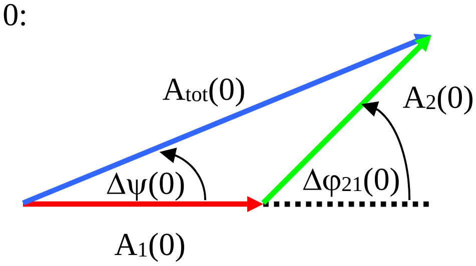
<u>Beats Phenomenon</u>

Linearly superpose (*i.e.* *<u>add</u>*) two signals with amplitudes $A_1(t)$ and $A_2(t)$, and which have similar/comparable frequencies, $\omega_2(t) \sim \omega_1(t)$, with instantaneous phase of the second signal relative to the first of $\Delta\varphi_{21}(t)$:

$$A_1(t) = A_{10}\cos\left(\omega_1(t)t\right) \qquad A_2(t) = A_{20}\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)$$
$$A_{tot}(t) = A_1(t) + A_2(t) = A_{10}\cos\left(\omega_1(t)t\right) + A_{20}\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)$$

Note that at the amplitude level, there is nothing explicitly overt and/or obvious in the above mathematical expression for the overall/total/resultant amplitude, $A_{tot}(t)$ that *easily* explains the phenomenon of beats associated with adding together two signals that have comparable amplitudes and frequencies.

However, let us consider the (instantaneous) <u>phasor relationship</u> between the individual amplitudes for the two signals, $A_1(t)$ and $A_2(t)$ respectively. Their relative initial phase difference at time $t = 0$ is $\Delta\varphi_{21}(t=0)$ and the resultant/total amplitude, $A_{tot}(t=0)$ is shown in the figure below, for time, $t = 0$:



From the law of cosines, the magnitude of the total amplitude, $A_{tot}(t)$ at an arbitrary time, t is obtained from the following:

$$A_{tot}^2(t) = A_1^2(t) + A_2^2(t) - 2A_1(t)A_2(t)\cos\left[\pi - \left((\omega_2(t) - \omega_1(t))t + \Delta\phi_{21}(t)\right)\right]$$

$$A_{tot}^2(t) = A_1^2(t) + A_2^2(t) + 2A_1(t)A_2(t)\cos\left((\omega_2(t) - \omega_1(t))t + \Delta\phi_{21}(t)\right)$$

Thus,

$$A_{tot}(t) = \sqrt{A_1^2(t) + A_2^2(t) + 2A_1(t)A_2(t)\cos\left((\omega_1(t) - \omega_2(t))t + \Delta\phi_{21}(t)\right)}$$

$$= \sqrt{\begin{array}{l} A_{10}^2\cos^2\left(\omega_1(t)t\right) + A_{20}^2\cos^2\left(\omega_2(t)t\right) \\ +2A_{10}A_{20}\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)\cos\left((\omega_2(t) - \omega_1(t))t + \Delta\phi_{21}(t)\right) \end{array}}$$

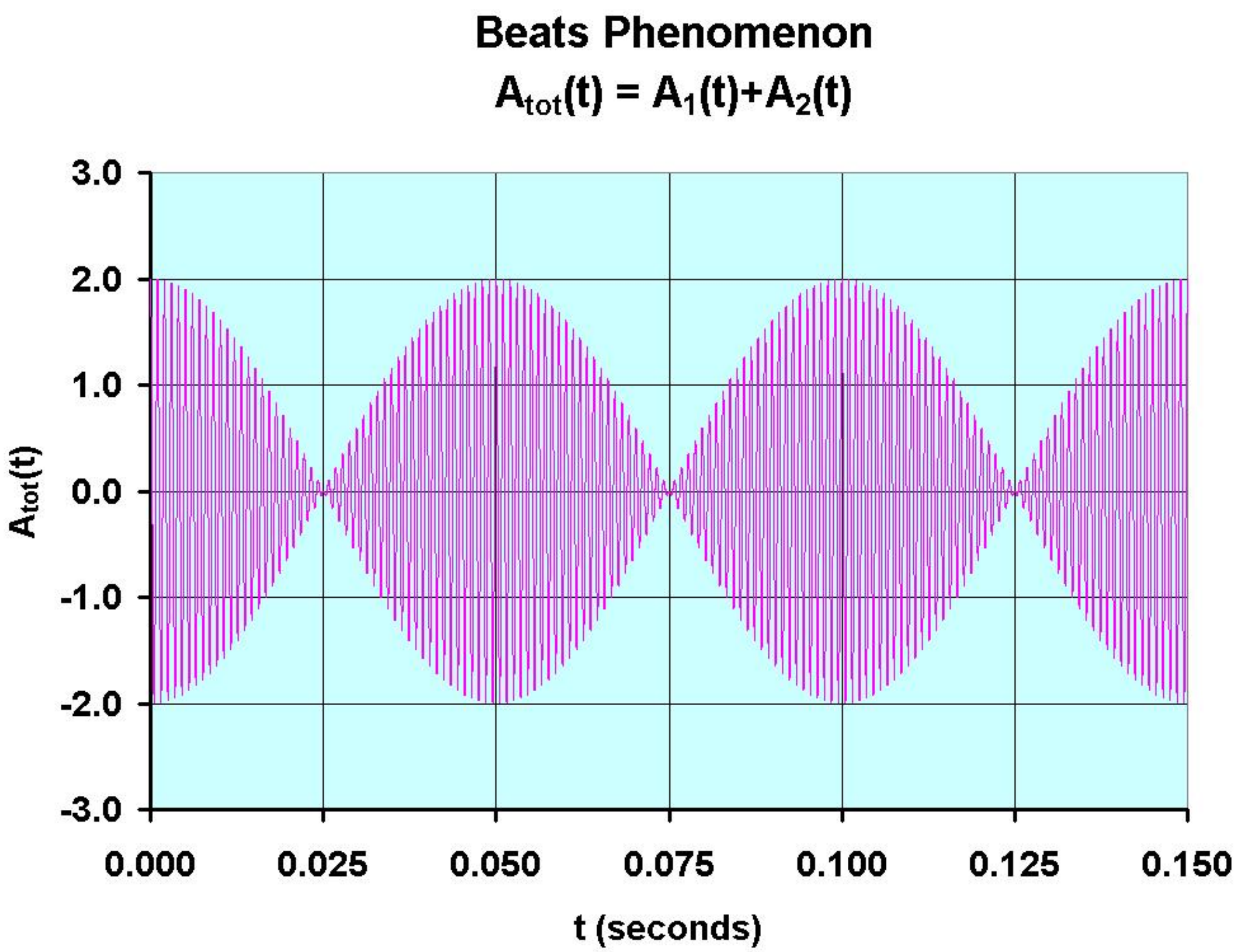
For equal amplitudes, $A_{10} = A_{20} = A_0$, zero relative initial phase, $\Delta\varphi_{21} = 0$ and constant (i.e. time-independent) frequencies, ω_2 and ω_1, this expression reduces to:

$$A_{tot}(t) = A_0\sqrt{\cos^2\omega_1 t + \cos^2\omega_2 t + 2\cos\omega_1 t\cos\omega_2 t\cos\left((\omega_2 - \omega_1)t\right)}$$

The phase of the total amplitude, $A_{tot}(t)$ relative to that of the first amplitude $A_1(t)$, at an arbitrary time, t is $\Delta\psi(t)$ and is obtained from the projections of the total amplitude phasor, $A_{tot}(t)$ onto the y- and x- axes of the 2-D phasor plane:

$$\tan(\Delta\psi) = \frac{A_2(t)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\sin\Delta\varphi_{21}(t)}{A_1(t)\cos\left(\omega_1(t)t\right) + A_2(t)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\cos\Delta\varphi_{21}(t)}$$

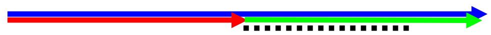
The total amplitude, $A_{tot}(t) = A_1(t) + A_2(t)$ vs. time, t is shown in the figure below, for time-independent/constant frequencies of $f_1 = 1000$ Hz and $f_2 = 980$ Hz, equal amplitudes of unit strength, $A_{10} = A_{20} = 1.0$ and zero relative phase, $\Delta\varphi_{21} = 0.0$



Clearly, the beats phenomenon can be seen in the above waveform of total amplitude, $A_{tot}(t) = A_1(t) + A_2(t)$ vs. time, t. From the above graph, it is obvious that the beat period, $\tau_{beat} = 1/f_{beat} = 0.050$ sec = 1/20th sec, corresponding to a beat frequency, $f_{beat} = 1/\tau_{beat} = 20$ Hz, which is simply the frequency difference, $f_{beat} \equiv | f_1 - f_2|$ between $f_1 = 1000$ Hz and $f_2 = 980$ Hz. Thus, the beat period, $\tau_{beat} = 1/f_{beat} = 1/| f_1 - f_2|$. When $f_1 = f_2$, the beat period becomes infinitely long, and no beats are heard.

In terms of the phasor diagram, as time progresses, the individual amplitudes $A_1(t)$ and $A_2(t)$ actually precess at (angular) rates of $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$ radians per second respectively, completing one revolution in the phasor diagram, for each cycle/each period of $\tau_1 = 2\pi/\omega_1 = 1/f_1$ and $\tau_2 = 2\pi/\omega_2 = 1/f_2$, respectively. If at time t = 0 the two phasors are precisely in phase with each other (i.e. with initial relative phase $\Delta\varphi_{21} = 0.0$), then the resultant/total amplitude, $A_{tot}(t = 0) = A_1(t = 0) + A_2(t = 0)$ will be as shown in the figure below.

$A_{tot}(t=0) = A_1(t=0) + A_2(t=0)$



$A_1(t=0)$ $A_2(t=0)$

As time progresses, if $\omega_1 \neq \omega_2$, (phasor 1 with angular frequency $\omega_1 = 2\pi f_1 = 2*1000\pi = 2000\pi$ radians/sec and $\omega_2 = 2\pi f_2 = 2*980\pi = 1960\pi$ radians/sec in our example above) phasor 1, with higher angular frequency will precess more rapidly than phasor 2 (by the difference in angular frequencies, $\Delta\omega = (\omega_1 - \omega_2) = (2000\pi - 1960\pi) = 40\pi$ radians/second). Thus, as time increases, phasor 1 will ***lead*** phasor 2; eventually (at time $t = \frac{1}{2}\tau_{beat} = 0.025 = 1/40^{th}$ sec in our above example) phasor 2 will be exactly $\Delta\varphi = \pi$ radians, or 180 degrees behind in phase relative to phasor 1. Phasor 1 at time $t = \frac{1}{2}\tau_{beat} = 0.025$ sec $= 1/40^{th}$ sec will be oriented exactly as it was at time t = 0.0 (having precessed exactly $N_1 = \omega_1 t/2\pi = 2\pi f_1 t/2\pi = f_1 t = 25.0$ revolutions in this time period), however phasor 2 will be pointing in the opposite direction at this instant in time (having precessed only $N_2 = \omega_2 t/2\pi = 2\pi f_2 t/2\pi = f_2 t = 24.5$ revolutions in this same time period), and thus the total amplitude $A_{tot}(t = \frac{1}{2}\tau_{beat}) = A_1(t = \frac{1}{2}\tau_{beat}) + A_2(t = \frac{1}{2}\tau_{beat})$ will be zero (if the magnitudes of the two individual amplitudes are precisely equal to each other), or minimal (if the magnitudes of the two individual amplitudes are not precisely equal to each other), as shown in the figure below.

$A_{tot}(t = \frac{1}{2}\tau_{beat}) = A_1(t = \frac{1}{2}\tau_{beat}) + A_2(t = \frac{1}{2}\tau_{beat}) = 0$

$A_2(t = \frac{1}{2}\tau_{beat}) = -A_1(t = \frac{1}{2}\tau_{beat})$

As time progresses further, phasor 2 will continue to lag farther and farther behind, and eventually (at time $t = \tau_{beat} = 0.050$ sec $= 1/20^{th}$ sec in our above example) phasor 2, having precessed through $N_2 = 49.0$ revolutions will now be exactly $\Delta\varphi = 2\pi$ radians, or 360 degrees (or one full revolution) behind in phase relative to phasor 1 (which has precessed through $N_1 = 50.0$ full revolutions), thus, the net/overall result is the same as being exactly in phase with phasor 1! At this point in time, $A_{tot}(t = \tau_{beat}) = A_1(t = \tau_{beat}) + A_2(t = \tau_{beat}) = 2A_1(t = \tau_{beat}) = 2A_1(t = \tau_{beat})$, and the phasor diagram looks precisely like that at time t = 0.

Thus, it should (hopefully) now be clear to the reader that the phenomenon of beats is manifestly that of time-dependent alternating constructive/destructive interference between two periodic signals of comparable frequency, at the amplitude level. This is by no means a trivial point, as often the beats phenomenon is discussed in physics textbooks in the context of intensity, $I_{tot}(t) = |A_{tot}(t)|^2 = |A_1(t) + A_2(t)|^2$. From the above discussion, the physics origin of the beats phenomenon has absolutely ***nothing*** to do with the ***intensity*** of the overall/ resultant signal.

The primary reason that the phenomenon of beats is discussed more often in terms of intensity, rather than amplitude is that the physics is perhaps easier to understand from the intensity perspective – at least mathematically, things appear more obvious, physically:

$$I_{tot}(t) = \left|A_{tot}(t)\right|^2 = \left|A_1(t) + A_2(t)\right|^2$$

$$I_{tot}(t) = A_{10}^2 \cos^2\left(\omega_1(t)t\right) + A_{20}^2 \cos^2\left(\omega_2(t)t + \Delta\phi_{21}(t)\right) + 2A_{10}A_{20}\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)$$

Let us define:

$$\vartheta_1(t) \equiv \omega_1(t)t \qquad \vartheta_2(t) \equiv \left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)$$

And then let us use the mathematical identity:

$$\cos\vartheta_1 \cos\vartheta_2 \equiv \tfrac{1}{2}\left[\cos\left(\vartheta_2 + \vartheta_1\right) + \cos\left(\vartheta_2 - \vartheta_1\right)\right]$$

Thus:

$$I_{tot}(t) = A_{10}^2 \cos^2\left(\omega_1(t)t\right) + A_{20}^2 \cos^2\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)$$
$$+A_{10}A_{20}\left[\cos\left((\omega_2(t) + \omega_1(t))t + \Delta\phi_{21}(t)\right) + \cos\left((\omega_2 - \omega_1)t - \Delta\phi_{21}(t)\right)\right]$$

The let us define:

$$\Omega_{21}(t) \equiv \left(\omega_2(t) + \omega_1(t)\right) \qquad \Delta\omega_{21}(t) \equiv \left|\omega_2(t) - \omega_1(t)\right|$$

We then obtain:

$$I_{tot}(t) = A_{10}^2 \cos^2\left(\omega_1(t)t\right) + A_{20}^2 \cos^2\left(\omega_2(t)t + \Delta\phi_{21}(t)\right) + A_{10}A_{20}\left[\cos\left(\Omega_{21}(t)t + \Delta\phi_{21}(t)\right) + \cos\left(\Delta\omega_{21}(t)t - \Delta\phi_{21}(t)\right)\right]$$

Using the identity:

$$\cos^2\vartheta = \cos\vartheta\cos\vartheta \equiv \tfrac{1}{2}\left[\cos 0 + \cos 2\vartheta\right] = \tfrac{1}{2}\left[1 + \cos 2\vartheta\right]$$

We then obtain the additional relation, which is not usually presented and/or discussed in physics textbooks:

$$I_{tot}(t) = \tfrac{1}{2}A_{10}^2\left[1 + \cos^2 2\left(\omega_1(t)t\right)\right] + \tfrac{1}{2}A_{20}^2\left[1 + \cos^2 2\left(\omega_2(t)t + \Delta\phi_{21}(t)\right)\right]$$
$$+A_{10}A_{20}\left[\cos\left(\Omega_{21}(t)t + \Delta\phi_{21}(t)\right) + \cos\left(\Delta\omega_{21}(t)t - \Delta\phi_{21}(t)\right)\right]$$

This latter formula shows that there are a.) DC (i.e. zero frequency) components/constant terms associated with both amplitudes, A_{10} and A_{20}, b.) 2nd harmonic components with $2f_1$ and $2f_2$, as well as c.) a component associated with the sum of the two frequencies, $\Omega_{21} = f_1 + f_2$, and d.) a component associated with the difference of the two frequencies, $\Delta f_{21} = f_1 - f_1$. This is a remarkably similar result to that associated with the output response from a system with a quadratic non-linear response to a pure/single-frequency sine-wave input!

A Special/Limiting Case – Amplitude Modulation:

When $A_{10} >> A_{20}$ and $f_1 >> f_2$, then the *exact* expression for the total amplitude,

$$A_{tot}(t) = \sqrt{A_1^2(t) + A_2^2(t) + 2A_1(t)A_2(t)\cos\left((\omega_1(t) - \omega_2(t))t + \Delta\varphi_{21}(t)\right)}$$
$$= \sqrt{\begin{array}{l} A_{10}^2 \cos^2\left(\omega_1(t)t\right) + A_{20}^2 \cos^2\left(\omega_2(t)t\right) \\ +2A_{10}A_{20}\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\cos\left((\omega_2(t) - \omega_1(t))t + \Delta\varphi_{21}(t)\right) \end{array}}$$

can be *approximated* by the following expression(s), neglecting terms of order $m^2 \equiv (A_{20}/A_{10})^2 << 1$ under the radical sign, and, defining $\Delta\omega_{21}(t) \equiv (\omega_1(t) - \omega_2(t))$, and noting that for $f_1 >> f_2$, $\Delta\omega_{21}(t) \equiv (\omega_1(t) - \omega_2(t)) \cong \omega_1(t)$:

$$A_{tot}(t) = A_{10}\sqrt{\begin{array}{l} \cos^2\left(\omega_1(t)t\right) + \left(A_{20}/A_{10}\right)^2 \cos^2\left(\omega_2(t)t\right) \\ +2\left(A_{20}/A_{10}\right)\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\cos\left((\omega_2(t) - \omega_1(t))t + \Delta\varphi_{21}(t)\right) \end{array}}$$
$$\simeq A_{10}\sqrt{\cos^2\left(\omega_1(t)t\right) + 2\left(A_{20}/A_{10}\right)\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\cos\left(\Delta\omega_{21}(t)t + \Delta\varphi_{21}(t)\right)}$$
$$\simeq A_{10}\sqrt{\cos^2\left(\omega_1(t)t\right) + 2\left(A_{20}/A_{10}\right)\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t + \Delta\varphi_{21}(t)\right)\cos\left(\omega_1(t)t + \Delta\varphi_{21}(t)\right)}$$

However, when $f_1 >> f_2$, the relative phase difference $\Delta\varphi_{21}(t)$ changes by 2π radians (essentially) for *each* cycle of the frequency, f_1. Hence we can safely set to zero this phase difference, i.e. $\Delta\varphi_{21}(t) = 0$ because its (time-averaged) effect in the $f_1 >> f_2$ limit is negligible. Using the Taylor series expansion $\sqrt{1+\varepsilon} \simeq 1 + \frac{1}{2}\varepsilon$ for the case when $\varepsilon << 1$, the expression for the total amplitude then becomes:

$$A_{tot}(t) \simeq A_{10}\sqrt{\cos^2\left(\omega_1(t)t\right) + 2\left(A_{20}/A_{10}\right)\cos\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t\right)\cos\left(\omega_1(t)t\right)}$$
$$\simeq A_{10}\sqrt{\cos^2\left(\omega_1(t)t\right) + 2\left(A_{20}/A_{10}\right)\cos^2\left(\omega_1(t)t\right)\cos\left(\omega_2(t)t\right)}$$
$$\simeq A_{10}\cos\left(\omega_1(t)t\right)\sqrt{1 + 2\left(A_{20}/A_{10}\right)\cos\left(\omega_2(t)t\right)}$$
$$\simeq A_{10}\cos\left(\omega_1(t)t\right)\left(1 + \left(A_{20}/A_{10}\right)\cos\left(\omega_2(t)t\right)\right)$$
$$\simeq A_{10}\cos\left(\omega_1(t)t\right)\left(1 + m\cos\left(\omega_2(t)t\right)\right)$$

The ratio $m \equiv (A_{20}/A_{10}) << 1$ is known as the (amplitude) *modulation depth* of the high-frequency *carrier* wave $A_1(t)$, with amplitude $A_{10} >> A_{20}$ and frequency $f_1 >> f_2$, *modulated* by the low frequency wave $A_2(t)$, with amplitude A_{20} and frequency f_2. This is the underlying principle of *AM radio* – *AM* stands for ***A**mplitude **M**odulation*…

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