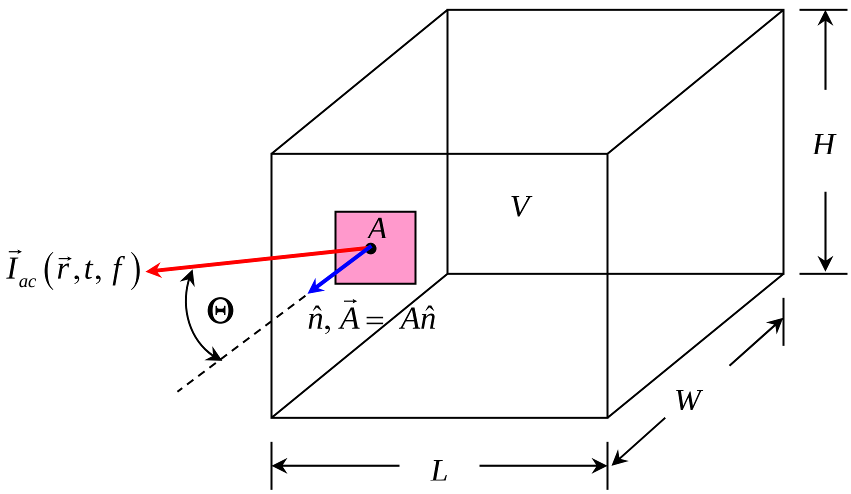
Derivation of the Sabine Equation: Conservation of Energy

Consider a large room of volume $V = H\times W\times L$ (m^3) with perfectly reflecting walls, filled with a uniform, steady-state (*i.e.* equilibrium) acoustic energy density $w_a(\vec{r},t,f)$ at given frequency *f* (*Hz*) within the volume *V* of the room. Uniform energy density means that a given time *t*: $w_a(\vec{r},t,f) = w_a(t,f) = constant$ (*SI* units: *Joules*/m^3). The large room also has a small opening of area *A* (m^2) in it, as shown in the figure below:



In the steady-state, the rate of acoustical energy W_a input *e.g.* by a point sound source within the large room equals the rate at which acoustical energy is "leaking" out of the room through the hole of area *A*, *i.e.* the acoustical power input by the sound source in the room into the room = the acoustical power leaving the room through the hole of area *A*. In this idealized model of a room with perfectly reflecting walls, the hole of area *A* thus represents absorption of sound in a real room with finite reflectivity walls, *i.e.* walls that have some absorption associated with them.

Suppose at time $t = 0$ the sound source in the room {located far from the hole} is turned off. Since the sound energy density is uniform in the room, the sound energy contained in the room $W_a(t,f) = \int_V w_a(\vec{r},t,f)\,d^3r = w_a(t,f)\int_V d^3r = w_a(t,f)V$ will thus decrease with time, since acoustical energy is (slowly) leaking out of the room through the opening of area *A*.

The instantaneous acoustical power at the frequency f passing through the hole of area *A* is the instantaneous time-rate of change of the acoustic energy in the room:

$$P_a(t,f) = \frac{\partial W_a(t,f)}{\partial t}$$

However, the instantaneous acoustical power loss at the frequency f associated with the flux of acoustic energy passing through the hole of area *A* is also $P_a(t,f) = -\int_A \vec{I}_a(\vec{r},t,f)\bullet d\vec{A}$ where $\vec{I}_a(\vec{r},t,f)$ is the instantaneous 3-D vector sound intensity at the point $\vec{r}$ at frequency *f* (*SI* units: *Watts*/m^2) and $d\vec{A} = dA\hat{n}$ is a infinitesimal vector area element associated with the hole of area *A*, and $\hat{n}$ is the outward-pointing unit normal to the hole of area *A*, as shown in the above figure.

Thus:

$$P_a(t,f) = \frac{\partial W_a(t,f)}{\partial t} = -\int_A \vec{I}_a(\vec{r},t,f) \bullet d\vec{A} = -\int_A \vec{I}_a(\vec{r},t,f) \bullet \hat{n} dA = -\int_A \left| I_a(t,f) \right| \cos\Theta(t)\, dA$$

where $\cos\Theta(t)$ is the instantaneous <u>direction</u> <u>cosine</u> , and thus $\Theta(t)$ is the instantaneous 3-D opening angle between the two vectors $\vec{I}_a(t,f)$ and $\vec{A} = A\hat{n}$ (as shown in the above figure), and $\vec{I}_a(\vec{r},t,f) \bullet \hat{n} = \left| I_a(\vec{r},f) \right| \cos\Theta(t)$. In the <u>steady-state</u>, the <u>*magnitude*</u> of the 3-D vector sound intensity is <u>constant</u> in time at any given point $\vec{r}$ inside the volume *V* of the room, and on/at the opening of the hole of area *A*, however the <u>*direction*</u> of the 3-D vector sound intensity at any given point $\vec{r}$ fluctuates <u>randomly</u> from one moment to the next. At/on the surface of the opening of the hole of area *A*, the direction of the 3-D vector sound intensity associated with energy leaking out of the room of volume *V* is such that the direction of the 3-D sound intensity points randomly from moment-to-moment in the forward-going hemisphere, *i.e.* is contained within a solid angle $d\Omega$ associated only with the forward half of 4π steradians (since sound energy is leaking <u>out</u> of the room – sound energy is not coming <u>into</u> the room from the outside).

We are <u>not</u> interested in following the instantaneous, moment-to-moment/short-time scale fluctuations in the 3-D vector sound intensity $\vec{I}_a(\vec{r},t,f)$, but we <u>are</u> interested in the mean power loss, time-averaged over these moment-to-moment fluctuations. For randomly fluctuating direction in $\vec{I}_a(\vec{r},t,f)$, the <u>mean</u> power loss through the hole of area *A*, time-averaging over such moment-to-moment fluctuations is:

$$\left\langle P_a(t,f) \right\rangle = \frac{\partial \left\langle W_a(t,f) \right\rangle}{\partial t} = -\left\langle \int_A \vec{I}_a(\vec{r}_{hole},t,f) \bullet d\vec{A} \right\rangle = -\int_A \left| I_a(\vec{r}_{hole} f) \right| \left\langle \cos\Theta(t) \right\rangle dA$$

The random, fluctuating moment-to-moment direction in the 3-D vector sound intensity $\vec{I}_a(\vec{r}_{hole},t,f) = \underbrace{\left| \vec{I}_a(\vec{r}_{hole},f) \right|}_{=\ constant} \cos\Theta(t)$ means that the fluctuating, moment-to-moment $\cos\Theta(t)$ is random at the hole opening of area *A*. What this means physically is that the <u>probability</u> <u>density</u> distribution $d\mathcal{P}(\cos\Theta)/d\cos\Theta = 1/2$ is <u>flat/uniform</u> in the $\cos\Theta$ variable, as shown in the figure below:

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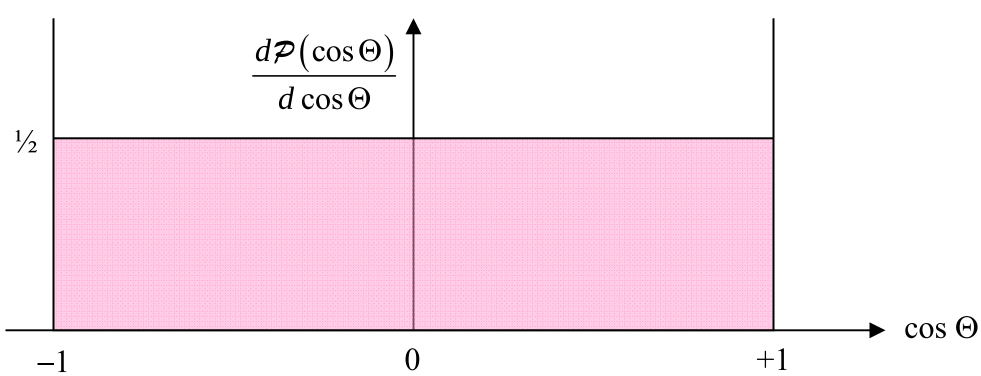


Since probability $\mathcal{P}$ is (always) conserved, and defining $x \equiv \cos\Theta$, we must have:

$$\int_{-1}^{+1}\left(\frac{d\mathcal{P}(\cos\Theta)}{d\cos\Theta}\right)d\cos\Theta = \int_{-1}^{+1}\left(\frac{d\mathcal{P}(x)}{dx}\right)dx = \int_{-1}^{+1}\frac{1}{2}dx = \frac{1}{2}\int_{-1}^{+1}dx = \frac{1}{2}x\Big|_{-1}^{+1} = \frac{1}{2}\cdot 2 = 1$$

However, for our physical situation *here*, only the forward half of this probability distribution is occupied $(0 \le \cos\Theta \le 1)$ - {sound energy is leaking out of the hole, not into it}. Thus, the time-averaged value of $\cos\Theta(t)$, for a flat random distribution in $\cos\Theta$ over the forward hemisphere is:

$$\langle\cos\Theta(t)\rangle = \int_0^1\left(\frac{d\mathcal{P}(\cos\Theta)}{d\cos\Theta}\right)\cos\Theta\, d\cos\Theta = \int_0^1\left(\frac{1}{2}\right)\cos\Theta\, d\cos\Theta = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^1 = \frac{1}{4}$$

and hence:

$$\begin{aligned}\langle P_a(t,f)\rangle &= \frac{\partial\langle W_a(t,f)\rangle}{\partial t} = -\left\langle\int_A \vec{I}_a(\vec{r},t,f)\bullet d\vec{A}\right\rangle = -\int_A |I_a(\vec{r},f)|\langle\cos\Theta(t)\rangle dA \\ &= -\frac{1}{4}\int_A |I_a(\vec{r},f)|dA = -\frac{1}{4}|I_a(\vec{r}_{hole},f)|\int_A dA = -\frac{1}{4}|I_a(\vec{r}_{hole},f)|A\end{aligned}$$

A clarificational note: In the steady-state, note that the time interval Δt_{avg} needed for averaging over the moment-to-moment fluctuations in the instantaneous direction of the 3-D sound intensity $\vec{I}_a(\vec{r},t,f)$ is much less than the characteristic time constant τ_W associated with sound energy leaking out of the room of volume *V* through the hole of area *A*, *i.e.* $\Delta t_{avg} \ll \tau_W$.

For a large room, we {can safely} assume that the nature of sound propagation is very similar to that in "free air" – *i.e.* the great outdoors. Then the instantaneous 3-D vector sound intensity $\vec{I}_a(\vec{r},t,f)$ is related to the instantaneous scalar acoustic energy density $w_a(\vec{r},t,f)$ (*Joules*/*m*^{3}) by the relation $\vec{I}_a(\vec{r},t,f) = \vec{c}w_a(\vec{r},t,f)$ where $\vec{c}$ = velocity vector associated with propagation of sound in free-air with $|\vec{c}| \simeq 344$ *m/s* at *NTP*. Thus, from the above discussion on averaging out random, moment-to-moment fluctuations in the direction of 3-D sound intensity at the hole of area *A*, we see that the time-averaged version of this relation also holds:

$$\langle\vec{I}_a(\vec{r},t,f)\rangle = \langle\vec{c}w_a(\vec{r},t,f)\rangle \Rightarrow \langle I_a(\vec{r}_{hole},t,f)\rangle = c\langle w_a(\vec{r}_{hole},t,f)\rangle$$

Note further that in the steady-state, at time *t* the acoustic energy $W_a(t,f)$ contained within the room of volume *V* is related to the {uniform} acoustic energy density $w_a(\vec{r},t,f)$ by:

$$W_a(t,f) = \int_V w_a(\vec{r},t,f)d^3r = w_a(t,f)\int_V d^3r = w_a(t,f)V$$

This relation also holds for time-averaged quantities:

$$\langle W_a(t,f)\rangle = \int_V \langle w_a(\vec{r},t,f)\rangle d^3r = \langle w_a(t,f)\rangle\int_V d^3r = \langle w_a(t,f)\rangle V$$

Since the acoustic energy leaking out of the hole comes from inside the room, by energy conservation, we see that:

$$\langle P_a(t,f)\rangle = \frac{\partial\langle W_a(t,f)\rangle}{\partial t} = -\left\langle \int_A \vec{I}_a(\vec{r},t,f)\bullet d\vec{A}\right\rangle = -\frac{1}{4}\langle I_a(\vec{r}_{hole},t,f)\rangle A = -\frac{1}{4}c\langle w_a(\vec{r}_{hole},t,f)\rangle A$$
$$= -\frac{cA}{4V}\langle W_a(t,f)\rangle$$

or: $$\frac{\partial\langle W_a(t,f)\rangle}{\partial t} = -\frac{cA}{4V}\cdot\langle W_a(t,f)\rangle = -\frac{1}{\tau_W}\langle W_{ac}(t,f)\rangle$$

where we have defined the characteristic time constant $\tau_W \equiv \frac{4V}{cA}$ (*SI* units: *seconds*).

The equation $\frac{\partial\langle W_a(t,f)\rangle}{\partial t} = -\frac{1}{\tau_W}\langle W_a(t,f)\rangle$ is a linear, first-order homogeneous differential equation {known as the diffusion, or heat equation} which, for our situation/our initial conditions (at $t = 0$) has the well-known solution of the form:

$$\langle W_a(t,f)\rangle = \langle W_a^o(f)\rangle e^{-t/\tau_W}$$

where $\langle W_a^o(f)\rangle$ is the time-averaged value of the acoustic energy contained in the room at the frequency *f* at time $t = 0$. Thus, at time $t = \tau_W$: $\langle W_a(f, t=\tau_W)\rangle = \langle W_a^o(f)\rangle e^{-\tau/\tau_W} = \langle W_a^o(f)\rangle e^{-1}$ *i.e.* the {time-averaged} acoustic energy at frequency *f* decreases to 1/*e* = 1/2.7183 = 0.3679 of its initial value in a time interval $t = \tau_W$.

For a large room, since $\langle I_a(f,t)\rangle = c\langle w_a(f,t)\rangle = c\langle W_a(f,t)\rangle/V$, we can equivalently rewrite the solution for the time-averaged acoustic energy in terms of the time-averaged sound intensity as: $\langle I_a(t,f)\rangle = \langle I_a^o(f)\rangle e^{-t/\tau_W}$ where $\langle I_a^o(f)\rangle$ is the time-averaged sound intensity at the frequency *f* at time $t = 0$, and instead ask: how long does it take for the time-averaged sound intensity to decay to one-millionth (10^{-6}) of its initial value, *i.e.* what is the ***reverberation time*** T_{60}? This occurs when:

$$\langle I_a(t=T_{60},f)\rangle = \langle I_a^o(f)\rangle e^{-T_{60}/\tau_W} = 10^{-6}\langle I_a^o(f)\rangle$$

i.e. this occurs when $e^{-T_{60}/\tau_W} = 10^{-6}$. Take the natural log of both sides of this relation: $\ln\left(e^{-T_{60}/\tau_W}\right) = \ln\left(10^{-6}\right)$. But $\ln\left(e^{-T_{60}/\tau_W}\right) = -T_{60}/\tau_W$. Thus: $-T_{60}/\tau_W = \ln\left(10^{-6}\right)$ or:

$T_{60} = -\tau_W \ln\left(10^{-6}\right)$ and since $\tau_W \equiv \frac{4V}{cA}$, we thus find that the ***reverberation time*** T_{60} is:

$$T_{60} = -\tau_W \ln\left(10^{-6}\right) = \left\{\frac{4V}{cA}\right\}\ln\left(10^{-6}\right) = \left\{-\frac{4\cdot\ln\left(10^{-6}\right)}{c}\right\}\frac{V}{A} = \kappa\frac{V}{A}$$

The numerical value of this "universal" constant, κ is:

$$\kappa \equiv -\frac{4\cdot\ln\left(10^{-6}\right)}{c} = +\frac{4\cdot 13.8155}{343\,m/s} = +\frac{55.262}{343\,m/s} = 0.1611\,s/m \;\left(=0.049\,s/ft\right)$$

Thus, the ***Sabine equation*** is: $\boxed{T_{60} = 0.161\frac{V}{A}\ (metric\ units) = 0.049\frac{V}{A}\ (english\ units)}$.

We also see that: $\boxed{T_{60} = -\tau_W \ln\left(10^{-6.0}\right) = 13.8155\,\tau_W}$.

Similarly, we can also show that the ***reverberation time*** T_{30}, defined as the time it takes for the time-averaged sound intensity to decay to one-thousandth (10^{-3}) of its initial value is given by:

$$\boxed{T_{30} = -\tau_W \ln\left(10^{-3.0}\right) = 6.9078\,\tau_W = \tfrac{1}{2}T_{60}}$$

How do we physically measure/determine the ***reverberation time*** T_{60} (and/or T_{30})?

<u>Method I</u>:

Note that since:

$$\left\langle I_a\left(t=T_{60},f\right)\right\rangle = \left\langle I_a^o\left(f\right)\right\rangle e^{-T_{60}/\tau_W} = 10^{-6}\left\langle I_a^o\left(f\right)\right\rangle$$
$$\left\langle I_a\left(t=T_{30},f\right)\right\rangle = \left\langle I_a^o\left(f\right)\right\rangle e^{-T_{30}/\tau_W} = 10^{-3}\left\langle I_a^o\left(f\right)\right\rangle$$

Then, from the above {time-averaged} sound intensity level formulae:

$$\left\langle SIL\left(t=T_{60},f\right)\right\rangle = 10\log_{10}\left(\left\langle I_a\left(t=T_{60},f\right)\right\rangle/\left\langle I_o\right\rangle\right)$$
$$\left\langle SIL\left(t=T_{30},f\right)\right\rangle = 10\log_{10}\left(\left\langle I_{ac}\left(t=T_{30},f\right)\right\rangle/\left\langle I_o\right\rangle\right)$$

where $\left\langle I_o\right\rangle = 10^{-12}\,Watts/m^2$ is the (time-averaged) reference sound intensity (at the threshold of human hearing).

At time $t=0$ we have:

$$\left\langle SIL\left(t=0,f\right)\right\rangle = 10\log_{10}\left(\left\langle I_{ac}\left(t=0,f\right)\right\rangle/\left\langle I_o\right\rangle\right)$$
$$= 10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle/\left\langle I_o\right\rangle\right)$$

Thus, the $t = 0$ to $t = T_{60}$ difference in {time-averaged} sound intensity levels is:

$$
\begin{aligned}
\Delta\left\langle SIL\left(\Delta t = T_{60}, f\right)\right\rangle &\equiv \left\langle SIL\left(t = T_{60}, f\right)\right\rangle - \left\langle SIL\left(t = 0, f\right)\right\rangle \\
&= 10\log_{10}\left(\left\langle I_a\left(t = T_{60}, f\right)\right\rangle / \left\langle I_o\right\rangle\right) - 10\log_{10}\left(\left\langle I_a\left(t = 0, f\right)\right\rangle / \left\langle I_o\right\rangle\right) \\
&= 10\log_{10}\left(10^{-6}\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right) - 10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right) \\
&= 10\log_{10}\left(10^{-6}\right) + \cancel{10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right)} - \cancel{10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right)} \\
&= 10\log_{10}\left(10^{-6}\right) = -60\ dB
\end{aligned}
$$

Likewise, the $t = 0$ to $t = T_{30}$ difference in {time-averaged} sound intensity levels is:

$$
\begin{aligned}
\Delta\left\langle SIL\left(\Delta t = T_{30}, f\right)\right\rangle &\equiv \left\langle SIL\left(t = T_{30}, f\right)\right\rangle - \left\langle SIL\left(t = 0, f\right)\right\rangle \\
&= 10\log_{10}\left(\left\langle I_a\left(t = T_{30}, f\right)\right\rangle / \left\langle I_o\right\rangle\right) - 10\log_{10}\left(\left\langle I_a\left(t = 0, f\right)\right\rangle / \left\langle I_o\right\rangle\right) \\
&= 10\log_{10}\left(10^{-3}\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right) - 10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right) \\
&= 10\log_{10}\left(10^{-3}\right) + \cancel{10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right)} - \cancel{10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right)} \\
&= 10\log_{10}\left(10^{-3}\right) = -30\ dB
\end{aligned}
$$

Now, for a large room (*i.e.* in a "free field" situation), the sound intensity level (*SIL*) will be within ~ 0.1 *dB* of the sound pressure level (*SPL*) over the human auditory range (~ 20 *Hz* – 20 *KHz*). Thus, using an accurately-calibrated *SPL* meter, technically speaking, the ***reverberation time*** $T_{60} = \Delta t = t_2 - t_1$ is the measured time interval corresponding to a decrease of:

$\Delta\left\langle SPL\left(\Delta t = T_{60}, f\right)\right\rangle = -60\ dB$ measured from $\Delta\left\langle SPL\left(t = t_1, f\right)\right\rangle = -5\ dB$ to $\Delta\left\langle SPL\left(t = t_2, f\right)\right\rangle = -65\ dB$ referenced to $\left\langle SPL\left(t = 0, f\right)\right\rangle$.

Similarly, technically speaking, the ***reverberation time*** $T_{30} = \Delta t = t_2' - t_1'$ is the measured time interval corresponding to a decrease of:

$\Delta\left\langle SPL\left(\Delta t = T_{30}, f\right)\right\rangle = -30\ dB$ measured from $\Delta\left\langle SPL\left(t = t_1', f\right)\right\rangle = -5\ dB$ to $\Delta\left\langle SPL\left(t = t_2', f\right)\right\rangle = -35\ dB$ referenced to $\left\langle SPL\left(t = 0, f\right)\right\rangle$.

For example, if $\left\langle SPL\left(t = 0, f\right)\right\rangle = 100\ dB$, then $\left\langle SPL\left(t = t_1, f\right)\right\rangle = \left\langle SPL\left(t = t_1', f\right)\right\rangle = 95\ dB$ and $\left\langle SPL\left(t = t_2, f\right)\right\rangle = 35\ dB$, whereas $\left\langle SPL\left(t = t_2', f\right)\right\rangle = 65\ dB$, thus:

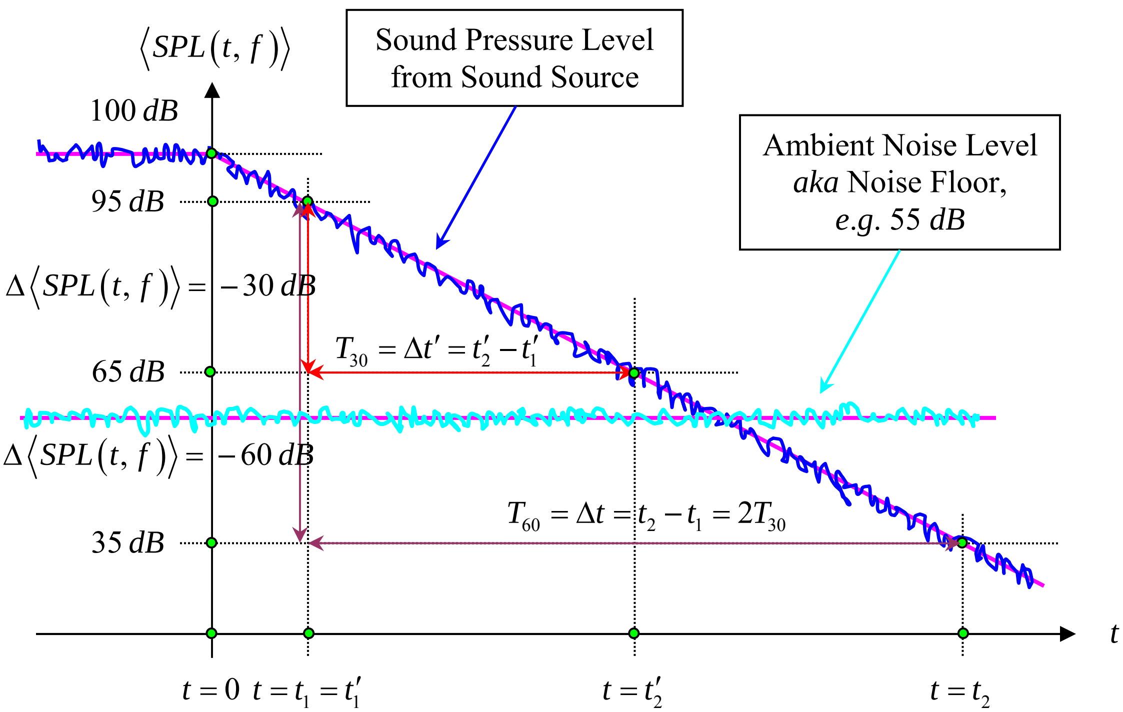
$$\Delta\left\langle SPL\left(\Delta t = T_{60}, f\right)\right\rangle = \left\langle SPL\left(t = t_2, f\right)\right\rangle - \left\langle SPL\left(t = t_1, f\right)\right\rangle = 35\ dB - 95\ dB = -60\ dB$$

and:

$$t_1 = t_1'$$

$$\Delta\left\langle SPL\left(\Delta t = T_{30}, f\right)\right\rangle = \left\langle SPL\left(t = t_2', f\right)\right\rangle - \left\langle SPL\left(t = t_1', f\right)\right\rangle = 65\ dB - 95\ dB = -30\ dB$$

These relations are shown in the figure below:



Note that the ambient noise level (*aka* noise floor) of the room may (often) be such that it is significantly above $\langle SPL(t = t_2, f)\rangle = 35\ dB$ (*e.g.* $55\ dB$), hence measuring the T_{60} reverberation time will ***not*** be possible, whereas measuring the T_{30} reverberation time ***will*** be possible (as long as the noise floor of the room is below $\langle SPL(t = t_2', f)\rangle = 65\ dB$). The T_{60} reverberation time can then be ***calculated*** from the ***measured*** T_{30} reverberation time using the simple relation $T_{60} = 2T_{30}$.

Note that one can also simply increase the sound power input to the room, such that with *e.g.* $\langle SPL(t = 0, f)\rangle = 120\ dB$, then: $\langle SPL(t = t_1, f)\rangle = \langle SPL(t = t_1', f)\rangle = 115\ dB$, $\langle SPL(t = t_2, f)\rangle = 55\ dB$, and $\langle SPL(t = t_2', f)\rangle = 85\ dB$, thus:

$$\Delta\langle SPL(\Delta t = T_{60}, f)\rangle = \langle SPL(t = t_2, f)\rangle - \langle SPL(t = t_1, f)\rangle = 55\ dB - 115\ dB = -60\ dB$$

and: $t_1 = t_1'$

$$\Delta\langle SPL(\Delta t = T_{30}, f)\rangle = \langle SPL(t = t_2', f)\rangle - \langle SPL(t = t_1', f)\rangle = 85\ dB - 115\ dB = -30\ dB$$

Note further that there is nothing sacred about using T_{60} or T_{30}; We can also define other reverberation times: $T_{10}, T_{20}, T_{40}, T_{50}$ ... which correspond to relative decreases of SIL's/SPL's of $-10, -20, -40, -50\ ...\ dB$ respectively… The general relation for an arbitrary definition of the reverberation time is: $\boxed{T_{xx} \equiv -\tau_W \ln\left(10^{-(xx/10)}\right)}$, with: $\boxed{\Delta\langle SPL(\Delta t = T_{xx}, f)\rangle = -xx\ dB}$.

Method II:

Note that since: $\left\langle I_a\left(t=\tau_W, f\right)\right\rangle = \left\langle I_a^o\left(f\right)\right\rangle e^{-\tau_W/\tau_W} = \left(1/e\right)\left\langle I_a^o\left(f\right)\right\rangle = 0.3679\left\langle I_a^o\left(f\right)\right\rangle$

we can use a ***true sound intensity meter*** (*e.g*. Bruel & Kjaer 2260E) to measure the time interval $\Delta t = \tau_W - 0 = \tau_W$ that the time-averaged sound intensity falls to 1/*e* = 0.3679 of its initial (*t* = 0) value. We can then subsequently calculate T_{60} and T_{30} using the relations:

$$\boxed{T_{60} = -\tau_W \ln\left(10^{-6.0}\right) = 13.8155\,\tau_W} \quad \text{and:} \quad \boxed{T_{30} = -\tau_W \ln\left(10^{-3.0}\right) = 6.9078\,\tau_W = \tfrac{1}{2}T_{60}}$$

However, note that if the typical T_{60} reverberation time is ~ 1.4 sec, then $T_{30} = \tfrac{1}{2}T_{60} \sim 0.7\,\text{sec}$ and $\tau_W \sim 0.1\,\text{sec}$! Hence, experimentally it is much better to (directly) measure T_{60} and/or T_{30} if the time resolution of the measurement, $\sigma_t \sim \tau_W \sim 0.1\,\text{sec}$.

Note further that the following is also a difficult quantity to (accurately) measure:

$$\begin{aligned}
\Delta\left\langle SIL\left(\Delta t=\tau_W, f\right)\right\rangle &\equiv \left\langle SIL\left(t=\tau_W, f\right)\right\rangle - \left\langle SIL\left(t=0, f\right)\right\rangle \simeq \Delta\left\langle SPL\left(\Delta t=\tau_W, f\right)\right\rangle \\
&= 10\log_{10}\left(\left\langle I_a\left(t=\tau_W, f\right)\right\rangle / \left\langle I_o\right\rangle\right) - 10\log_{10}\left(\left\langle I_a\left(t=0, f\right)\right\rangle / \left\langle I_o\right\rangle\right) \\
&= 10\log_{10}\left(\left(1/e\right)\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right) - 10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right) \\
&= 10\log_{10}\left(1/e\right) + \cancel{10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right)} - \cancel{10\log_{10}\left(\left\langle I_a^o\left(f\right)\right\rangle / \left\langle I_o\right\rangle\right)} \\
&= 10\log_{10}\left(1/e\right) = -4.343\,dB
\end{aligned}$$

However, additionally note that one could very easily carry out a least-squares fit of the *SIL*(*t*) (or *SPL*(*t*)) data to a decaying exponential in order to obtain an accurate determination of τ_W.

Method III:

For a large room (*i.e*. in a "free field" situation), the (time-averaged) sound intensity $\left\langle I_a\left(t, f\right)\right\rangle$ is related to the time-averaged square of the instantaneous over-pressure by the relation:

$$\left\langle I_a\left(t, f\right)\right\rangle = \left\langle p^2\left(t, f\right)\right\rangle / \rho_o c$$

where $\rho_o = 1.204\,kg/m^3$ is the density of air and $c = 343\,m/s$ is the longitudinal speed of propagation of sound in air {at NTP}.

Then since: $\left\langle I_a\left(t, f\right)\right\rangle = \left\langle I_a^o\left(f\right)\right\rangle e^{-t/\tau_W}$, then: $\left\langle p^2\left(t, f\right)\right\rangle = \left\langle p^2\left(t=0, f\right)\right\rangle e^{-t/\tau_W}$

For a large room (*i.e*. in a "free field" situation), the time-average square of the over-pressure ***amplitude*** is related to the over-pressure amplitude by:

$$\left\langle p^2\left(t, f\right)\right\rangle = \left\langle p^2\left(t=0, f\right)\right\rangle e^{-t/\tau_W} = \tfrac{1}{2}\,p_o^2 e^{-t/\tau_W}$$

where p_o is the over-pressure amplitude at $t = 0$, and $p(t, f) = p(t = 0, f)e^{-t/\tau_p} = p_o \cos\omega t \cdot e^{-t/\tau_p}$ is the instantaneous over-pressure, and τ_p is the characteristic time constant associated with the exponential decay of the over-pressure amplitude with time.

Now, time-averaging over the rapid oscillations associated with $\cos\omega t = \cos 2\pi ft$, but ***not*** the slow-decay of the exponential, we have:

$$\left\langle p^2(t, f)\right\rangle = \left\langle p_o^2 \cos^2 \omega t \cdot e^{-2t/\tau_p}\right\rangle = p_o^2 \left\langle \cos^2 \omega t\right\rangle e^{-2t/\tau_p} = \tfrac{1}{2} p_o^2 e^{-2t/\tau_p}$$

Comparing this result with the above, we see that a relation exists between the two time constants τ_W and τ_p:

$$\boxed{\tau_W = \tau_p/2} \text{ or: } \boxed{\tau_p = 2\tau_W}$$

Hence, we obtain the following relations:

$$\boxed{T_{60} = -\tau_W \ln\left(10^{-6.0}\right) = 13.8155\,\tau_W = 13.8155\cdot\left(\tau_p/2\right) = 6.90776\,\tau_p}$$

and:

$$\boxed{T_{30} = -\tau_W \ln\left(10^{-3.0}\right) = 6.9078\,\tau_W = 6.9078\cdot\left(\tau_p/2\right) = 3.45388\,\tau_p = \tfrac{1}{2}T_{60}}$$

We have developed for the UIUC Physics 406 (and Physics 193) POM course a method which has enabled us to determine the over-pressure decay time constant τ_p to high accuracy by:

a.) 24-bit digital recording the time-dependent signal output from a (reference) pressure mic situated somewhere in the large room, stimulated by either a single frequency, or white/pink noise emanating from a sound source located somewhere in the large room.

b.) offline analyzing the pressure mic's time-dependent 24-bit digital signal data (*.wav format), using digital filtering techniques to window around the single frequency (in order to reject ambient noise), or use *e.g.* digital filters to window pink/white noise signals in 31 1/3-octave bands across the full audio spectrum, calculating the standard deviations $\sigma_p(t)$ of the over-pressure signals (which are linearly proportional to the over-pressure amplitudes) of the filtered signals in a short, running time window Δt (with $\tau_p\,(=2\tau_W) \gg \Delta t \gg \tau = 1/f$), and then:

c.) carrying out least-squares fits to decaying exponentials $\sigma_p(t) = \sigma_p^o e^{-t/\tau_p}$ associated with each of the windowed standard deviations, *e.g.* in the first ½ second (second ½ second) time interval – often there is more than one time constant, *e.g.* if the ceiling and floor of the large room are more absorptive than the walls of the room…

If interested, please see/read *e.g.* Serin Yoon's Fall 2012, Nathan Oliveira and Frank Horger's Fall 2010 and also Eric Egner's Fall 2007 UIUC 193 POM Final Reports, available on-line at the following URL:

http://courses.physics.illinois.edu/phys193/193_student_projects.html