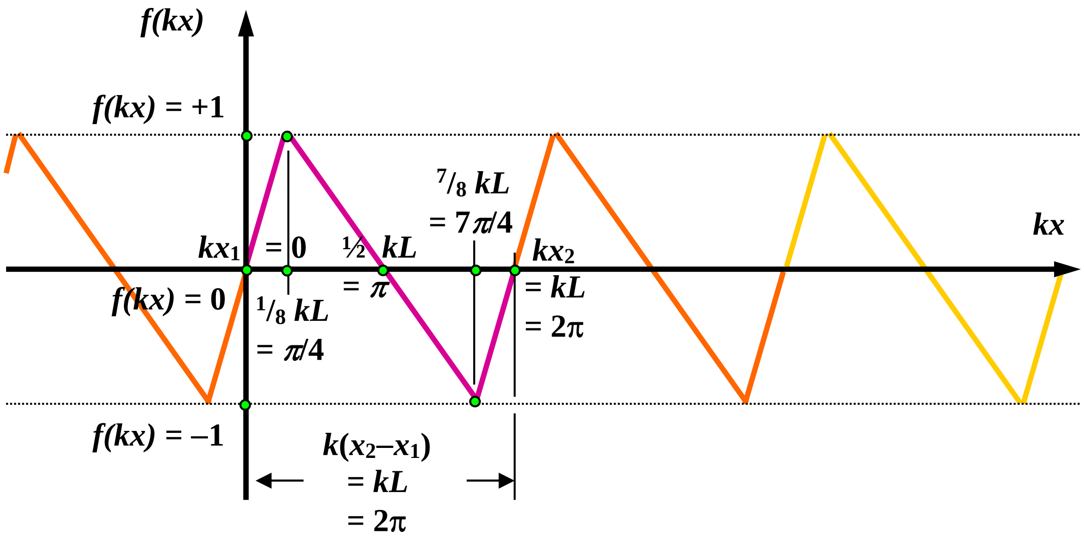
As indicated in the above figure, when an open string vibrates in the fundamental mode, this occurs only on a half-length of the fundamental (here the right-hand half) – the left-hand half of the fundamental doesn't physically exist in stringed instruments.

The pick (here) is used to excite an open guitar string at its *midpoint* – at the 12th fret (i.e. 1st octave location), which is an *anti-node* of the fundamental (i.e. a point of maximum *transverse* displacement). This position is a distance of $L_{pick} = \lambda/4$ from the bridge of the guitar. At this loction, the pick stretches the string transversely from its zero-displacement equilibrium position. Before the pick is released from stretching the string, the energy associated with the stretching of the string into this shape is entirely in the form of mechanical *potential* energy. At the precise instant the pick disengages from the stretched string, the shape of the string *is* a symmetric (i.e. isosceles) triangle. Immediately after the pick releases the string, the string begins to vibrate, converting the mechanical potential energy back and forth into kinetic energy (and also radiating some of this energy away as sound waves). However *because* of energy conservation, all the energy initially contained in each of the harmonics is also preserved (see Parseval's theorem) and thus, the initial shape of the string at the instant it was released from the disengagement of the pick is also preserved. In other words, the transverse *shape* of the string the instant before it is released *dictates* its harmonic sound-content afterward!

Guitar players do not normally play at this location on the guitar, because picking the strings of the guitar with the fingerboard/fretboard immediately underneath is difficult. However, those guitarists who *have* tried playing there know that the resulting sound output from the guitar is quite mellow, because picking the strings at this location predominantly excites the fundamental at frequency, f. The second harmonic, one octave above at frequency, $2f$ is *completely* absent in picking the strings at this location on the guitar, because the second harmonic has a *node* at $L_{pick} = \lambda/4$ - i.e. it *cannot* be excited by picking here! In fact *none* of the even-n harmonics - at $2f$, $4f$, $6f$, $8f$, $10f$, ... etc. can be excited by picking at $L_{pick} = \lambda/4$ because they *all* have nodes at this point! In addition to the fundamental, only the odd-n harmonics of the fundamental can be excited by playing at the 12th fret of the guitar - in fact the odd-n harmonics all have anti-nodes at this point!

E. Fourier Analysis of a Periodic Sawtooth (Asymmetrical Triangle) Wave

Next, we consider a spatially-periodic bipolar *sawtooth* wave, i.e. an *asymmetrical* bipolar triangle wave of *unit* amplitude, as shown in the figure below:



Mathematically, this *odd*-symmetry waveform, on the "generic" interval $0 \le \theta < 2\pi$ (i.e. one cycle of this waveform) is described as:

$$f(\theta) = f(kx) = +(4/\pi)\theta \quad \text{for} \quad 0 \le \theta < \pi/4$$

and:

$$f(\theta) = f(kx) = -(4/3\pi)\theta + 4/3 \quad \text{for} \quad \pi/4 \le \theta < 7\pi/4$$

and:

$$f(\theta) = f(kx) = +(4/\pi)\theta - 8 \quad \text{for} \quad 7\pi/4 \le \theta < 2\pi$$

Where we used the straight line equation, $y = mx + b$ to determine the slopes, m and the intercepts, b associated with each of the three line segments in the above waveform on this θ-interval.

We determine the Fourier coefficients, a_0, a_n and b_n from the inner products:

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\langle f(\theta), 1\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} f(\theta)d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} f(\theta)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} f(\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} (\tfrac{4}{\pi})\theta d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4}\left((\tfrac{-4}{3\pi})\theta + \tfrac{4}{3}\right)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi}\left((\tfrac{4}{\pi})\theta - 8\right)d\theta\right] \\
&= \frac{1}{\pi}\left[(\tfrac{4}{\pi})\tfrac{1}{2}\theta^2 \Big|_0^{\pi/4} + [(\tfrac{-4}{3\pi})\tfrac{1}{2}\theta^2 + \tfrac{4}{3}\theta]\Big|_{\pi/4}^{7\pi/4} + [(\tfrac{4}{\pi})\tfrac{1}{2}\theta^2 - 8\theta]\Big|_{7\pi/4}^{2\pi}\right] \\
&= \frac{1}{\pi}\left[\tfrac{1}{\pi}\left[\tfrac{1}{4}\pi^2 - \tfrac{9}{4}\pi^2 + \tfrac{1}{4}\pi^2 + 4\pi^2 - \tfrac{9}{4}\pi^2\right] + \left[3\pi - \pi - 8\pi + 6\pi\right]\right] = 0
\end{aligned}$$

Since this waveform has no d.c. offset, $a_0 = 0$. The Fourier coefficients, a_n and b_n are:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left\langle f(\theta),\cos(\theta_n)\right\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} f(\theta)\cos(n\theta)d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} f(\theta)\cos(n\theta)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} (\tfrac{4}{\pi})\theta\cos(n\theta)d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} \left((\tfrac{-4}{3\pi})\theta + \tfrac{4}{3}\right)\cos(n\theta)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} \left((\tfrac{4}{\pi})\theta - 8\right)\cos(n\theta)d\theta\right]
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\left\langle f(\theta),\sin(\theta_n)\right\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} f(\theta)\sin(n\theta)d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} f(\theta)\sin(n\theta)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=\pi/4} (\tfrac{4}{\pi})\theta\sin(n\theta)d\theta + \int_{\theta=\pi/4}^{\theta=7\pi/4} \left((\tfrac{-4}{3\pi})\theta + \tfrac{4}{3}\right)\sin(n\theta)d\theta + \int_{\theta=7\pi/4}^{\theta=2\pi} \left((\tfrac{4}{\pi})\theta - 8\right)\sin(n\theta)d\theta\right]
\end{aligned}
$$

Now the *indefinite* integrals:

$$\int \cos(n\theta)d\theta = +\frac{\sin(n\theta)}{n} \qquad \int \sin(n\theta)d\theta = -\frac{\cos(n\theta)}{n}$$

$$\int \theta\cos(n\theta)d\theta = \frac{\cos(n\theta)}{n^2} + \frac{\theta\sin(n\theta)}{n} \qquad \int \theta\sin(n\theta)d\theta = \frac{\sin(n\theta)}{n^2} - \frac{\theta\cos(n\theta)}{n}$$

Using these relations in the above formulae for determining the Fourier coefficients, a_n and b_n, for $n > 0$. We obtain, after much algebra and using the fact(s) that $\sin(7n\pi/4) = -\sin(n\pi/4)$, and $\cos(7n\pi/4) = +\cos(n\pi/4)$ that the Fourier coefficients:

$a_n = 0$ for *all* $n > 0$

and:

$b_n = (2/3)*(4/n\pi)^2 \sin(n\pi/4)$ for *all* $n > 0$

The *odd* Fourier coefficients, $b_n = +(2/3)*(4/n\pi)^2/\sqrt{2}$ for n = 1, 9, 17, 25, ... etc.
The *even* Fourier coefficients, $b_n = +(2/3)*(4/n\pi)^2$ for n = 2, 10, 18, 26, ... etc.
The *odd* Fourier coefficients, $b_n = +(2/3)*(4/n\pi)^2/\sqrt{2}$ for n = 3, 11, 19, 27, ... etc.
The *even* Fourier coefficients, $b_n = 0$ for n = 4, 12, 20, 28, ... etc.
The *odd* Fourier coefficients, $b_n = -(2/3)*(4/n\pi)^2/\sqrt{2}$ for n = 5, 13, 21, 29, ... etc.
The *even* Fourier coefficients, $b_n = -(2/3)*(4/n\pi)^2$ for n = 6, 14, 22, 30, ... etc.
The *odd* Fourier coefficients, $b_n = -(2/3)*(4/n\pi)^2/\sqrt{2}$ for n = 7, 15, 23, 31, ... etc.
The *even* Fourier coefficients, $b_n = 0$ for n = 8, 16, 24, 32, ... etc.

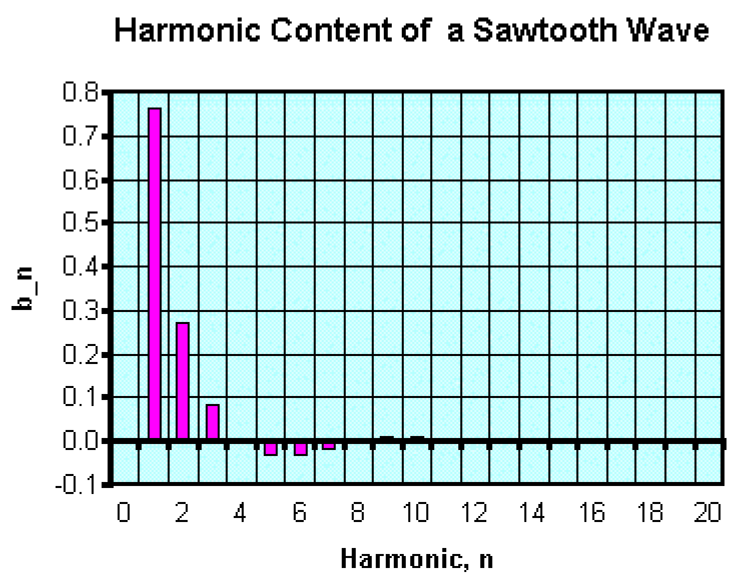
All of the *even*-reflection symmetry Fourier coefficients, $a_n = 0$ because the sawtooth waveform has overall *odd*-reflection symmetry.

Thus, for the sawtooth form of a triangle wave, *both* even-n *and* odd-n b_n-harmonics are present! The reason for this is that while the *overall* sawtooth waveform still has *odd* reflection symmetry about its midpoint ($\theta = \pi$), i.e. that for $0 \le \theta \le 2\pi$, $f(\theta > \pi) = -f((2\pi - \theta) < \pi)$, the sawtooth waveform no longer has any *local* reflection symmetry properties about its peaks - e.g. about $\theta = \pi/4$ and/or about $\theta = 7\pi/4$, i.e. locally, for $0 \le \theta \le \pi/2$, $f(\theta > \pi/4) \ne \pm f((\pi/2 - \theta) < \pi/4)$, and for $3\pi/2 \le \theta \le 2\pi$, $f(\theta > 7\pi/4) \ne \pm f((2\pi - \theta) < 7\pi/4)$. Because of this, both odd-n and even-n terms in the Fourier coefficients, b_n are needed for the overall odd-reflection symmetry *sin* $(n\theta)$ functions associated with the Fourier series expansion for the bipolar sawtooth waveform.

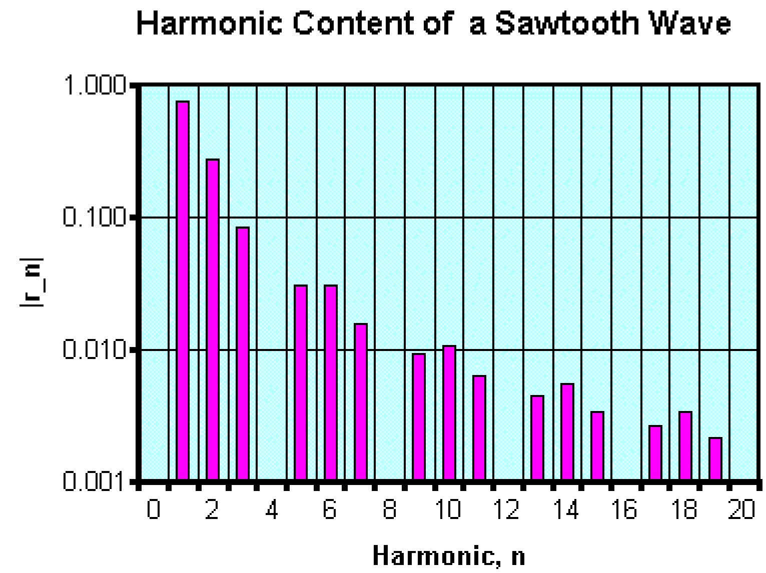
The Fourier series for the bipolar sawtooth wave of <u>*unit*</u> amplitude, is thus given by:

$$f(\theta)\Big|_{\substack{sawtooth\\ -wave}} = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos\theta_n + \sum_{n=1}^{n=\infty} b_n \sin\theta_n = \tfrac{2}{3}\sum_{n=1}^{n=\infty}\left(\tfrac{4}{n\pi}\right)^2 \sin(\tfrac{n\pi}{4})\sin(n\theta)$$

The numerical values of the Fourier coefficients, b_n for the bipolar sawtooth wave are shown in the figure below for the first 20 harmonics.

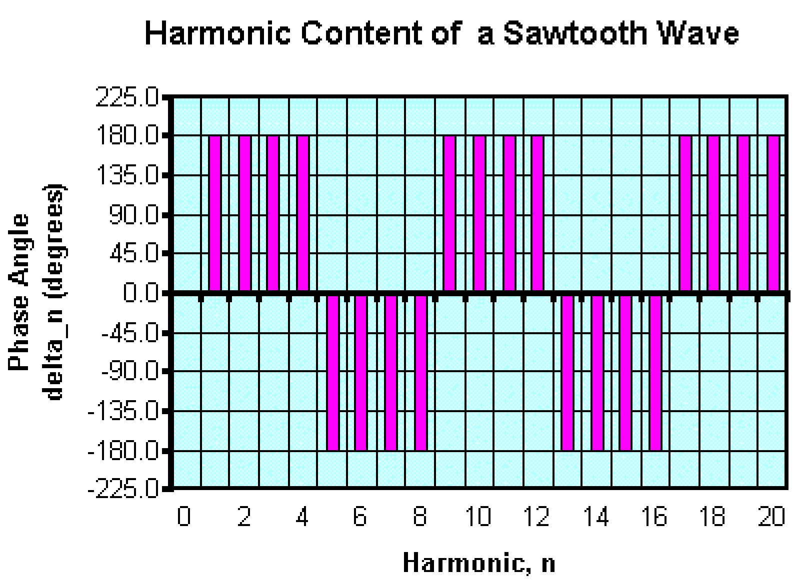


The *magnitudes* of the amplitudes of the harmonics, $|r_n| = |b_n|$ for the bipolar sawtooth wave, again decrease with increasing harmonic #, n, as $\sim 1/n^2$, as for the bipolar triangle wave. We show the numerical values of the $|r_n|$ for the first 20 harmonics of the bipolar sawtooth wave in the figure below. Note that this is a *semi-log* plot of $|r_n|$ vs. n.



We can compare e.g. the relative strength of the third harmonic to the fundamental for the bipolar sawtooth wave to that for the third harmonic associated with the bipolar triangle wave. For the sawtooth wave, $|r_3| / |r_1|$ = 11.1%, while for the triangle wave, we also have $|r_3| / |r_1|$ = 11.1% - i.e. the same value of harmonic amplitude ratio! In fact the ratios $|r_n| / |r_1|$ for ***all*** odd-n harmonics are ***identical*** for triangle vs. sawtooth waves!

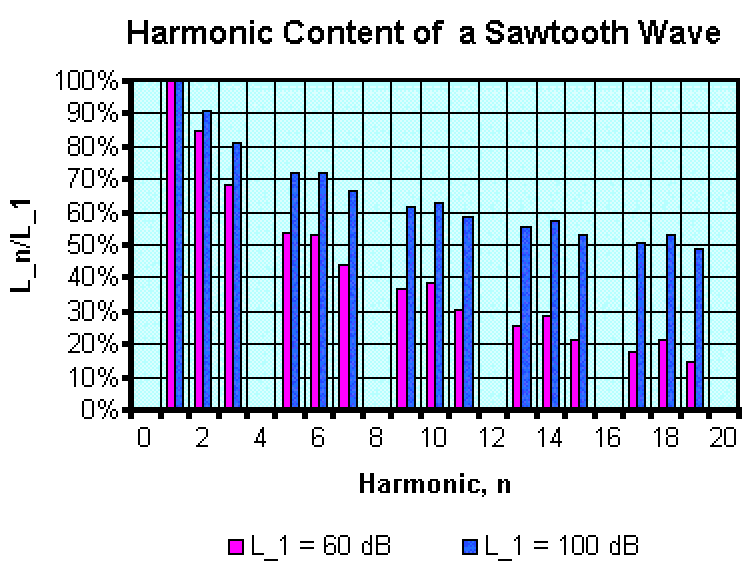
The phase angles, δ_n of the harmonics associated with the bipolar sawtooth wave are shown in the figure below for the first 20 harmonics.



Note that the first four harmonics - the fundamental (aka first harmonic), the second, third and fourth (even though it has zero strength) harmonics all have the same phase angle, $\delta_n = +180^o$. The next four harmonics have the opposite phase angle, $\delta_n = -180^o$, the next four after that are in phase again with the first four harmonics, and so on. This behavior of the groups-of-four phase angle arises from the *sin* ($n\pi/4$) term in the Fourier coefficients, b_n for the sawtooth waveform.

The *sound* of an audio sawtooth wave to the human ear is *brighter* than the triangle wave, due to the existence of the *second* harmonic in the sawtooth wave, which is *absent* in the triangle wave. If the loudness of the fundamental, $L_1 = 60\ dB$ ($100\ dB$), then the loudness of the second harmonic is $L_2 = 50.9\ dB$ ($91.0\ dB$), corresponding to a loudness ratio of $L_2 / L_1 =$ 84.9% (91.0%), respectively. For the third harmonic associated with the sawtooth wave, $L_3 = 40.9\ dB$ ($80.9\ dB$), corresponding to a loudness ratio of $L_3 / L_1 =$ 68.2% (80.9%), respectively. Interestingly enough, these loudness results for the third harmonic of the sawtooth wave are also *precisely* those for the triangle wave, as are all the odd-*n* loudness results! The sawtooth wave differs from the triangle wave primarily because of the additional presence of the even-*n* harmonics, however note also that the *phase relations* for the odd-*n* harmonics are *not* the same for these two waves. As we have mentioned before, the human ear is *not* sensitive to such phase relations.

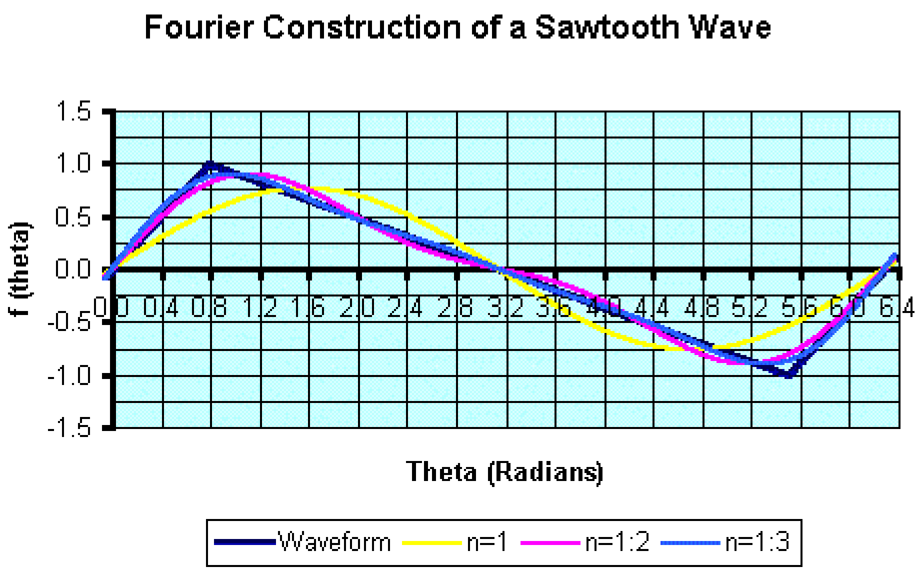
The following figure shows the loudness ratios, L_n / L_1 for the first twenty harmonics (i.e. $n < 20$) associated with the bipolar sawtooth wave, for loudness values of the fundamental of $L_1 = 60\ dB$ (~ quiet) and for $L_1 = 100\ dB$ (~ quite loud). This is what the human ear perceives as the loudness of the harmonics relative to that of the fundamental. Note that the decrease in the loudness ratio, L_n / L_1 with increasing harmonic #, n is again rather slow.



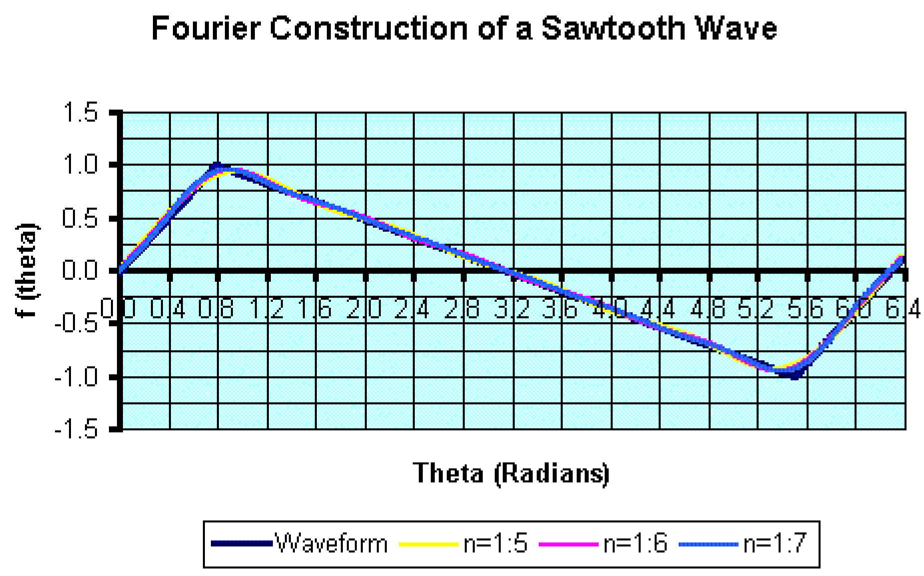
The following two figures show the "Fourier construction" of a periodic, bipolar, unit-amplitude sawtooth wave. The waveforms in these figures were generated using truncated, finite-term version(s) of the Fourier series expansion for this waveform:

$$f(\theta)\,|_{\substack{sawtooth\\-wave}} = \frac{2}{3}\sum_{n=1}^{n=\infty}\left(\frac{4}{n\pi}\right)^2 \sin(\tfrac{n\pi}{4})\sin(n\theta) = \frac{32}{3\pi^2}\left\{\frac{1}{\sqrt{2}}\sin(\theta)+\frac{1}{4}\sin(2n\theta)+\frac{1}{9\sqrt{2}}\sin(3n\theta)+0-...+...\right\}$$

The first figure shows the bipolar sawtooth wave (labelled as "Waveform") overlaid with three other waveforms: that associated with just the fundamental ("*n* = 1"), then the waveform associated with fundamental + 2nd harmonic ("*n* = 1:2"), then the waveform associated with fundamental + 2nd + 3rd harmonic ("*n* = 1:3").

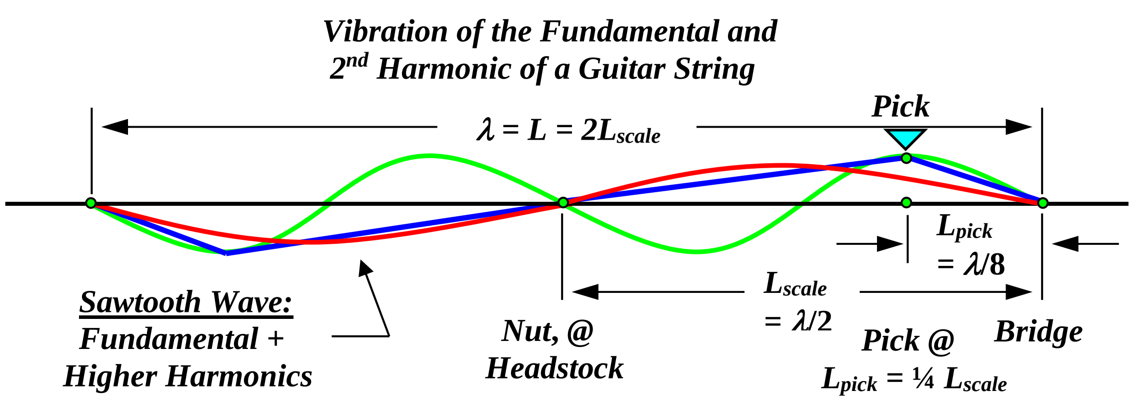


The second figure shows the bipolar sawtooth wave (labelled as "Waveform") overlaid with three other waveforms: that associated with the fundamental through the 5th harmonic ("*n* = 1:5"), then the waveform associated with fundamental through the 6th harmonic ("*n* = 1:6"), then the waveform associated with fundamental through the 7th harmonic ("*n* = 1:7")



Again, adding on higher harmonics to the lower-order harmonics associated with the sawtooth wave makes for only small visual changes in the overall waveform - primarily, just the peak(s) sharpen as the higher harmonics are added.

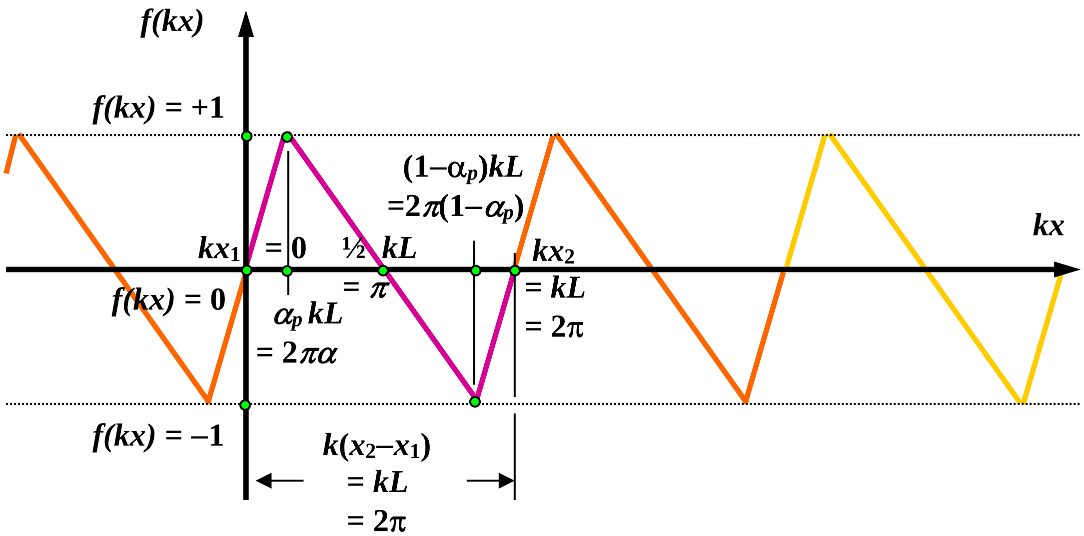
The sawtooth wave again has physical relevance in stringed instruments, such as the guitar or violin, when the strings are plucked at the <u>*one-quarter-point*</u> along the length of the string (as measured from the bridge), using either one's fingernail or a guitar pick, as shown in the figure below. This is the region along the strings where guitar players spend much of their time playing notes and/or chords on the guitar.



As can be seen from the figure, this picking location is *not* at the anti-node of the fundamental. It *is*, however at the anti-node associated with the second harmonic, an octave above the fundamental. The picking location is *near to*, but not *on* the anti-node of the third harmonic (e.g. see diagram, 2nd figure below). As mentioned earlier, this picking location is at the *node* of the fourth harmonic, which is the physical reason why *it* is not excited. As we mentioned earlier, the fundamental, 2nd and 3rd harmonics are in phase with each other for the sawtooth wave.

F. Fourier Analysis of a Generalized Sawtooth (Asymmetrical Triangle) Wave

We can generalize our above formalism for a bipolar sawtooth (asymmetrical triangle) wave of *unit* amplitude, for a sawtooth wave of any kind, as shown in the figure below:



We introduce the parameter, $\alpha_p \equiv \theta_p/2\pi = kx_p/2\pi = 2\pi x_p/2\pi\lambda = x_p/\lambda$ which physically represents the fractional location of the *first* peak in the sawtooth waveform, located at $\theta_p = kx_p = \alpha_p\, kL = 2\pi\alpha_p$. The parameter α_p *cannot* physically be larger than ½, because the sawtooth wave, $f(\theta)$ *must* be a single-valued function on the "generic" interval $0 \le \theta < 2\pi$, requiring that the first peak in the sawtooth waveform lie within the "generic" interval $0 \le \theta_p < \pi$, which in turn corresponds to a range allowed for the α_p-parameter of $0 < \alpha_p < ½$.

Mathematically, the *odd*-symmetry sawtooth waveform, on the "generic" interval $0 \le \theta < 2\pi$ (i.e. one cycle of this waveform) can then be described in terms of the α_p-parameter as:

$$f(\theta) = f(kx) = +\left({}^{1}/_{2\pi\,\alpha_p}\right)\theta \qquad \text{for} \qquad 0 \le \theta < 2\pi\alpha_p$$

and:

$$f(\theta) = f(kx) = -\left({}^{1}/_{\pi\,(1-2\alpha_p)}\right)\theta + \left({}^{1}/_{(1-2\alpha_p)}\right) \quad \text{for} \quad 2\pi\alpha_p \le \theta < 2\pi(1-\alpha_p)$$

and:

$$f(\theta) = f(kx) = +\left({}^{1}/_{2\pi\,\alpha_p}\right)\theta - \left({}^{1}/_{\alpha_p}\right) \qquad \text{for} \qquad 2\pi(1-\alpha_p) \le \theta < 2\pi$$

Where we used the straight line equation, $y = mx + b$ to determine the slopes, m and the intercepts, b associated with each of the three line segments in the above waveform on this θ-interval.

Again, we can determine the Fourier coefficients, a_0, a_n and b_n from the inner products:

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\left\langle f(\theta),1\right\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=2\pi\alpha} f(\theta)d\theta + \int_{\theta=2\pi\alpha}^{\theta=2\pi(1-\alpha)} f(\theta)d\theta + \int_{\theta=2\pi(1-\alpha)}^{\theta=2\pi} f(\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=2\pi\alpha} (\tfrac{1}{2\pi\alpha})\theta d\theta + \int_{\theta=2\pi\alpha}^{\theta=2\pi(1-\alpha)} \left((\tfrac{-1}{\pi(1-2\alpha)})\theta + \tfrac{1}{1-2\alpha}\right)d\theta + \int_{\theta=2\pi(1=\alpha)}^{\theta=2\pi} \left((\tfrac{1}{2\pi\alpha})\theta - \tfrac{1}{\alpha}\right)d\theta\right]
\end{aligned}
$$

Since this *bipolar* sawtooth waveform has no d.c. offset, we know that $a_0 = 0$.

The Fourier coefficients, a_n and b_n are:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left\langle f(\theta),\cos(\theta_n)\right\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\cos(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=2\pi\alpha} f(\theta)\cos(n\theta)d\theta + \int_{\theta=2\pi\alpha}^{\theta=2\pi(1-\alpha)} f(\theta)\cos(n\theta)d\theta + \int_{\theta=2\pi(1-\alpha)}^{\theta=2\pi} f(\theta)\cos(n\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=2\pi\alpha} (\tfrac{1}{2\pi\alpha})\theta\cos(n\theta)d\theta + \int_{\theta=2\pi\alpha}^{\theta=2\pi(1-\alpha)} \left((\tfrac{-1}{\pi(1-2\alpha)})\theta + \tfrac{1}{1-2\alpha}\right)\cos(n\theta)d\theta + \int_{\theta=2\pi(1=\alpha)}^{\theta=2\pi} \left((\tfrac{1}{2\pi\alpha})\theta - \tfrac{1}{\alpha}\right)\cos(n\theta)d\theta\right]
\end{aligned}
$$

and:

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\left\langle f(\theta),\sin(\theta_n)\right\rangle = \frac{1}{\pi}\int_{\theta=\theta_1}^{\theta=\theta_2} f(\theta)\sin(\theta_n)d\theta = \frac{1}{\pi}\int_{\theta=0}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=2\pi\alpha} f(\theta)\sin(n\theta)d\theta + \int_{\theta=2\pi\alpha}^{\theta=2\pi(1-\alpha)} f(\theta)\sin(n\theta)d\theta + \int_{\theta=2\pi(1-\alpha)}^{\theta=2\pi} f(\theta)\sin(n\theta)d\theta\right] \\
&= \frac{1}{\pi}\left[\int_{\theta=0}^{\theta=2\pi\alpha} (\tfrac{1}{2\pi\alpha})\theta\sin(n\theta)d\theta + \int_{\theta=2\pi\alpha}^{\theta=2\pi(1-\alpha)} \left((\tfrac{-1}{\pi(1-2\alpha)})\theta + \tfrac{1}{1-2\alpha}\right)\sin(n\theta)d\theta + \int_{\theta=2\pi(1=\alpha)}^{\theta=2\pi} \left((\tfrac{1}{2\pi\alpha})\theta - \tfrac{1}{\alpha}\right)\sin(n\theta)d\theta\right]
\end{aligned}
$$

Again, we will need to use the *indefinite* integrals:

$$\int\cos(n\theta)d\theta = +\frac{\sin(n\theta)}{n} \qquad \int\sin(n\theta)d\theta = -\frac{\cos(n\theta)}{n}$$

$$\int\theta\cos(n\theta)d\theta = \frac{\cos(n\theta)}{n^2} + \frac{\theta\sin(n\theta)}{n} \qquad \int\theta\sin(n\theta)d\theta = \frac{\sin(n\theta)}{n^2} - \frac{\theta\cos(n\theta)}{n}$$

And again using the fact(s) that:

$$sin(2n\pi(1-\alpha_p)) = -\,sin(2n\pi\alpha_p)$$

and that:

$$cos(2n\pi(1-\alpha_p)) = +\,cos(2n\pi\alpha_p)$$

We will again find that due to the intrinsic, overall odd-symmetry of the bipolar sawtooth waveform, that

$$a_n = 0 \text{ for } all\ n > 0$$

and that:

$$b_n = 2*\left[{}^{2\alpha p}/_{(1-2\alpha p)}\right]*\left({}^{1}/_{2n\pi\alpha p}\right)^2 \sin(2n\pi\alpha_p) \text{ for } all\ n > 0$$

The factor in brackets, $\left[{}^{2\alpha p}/_{(1-2\alpha p)}\right]$ has physical significance - it is the (absolute-value) ratio of the *slope* of the middle portion of the sawtooth waveform to the *slope* of (either) end-portion of the sawtooth waveform, i.e.

$$\left[{}^{2\alpha p}/_{(1-2\alpha p)}\right] = \left|-\left({}^{1}/_{\pi\,(1-2\alpha p)}\right)/\left({}^{1}/_{2\pi\,\alpha p}\right)\right|$$

Again, the physically allowed range for α_p is $0 < \alpha_p < ½$. Note that the endpoints of this interval are excluded, since both $\alpha_p = 0$ and $\alpha_p = ½$ correspond to the *sawtooth* waveform "evolving" into a *ramp* waveform, which physically cannot happen, because of the *boundary-condition* requirement that *each* of the *n* harmonic waves have *nodes* at the endpoints of the generic interval $0 \le \theta < 2\pi$ (*and* at $\theta = \pi$, for the guitar, at the nut). However note mathematically (referring to the above formula for the Fourier coefficient, b_n) that in fact when either $\alpha_p = 0$ and/or $\alpha_p = ½$, we discover that $b_n = 0$. Thus, the mathematics tells us, ***because*** of the boundary conditions, that ***no*** wave solutions exist for $\alpha_p = 0$ and/or $\alpha_p = ½$.

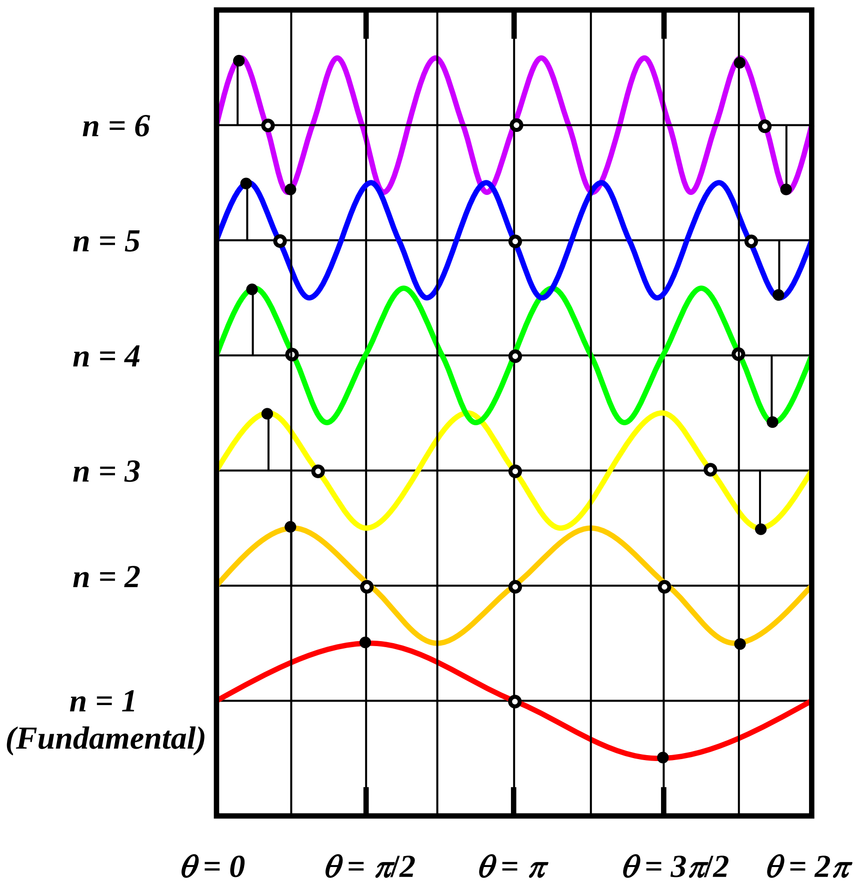
If the value of the α_p-parameter for the peak location(s) of the sawtooth wave is such that $\theta = 2\pi\alpha_p$ and $\theta = 2\pi(1-\alpha_p)$ correspond to peak positions along the sawtooth waveform that coincide with a *node* for a particular harmonic, *n*, then the Fourier coefficient, b_n will vanish for that harmonic. For physically-allowed values of the α_p-parameter, from the above formula for the Fourier coefficients, b_n we see that a particular Fourier coefficient, b_n will vanish whenever $\sin(2n\pi\alpha_p)$ vanishes, i.e. when $2n\pi\alpha_p = m\pi$ (where the integer m = 1, 2, 3, ... etc.), i.e. when $n = m/2\alpha_p$, or equivalently, when $\alpha_p = m/2n$ (< ½).

We have already encountered this phenomenon for the above specific case(s) of the bipolar triangle wave, with $\alpha_p = 1/4$, corresponding to $\theta_p = \pi/2$ and $\theta_p = 3\pi/2$, where *all* of the even-*n* Fourier harmonics, b_n vanished, because they had nodes at these θ-values; and the case of the bipolar sawtooth wave, with $\alpha_p = 1/8$, corresponding to $\theta_p = \pi/4$ and $\theta_p = 7\pi/4$, where the $n = 4^{th}$, 8^{th}, 12^{th}, 16^{th}, ... etc. Fourier harmonics, b_n vanished, because they too had nodes at these θ-locations.

Thus, for $n \ge 2$ (e.g. n = 2, 3, 4, 5, 6, ... etc.), whenever the value of the α_p-parameter is such that $\alpha_p = 1/2n$, corresponding to $\theta_p = 2\pi\alpha_p = 2\pi/2n = \pi/n$ and $\theta_p = 2\pi(1-\alpha_p) = 2\pi(1 - 1/2n)$, the Fourier coefficient, b_n will *vanish* for that harmonic associated with the bipolar sawtooth wave.

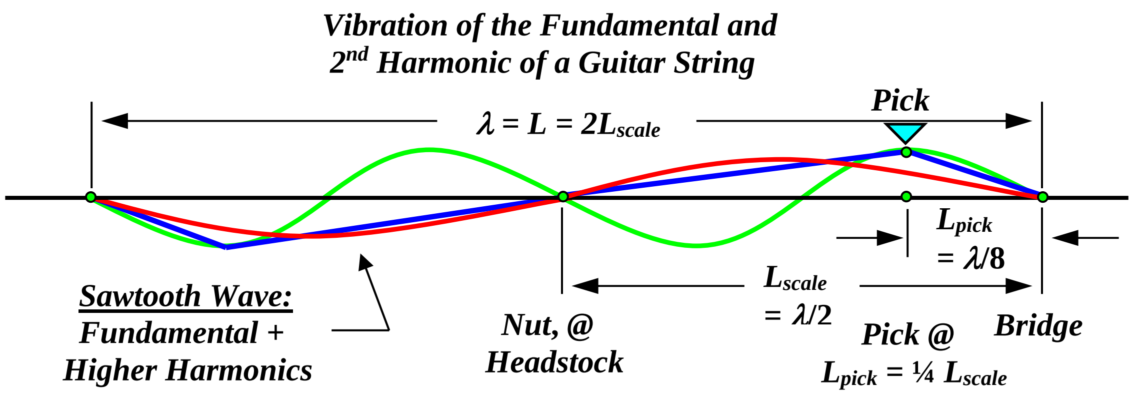
If the value of the α_p-parameter is such that $\theta_p = 2\pi\alpha_p$ and $\theta_p = 2\pi(1-\alpha)$ correspond to *anti-nodes* associated with one (or more) Fourier harmonics, b_n then the harmonic amplitudes, $|r_n| = |b_n|$ associated with the bipolar sawtooth wave will be particularly strong. This occurs when $sin(2n\pi\alpha_p) = 1$, i.e. when $(2n\pi\alpha_p) = (2m-1)\pi/2$ (where m is again an integer m = 1,2, 3... etc.), i.e. when $n = (2m-1)/4\alpha_p$, or equivalently, when $\alpha_p = (2m-1)/4n$ (with $0 < \alpha_p < ½$).

Again, we have already have experience with this phenomenon, in the above example of the bipolar sawtooth wave, where $\alpha_p = 1/8$, corresponding to $\theta_p = \pi/4$ and $\theta_p = 7\pi/4$, which are *anti-nodes* of the $n = 2^{nd}$, 6^{th}, 10^{th}, 14^{th}, ... etc. Fourier harmonics, b_n , but which also simultaneously correspond to *nodes* of the $n = 4^{th}$, 8^{th}, 12^{th}, 16^{th}, ... etc. Fourier harmonics, b_n, as shown in the figure below, for the first six harmonics:



Some of the *anti-nodes* (*nodes*) associated with each harmonic in the above figure are explicitly marked with a solid bullet (open circle), respectively. Note also that all of these harmonics are drawn as being in-phase with each other. If one imagines a vertical line drawn for the ($\theta_p = 2\pi\alpha_p$)-parameter (representing the peak location of the triangle wave) ranging between $0 < (\theta_p = 2\pi\alpha_p) < \pi$, the intersection of this line with each of the harmonics, will indicate whether or not that harmonic is in-phase or out-of-phase with the fundamental, and/or whether the harmonics are at a node or anti-node for this value of θ_p.

To connect these results with the physical world, we return to the example of the guitar. As shown (again) in the figure below, the scale length, L_{scale} of the guitar corresponds to half the wavelength, λ of the fundamental, for open-string notes played on the guitar, i.e. $L_{scale} = \frac{1}{2}\lambda$. For a pick position distance, L_{pick} (referenced from the bridge of the guitar), this is a fractional distance, $\beta_{pick} \equiv L_{pick} / L_{scale} = 2L_{pick} / \lambda$.



In the following table, we summarize the $\beta_{pick} \equiv L_{pick} / L_{scale}$ locations for the nodes and anti-nodes associated with the first 10 harmonics. Playing at the anti-node locations will result in enhancing that particular harmonic, while playing at the nodal-locations will cause that harmonic to be absent. Physically, values of $\beta_{pick} \equiv L_{pick} / L_{scale} < \frac{1}{2}$ correspond to playing between the bridge and the bottom end of the neck, at the body of the guitar. Smaller values of $\beta_{pick} \equiv L_{pick} / L_{scale}$ are closer to the bridge end of the guitar.

Harmonic # ***n***	**$\beta_{pick} \equiv L_{pick} / L_{scale}$ for Node**	**$\beta_{pick} \equiv L_{pick} / L_{scale}$ for Anti-Node**
1 (Fundamental)	–	$1/2$
2	$1/2$	$1/4$, $3/4$
3	$1/3$, $2/3$	$1/6$, $3/6=1/2$, $5/6$
4	$1/4$, $2/4 = 1/2$, $3/4$	$1/8$, $3/8$, $5/8$, $7/8$
5	$1/5$, $2/5$, $3/5$, $4/5$	$1/10$, $3/10$, $5/10=1/2$, $7/10$, $9/10$
6	$1/6$, $2/6=1/3$, $3/6=1/2$, $4/6=2/3$, $5/6$	$1/12$, $3/12=1/4$, $5/12$, $7/12$ ...
7	$1/7$, $2/7$, $3/7$, $4/7$, $5/7$, $6/7$	$1/14$, $3/14$, $5/14$, $7/14=1/2$, $9/14$ ...
8	$1/8$, $2/8=1/4$, $3/8$, $4/8=1/2$, $5/8$, ...	$1/16$, $3/16$, $5/16$, $7/16$, $9/16$, $11/16$, ..
9	$1/9$, $2/9$, $3/9=1/3$, $4/9$, $5/9$, ...	$1/18$, $3/18=1/6$, $5/18$, $7/18$, ...
10	$1/10$, $2/10=1/5$, $3/10$, $4/10$, $5/10$, ...	$1/20$, $3/20$, $5/20=1/4$, $7/20$, ...

Thus, from the above table, we can see that for playing on <u>*nodes*</u> associated with the n^{th} harmonic, that $\beta_{pick} \equiv L_{pick} / L_{scale} = m / n$, where n is the harmonic #, and m is an integer such that $m = 1, 2, 3, ... < n$. For playing on <u>*anti-nodes*</u> associated with the n^{th} harmonic, we see that $\beta_{pick} \equiv L_{pick} / L_{scale} = (2m - 1) / 2n$, where again, m is an integer such that $m = 1, 2, 3, ... < n$.

We can also relate the formulae for the β_{pick} node and anti-node locations with those we obtained above for the node and anti-node locations, in terms of the α_p-parameter, since both of these variables describe the (same) peak locations of the sawtooth waveform, where the pick is located along the length of the open string(s) guitar, referenced to the bridge end of the guitar.

For ***nodes*** associated with the n^{th} harmonic, we have:

$$\beta_{pick} \equiv L_{pick} / L_{scale} = m/n \quad (\text{with } 0 < \beta_{pick} < 1)$$

and:

$$\alpha_p = m/2n \qquad (\text{with } 0 < \alpha_p < ½)$$

For ***anti-nodes*** associated with the n^{th} harmonic, we have:

$$\beta_{pick} \equiv L_{pick} / L_{scale} = (2m-1)/2n \quad (\text{with } 0 < \beta_{pick} < 1)$$

and:

$$\alpha_p = (2m-1)/4n \qquad (\text{with } 0 < \alpha_p < ½)$$

where m is an integer such that $m = 1, 2, 3, ... < n$. Thus, we see that $2\alpha_p = \beta_{pick}$.

We can also see this from the definition of the α_p-parameter:

$$\alpha_p \equiv \theta_p/2\pi = kx_p/2\pi = 2\pi x_p/2\pi\lambda = x_p/\lambda$$

Since the location of the first peak of the triangle wave is x_p, referenced from the bridge of the guitar, then $x_p = L_{pick}$. Since the wavelength, λ of the fundamental is twice the scale length of the guitar, i.e. $\lambda = 2L_{scale}$, then:

$$\alpha_p = x_p/\lambda. = L_{pick} / 2L_{scale} = ½\, \beta_{pick}$$

Every guitarist knows that for maximum "twang", he or she can play notes close to the bridge. The harmonic content of the notes played here "brightens" up considerably in comparison to playing near the top of the neck, where it joins the body of the guitar, or e.g. playing notes at the 12th fret on the neck, as discussed above. The higher harmonics contribute more and more as the strings of the guitar are picked closer and closer to the bridge. Can we understand how this happens?

First, look at the diagram two figures that shows the first few harmonics (n = 1:6). Note that e.g. in the region below $\theta < \pi/8$, all of the harmonics shown have non-zero amplitudes, $|r_n| = |b_n|$. Since $\alpha_p \equiv \theta_p/2\pi = ½\, \beta_{pick} = L_{pick} / 2L_{scale}$, then for $\theta_p < \pi/8$, we have $\theta_p = 2\pi L_{pick} / 2L_{scale} = \pi L_{pick} / L_{scale} < \pi/8$, or $L_{pick} / L_{scale} = \beta_{pick} < 1/8$. Thus, picking in a region near the bridge which is within 1/8 of the overall scale length will tend to excite all of these harmonics. The ability to excite the fundamental from this picking location is reduced from that e.g. near the top of the neck, where it joins the body of the guitar. Thus, the fundamental is suppressed near the bridge. Likewise for the other harmonics, but deferentially, the fundamental is suppressed moreso than the other harmonics, the second harmonic suppressed less, the third harmonic, even less suppressed, and so on, near the bridge, for the first few harmonics, with $\beta_{pick} = L_{pick} / L_{scale} < 1/8$, or equivalently, $\alpha_p = ½\, \beta_{pick} < 1/16$.

Suppose we decide to pick very close to the bridge, such that the fractional distance, $\beta_{pick} = L_{pick} / L_{scale} << 1/8$, corresponding to $\alpha_p = ½\, \beta_{pick} << 1/16$. For definiteness' sake, let us choose $\beta_{pick} = L_{pick} / L_{scale} = ½ / 25 = 1/50 = 0.0200 << 1/8 = 0.1250$, corresponding to $\alpha_p = ½\, \beta_{pick} = ½ / 50 = 1/100 = 0.0100 << 1/16 = 0.0625$.

Now let us look at the generalized expression we obtained above for the *odd*-symmetry Fourier coefficients, b_n associated with the sawtooth wave:

$$b_n = 2*\left[{}^{2\alpha p}/_{(1-2\alpha p)}\right]*\left({}^{1}/_{2n\pi\alpha p}\right)^2 \sin(2n\pi\alpha_p) \text{ for } all\ n > 0$$

If we consider only the lower-order harmonics, e.g. $n \le 5$, then the argument of the *sine* function in the above formula, $(2n\pi\alpha_p) < 2*5*\pi/100 = \pi/10 = 0.314159...$

Now note that $sin(\pi/10) = sin(0.314159...) = 0.309017...$ The numerical value of $sin(\pi/10) = 0.309017...$ is within ~ 2% of the argument of the *sine* function, $\pi/10 = 0.314159...$ The reason this is so, can be understood from the Taylor series expansion of the *sin* (x) function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=1}^{n=\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

For small values of the argument, x of the *sin* (x) function, e.g. $x << 1$, then the higher-order terms in the Taylor series expansion of *sin* (x) are negligible, and thus *sin* (x) ~ x for $x << 1$. This also works reasonably well for $x < 1$ (not just $x << 1$), as we have seen above, as an approximation.

Thus, for $n \le 5$ and for $\beta_{pick} = L_{pick} / L_{scale} = ½ / 25 = 1/50 = 0.0200$, then

$$sin(2n\pi\alpha_p) \le sin(\pi/10) \sim \pi/10$$

or simply, $sin(2n\pi\alpha_p) \sim 2n\pi\alpha_p$ for $n \le 5$ and $\beta_{pick} = L_{pick} / L_{scale} = ½ / 25 = 0.0200$.

Then:

$$b_n \sim 2*\left[{}^{2\alpha p}/_{(1-2\alpha p)}\right]*\left({}^{1}/_{2n\pi\alpha p}\right)^2 2n\pi\alpha_p = 2*\left[{}^{2\alpha p}/_{(1-2\alpha p)}\right]*\left({}^{1}/_{2n\pi\alpha p}\right)$$

or:

$$b_n \sim 2*\left({}^{1}/_{n\pi}\right) *\left[{}^{1}/_{(1-2\alpha p)}\right]$$

Now:

$$\alpha_p = ½\, \beta_{pick} = ½ / 50 = 1/100 = 0.0100 << 1/16 = 0.0625.$$

Thus, $2\alpha_p = \beta_{pick} = 1/8 << 1$, and we can also approximate the factor $\left[{}^{1}/_{(1-2\alpha p)}\right]$ in the above approximate expression for the odd-function Fourier coefficients, b_n by taking the leading terms in the Taylor series expansion for the function $1/(1-\varepsilon)$ for $\varepsilon << 1$:

$$\frac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \varepsilon^4 + ... = \sum_{n=1}^{n=\infty} \varepsilon^n \quad for \quad -1 < \varepsilon < 1$$

Thus, for $\varepsilon << 1$, $1/(1-\varepsilon) \sim 1 + \varepsilon$. Thus for $2\alpha_p = 1/8 << 1$, the factor $\left[{}^{1}/_{(1-2\alpha p)}\right] \sim 1 + 2\alpha_p$.

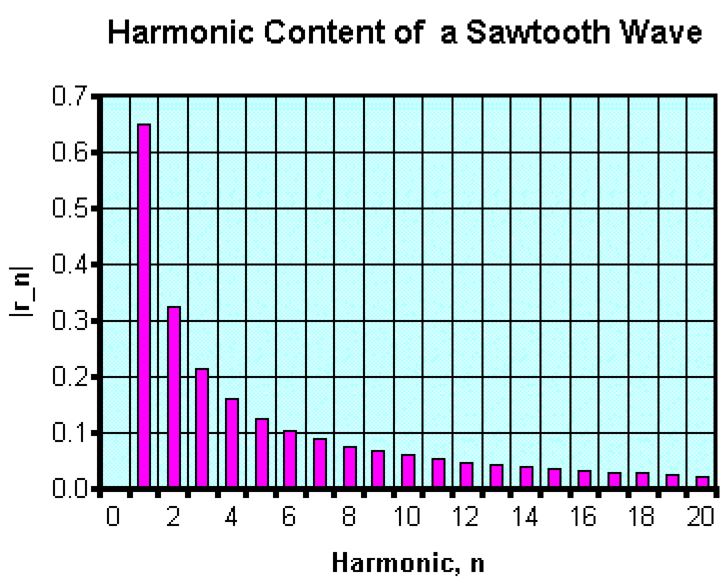
Then, for $n \leq 5$ and $\beta_{pick} = L_{pick} / L_{scale} = ½ / 25 = 0.0200$, we have (approximately) that:

$$b_n \sim 2*\left({}^{1}/_{n\pi}\right) *\left[{}^{1}/_{(1-2\alpha p)}\right] \sim 2*\left({}^{1}/_{n\pi}\right)*\left(1 + 2\alpha_p\right) \sim 2/n\pi$$

This (approximate) result for the low-order harmonic, odd-function Fourier coefficients, b_n, and thus the *magnitudes* of the harmonic amplitudes, $|r_n| = |b_n|$ shows that they decrease as $\sim 1/n$ for the harmonic #, n when picking notes very near to the bridge of the guitar.

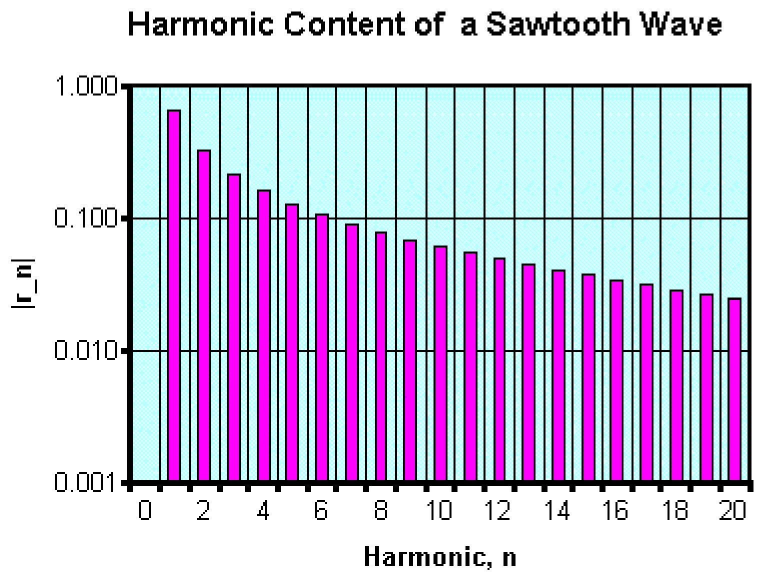
However, from the above discussions associated with the bipolar triangle and sawtooth waves, we found, for picking notes e.g. near the mid-point and/or the quarter point on the strings of the guitar, that the harmonic amplitudes, $|r_n| = |b_n|$ decreased as $\sim 1/n^2$ (not as $\sim 1/n$)!!! Therefore, picking notes on the strings very near to the bridge of the guitar, the tone is much brighter there because the low-order harmonics do not fall off in amplitude nearly as fast as they do when playing far away from the bridge!

In the following figure, we show the <u>*exact*</u> (i.e. no approximations-made) results for the magnitudes of the harmonic amplitudes, $|r_n| = |b_n|$ associated with the sawtooth wave for the case when $\beta_{pick} = L_{pick} / L_{scale} = ½ / 25 = 1/50 = 0.0200$.

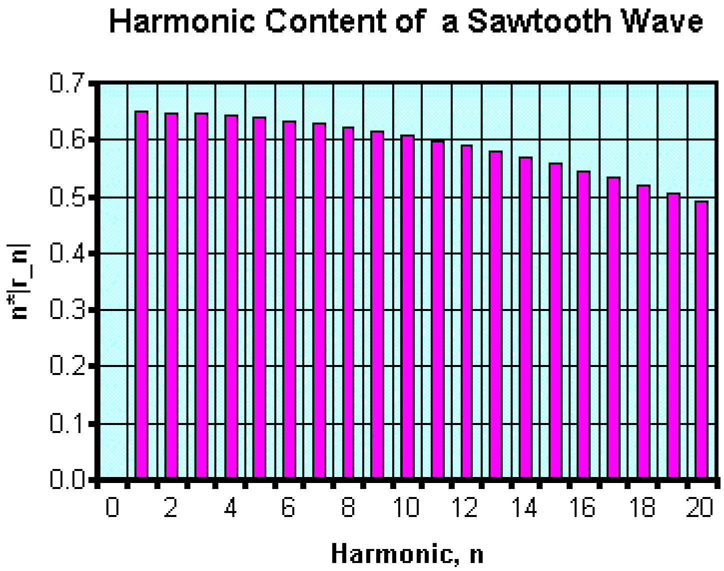


Note that because the picking notes on the guitar strings is done very near the bridge, the harmonics shown in the above figure are all in phase with each other, with phase angles, $\delta_n = tan^{-1}(b_n/a_n) = \pi = 180^o$. This can also be seen in the above figure showing the waveforms of the first six harmonics. If one imagines a vertical line drawn on this plot for the ($\theta_p = 2\pi\alpha_p$)-parameter (representing the peak location of the triangle wave) in the region of $\theta_p \sim 0$, the intersection of this line with each of the harmonics shows that these harmonics are indeed all in phase with each other.

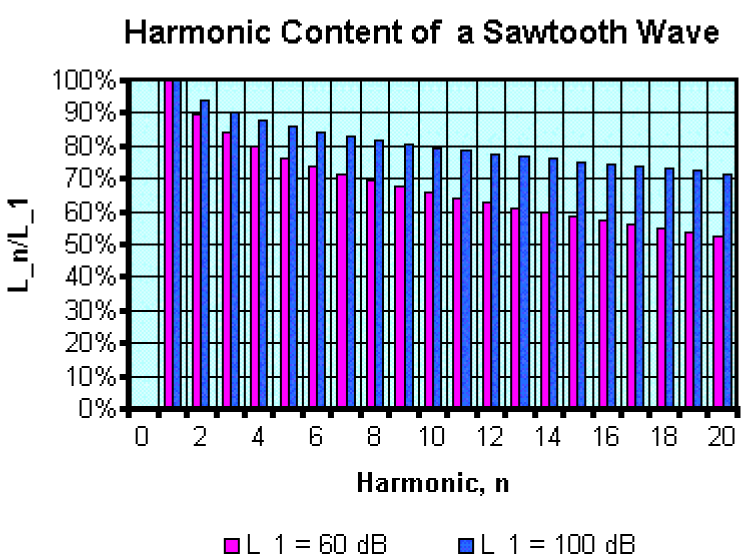
It can be seen that the harmonic amplitudes associated with a sawtooth wave for $\beta_{pick} = 0.02$, for picking guitar strings very close to the bridge, do not decrease with increasing harmonic #, *n* very rapidly, as we anticipated. Compare this result, and the following figure, which shows a semi-log plot of the harmonic amplitudes, with those above, for the triangle wave, with $\beta_{pick} = ½$, and for the sawtooth wave, with $\beta_{pick} = ¼$.



If the harmonic amplitudes, $|r_n|$ for $\beta_{pick} = 0.02$ fall off with increasing harmonic #, *n* as $|r_n| \sim 1/n$, Then the product of $n*|r_n|$ should be close to being a constant value, roughtly independent of the harmonic #, *n*. The following plot shows that for $\beta_{pick} = 0.02$, this is indeed the case, at least approximately so!



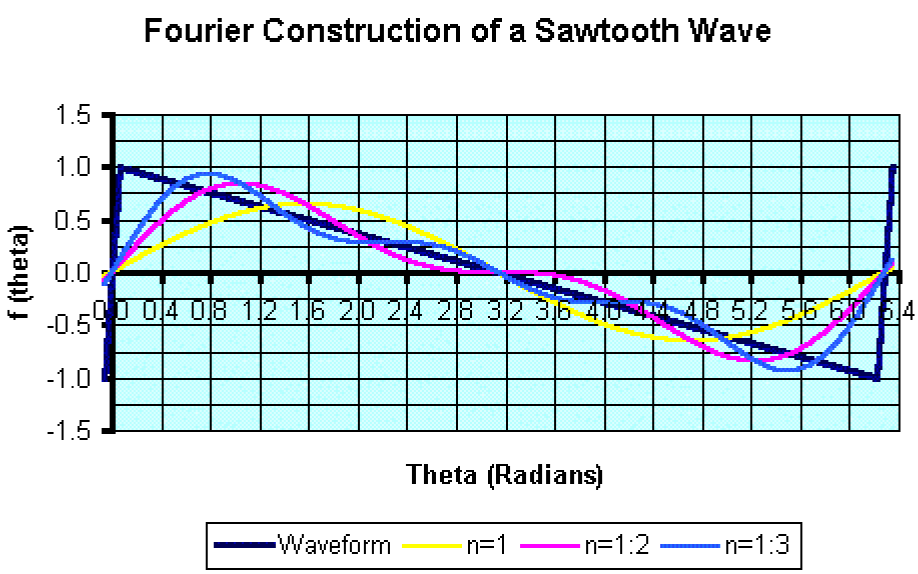
The following plot shows the loudness ratios, L_n / L_1 for the first twenty harmonics (i.e. $n < 20$) associated with the bipolar sawtooth wave, for $\beta_{pick} = 0.02$, for loudness values of the fundamental of $L_1 = 60\ dB$ (~ quiet) and for $L_1 = 100\ dB$ (~ quite loud). This is what the human ear perceives as the loudness of the harmonics relative to that of the fundamental. Note that the decrease in the loudness ratio, L_n / L_1 with increasing harmonic #, n is extremely slow, in comparison to that associated with the triangle wave, with $\beta_{pick} = ½$, and for the sawtooth wave, with $\beta_{pick} = ¼$.



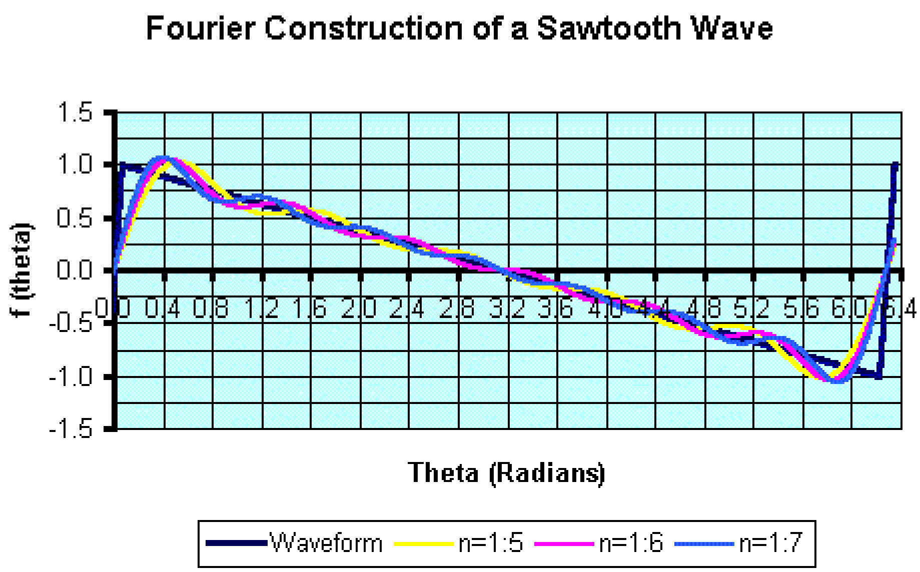
The following two figures show the "Fourier construction" of a periodic, bipolar, unit-amplitude sawtooth wave. The waveforms in these figures were generated using truncated, finite-term version(s) of the Fourier series expansion for this waveform:

$$f(\theta)\big|_{\substack{sawtooth\\-wave}} = \sum_{n=1}^{n=\infty} b_n \sin(n\theta) = 2\left[\frac{2\alpha_p}{(1-2\alpha_p)}\right]\sum_{n=1}^{n=\infty}\left(\frac{1}{2n\pi\alpha_p}\right)^2 \sin(2n\pi\alpha_p)\sin(n\theta)$$

The first figure shows the bipolar sawtooth wave (labelled as "Waveform") overlaid with three other waveforms: that associated with just the fundamental ("n = 1"), then the waveform associated with fundamental + 2nd harmonic ("n = 1:2"), then the waveform associated with fundamental + 2nd + 3rd harmonic ("n = 1:3"). It can be seen that using just these first three harmonics, that the replication of the sawtooth waveform is not very good, because of the extremely sharp/rapid changes in this waveform at its ends.



The second figure shows the bipolar sawtooth wave (labelled as "Waveform") overlaid with three other waveforms: that associated with the fundamental through the 5th harmonic ("n = 1:5"), then the waveform associated with fundamental through the 6th harmonic ("n = 1:6"), then the waveform associated with fundamental through the 7th harmonic ("n = 1:7")



Adding the additional harmonics up to n = 7 helps improve the agreement, but it can be seen that many more of the higher harmonics are needed to replicate the sharp break at the ends of the sawtooth waveform!

Fourier analysis of waveforms has many potential uses and applications. In these notes we have laid down the basics of Fourier analysis, given a few basic examples and connected them to various physical systems, such as the guitar. We shall also see other examples of the use of Fourier analysis elsewhere in this course.

Exercises:

1. Compute the Fourier coefficients, a_0, a_n and b_n for the "flipped" bipolar, triangle wave, in the time domain:
$$f(\theta) = f(kx) = -(2/\pi)\theta \quad \text{for} \quad 0 \le \theta < \pi/2$$
$$f(\theta) = f(kx) = +(2/\pi)\theta - 2 \quad \text{for} \quad \pi/2 \le \theta < 3\pi/2$$
$$f(\theta) = f(kx) = -(2/\pi)\theta + 4 \quad \text{for} \quad 3\pi/2 \le \theta < 2\pi$$
Compare these Fourier coefficients with those obtained above for the "unflipped" bipolar triangle wave.

2. Compute the Fourier coefficients, a_0, a_n and b_n for the "shifted" bipolar triangle wave, in the time domain:
$$f(\theta) = f(kx) = +(2/\pi)\theta - 1 \quad \text{for} \quad 0 \le \theta < \pi$$
$$f(\theta) = f(kx) = -(2/\pi)\theta + 3 \quad \text{for} \quad \pi \le \theta < 2\pi$$
Compare these Fourier coefficients with those obtained above for the "unflipped" and "flipped" bipolar triangle waves.

3. Work your way through the details of computing the Fourier coefficients, a_0, a_n and b_n for the above-discussed *specific* case of the bipolar sawtooth wave.

4. Concoct a waveform shape of your own interest, write out its mathematical representation, $f(\theta)$ over the interval $0 \le \theta < 2\pi$, and compute the Fourier coefficients, a_0, a_n and b_n associated with your waveform.

5. For each of the above exercises, use e.g. *MathLab*, or a spreadsheet program, such as *Excel* to make plots of the harmonic amplitudes, $|r_n|$, the loudness ratios, L_n / L_1 and Fourier contruction of the original waveform, for e.g. the first few harmonics.

References for Fourier Analysis and Further Reading:

1. Fourier Series and Boundary Value Problems, 2nd Edition, Ruel V. Churchill, McGraw-Hill Book Company, 1969.

2. Mathematics of Classical and Quantum Physics, Volumes 1 & 2, Frederick W. Byron, Jr. and Robert W. Fuller, Addison-Wesley Publishing Company, 1969.

3. Mathematical Methods of Physics, 2nd Edition, Jon Matthews and R.L. Walker, W.A. Benjamin, Inc., 1964.

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