Mathematical Musical Physics of the Wave Equation

The purpose of this particular set of lecture notes for this course is to investigate the mathematical physics (and the use) of the wave equation for describing wave behavior associated with different kinds of one, two and three-dimensional physical systems – which have relevance for various kinds of musical instruments. The wave equation mathematically describes the behavior of waves for a given physical system, and is "*generically*" given by:

$$\nabla^2\psi(\vec{r},t)-\frac{1}{v^2}\frac{\partial^2\psi(\vec{r},t)}{\partial t^2}=0$$

where $\psi(\vec{r},t)$ is the displacement amplitude of the wave at the (1-, 2-, or 3-D) space position, $\vec{r}$ at time, t from its equilibrium position; the symbol v represents the longitudinal speed of propagation of the wave and ∇^2 is the *L*aplacian operator (the form of which is relevant for the dimensionality and symmetry of the physical system under investigation). Formally-speaking, the above wave equation is a *linear*, *homogeneous* *2*nd-*order* differential equation.

For musical instrument applications, we are specifically interested in *standing wave solutions* of the wave equation (and not so much interested in investigating the *traveling wave solutions*). We have discussed the mathematical physics associated with traveling and standing waves in previous lecture notes for this course. Mathematically, standing wave solutions of the wave equation – are formally known as *eigen solutions*, *eigen modes*, and/or also known as *normal modes* – and such solutions result as a consequence of imposing *specific* *boundary conditions* in *space* on *either* the amplitude of the wave, $\psi(\vec{r},t)$ – which are formally known as *Dirichlet boundary conditions*, or imposing specific boundary conditions on the *spatial 1st derivatives* of $\psi(\vec{r},t)$, *e.g.* $\partial\psi(\vec{r},t)/\partial x|_{x=x_o}$ – which are formally known as *Neumann boundary conditions*, or imposing a combination of *both* – *i.e*. *mixed boundary conditions* on $\psi(\vec{r},t)$ *and* spatial 1st derivatives of $\psi(\vec{r},t)$ – which are formally known as *Cauchy boundary conditions*.

Since we are specifically interested in standing wave eigen-solutions of the wave equation, we will "*go for the jugular*" in these lecture notes and dispense with discussion of *the most general possible* solutions of the wave equation for a given physical problem, in order to keep the length of these lecture notes tractable. The reader is requested to bear this in mind!

The reader should also note that the above linear, homogeneous wave equation is a mathematical description/treatment of wave phenomena at a *first-order* level – actual (*i.e*. real) physical systems have *additional* physical processes that are simultaneously operative, *e.g*. dissipative processes – such as internal/external friction and/or damping - air viscosity, various other energy loss mechanisms, etc. and *e.g.* finite stiffness effects of vibrating systems, as opposed to the perfectly compliant material implicitly assumed in the (derivation of the) above wave equation. All these "higher-order" effects are (for the time being) temporarily neglected here – usually these effects are small, resulting in perturbations/small (but easily measurable/detectable) shifts in the normal modes of vibration of the mechanical system. When such higher-order effects are explicitly included, they add/contribute additional mathematical terms to the wave equation. Thus solution(s) to the more-sophisticated/realistic wave equation are correspondingly more complicated and tedious to obtain.

Furthermore, implicit in the mathematics of the above wave equation is the tacit assumption that the physical material(s) comprising the various systems we are about to discuss are *linear*, *homogeneous* and *isotropic* materials. There is nothing in the above form of the wave equation that explicitly takes into account the proper description of non-linear, non-homogeneous and/or anisotropic materials. The above wave equation is also really only valid strictly in the small-amplitude-of-oscillations/vibrations regime. However, modifications to explicitly include such effects *can* be incorporated into the wave equation to accommodate these additional properties of such physical systems, if needed. Again, the solution(s) to the more-sophisticated/realistic wave equation are correspondingly more complicated and tedious to obtain.

Here in these lecture notes, we wish to see the overall physics "forest", and thus temporarily neglect/ignore (some of) the details – the physics "trees".

A. Standing Waves In One-Dimensional Systems:

A1. Transverse Standing Waves on a Vibrating String – Fixed Ends:

One-dimensional wave behavior on a vibrating string is mathematically described by the 1-D wave equation:

$$\frac{\partial^2\psi(x,t)}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2\psi(x,t)}{\partial t^2}=0$$

where $\psi(\vec{r},t)$ is the instantaneous transverse displacement amplitude of the string at the point $\vec{r}$ at the time t. We can (trivially) rewrite this as:

$$\frac{\partial^2\psi(x,t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2\psi(x,t)}{\partial t^2}$$

Notice that the LHS of this equation has only space-derivatives of the one-dimensional variable, x associated with it, whereas the RHS of this equation has only time-derivatives associated with it. This suggests we try a *product solution* for the wave equation, *i.e.* $\psi(x,t)=U(x)T(t)$ where $U(x)$ contains only spatially x-dependent terms and $T(t)$ contains only temporally- (*i.e.* time)-dependent terms.

Formally, we insert $\psi(x,t)=U(x)T(t)$ into the above differential equation, and then explicitly carry out the differentiation:

$$\frac{\partial^2U(x)T(t)}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2U(x)T(t)}{\partial t^2}$$

$$T(t)\frac{\partial^2U(x)}{\partial x^2}=\frac{1}{v^2}U(x)\frac{\partial^2T(t)}{\partial t^2}$$

$$T(t)\frac{d^2U(x)}{dx^2}=\frac{1}{v^2}U(x)\frac{d^2T(t)}{dt^2}$$

$$\frac{1}{U(x)}\frac{d^2U(x)}{dx^2}=\frac{1}{v^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}=-k^2$$

In the second line above, we've explicitly used the fact(s) that $T(t)$ and $U(x)$, respectively are functions of time, *t* and space, *x* only. On the third line, again since $U(x)$ and $T(t)$, respectively are functions only of time and space, we formally convert partial derivatives of *x* and *t* to total derivatives of *x* and *t*, respectively. Finally, on the fourth line, we divide both sides of this equation by $\psi(x,t) = U(x)T(t)$ and then notice that the LHS is now a entirely function of $U(x)$ only and that the RHS is now entirely a function of $T(t)$. This last equation *must* be satisfied for *any/all* possible values of *x* and *t*, and the *only* way this can happen is if *both* sides of this latter equation are equal to a *constant*, which we (deliberately chose to) set equal to $-k^2$ (since we know what's going to happen next…:). We call the constant that arises from use of the separation of variables technique, the *separation constant*. Thus, here $-k^2$ is the separation constant.

Using the separation of variables technique, we actually wind up with *two* differential/wave equations:

$$\frac{d^2U(x)}{dx^2} = -k^2U(x)$$

$$\frac{1}{v^2}\frac{d^2T(t)}{dt^2} = -k^2T(t)$$

Now $v = f\lambda = (\omega/2\pi)(2\pi/k) = \omega / k$; thus $vk = \omega$. Rewriting the above two equations:

$$\frac{d^2U(x)}{dx^2} + k^2U(x) = 0$$

$$\frac{d^2T(t)}{dt^2} + \omega^2T(t) = 0$$

Both of these equations are of the same mathematical form – both are indeed wave equations. The space-domain version of this linear, homogeneous 2nd –order differential equation is known as the *Helmholtz equation*.

Thus far, we have not explicitly discussed any particular solution(s) of these wave equations – we know they are oscillatory in space and time, and again, we seek standing wave solutions.

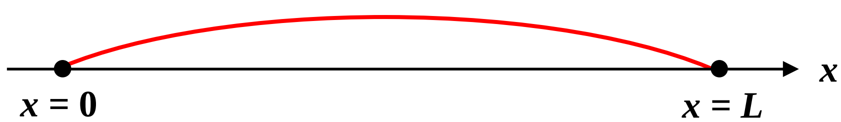
On most stringed instruments, such as the violin, viola, cello, guitar, mandolin, piano, etc. the ends of the string(s) – *e.g.* at the bridge and nut (headstock) end of a guitar are *ideally* rigidly attached to the body of the instrument in some manner. The string(s) on the instrument then have (active) length, *L* – the distance between the two fixed ends (this is known as the scale length). Defining our coordinate system such that the x-axis coincides with the equilibrium shape of the string, one end of the string at $x = 0$ (*e.g.* the bridge), the other end at $x = L$ (the nut/headstock on a guitar). Mathematically, the so-called boundary condition at the ends of the string(s) is that the displacement amplitude is zero at the ends of the string, *i.e.* $\psi(x=0,t) = \psi(x=L,t) = 0$ for fixed ends, independent of time, *t*.

Thus, this boundary condition is only relevant to the $U(x)$-wave equation, *i.e.* in reality the boundary condition for fixed ends is $U(x=0)=U(x=L)=0$. In general, there are only two allowed spatially periodic standing wave solutions to this wave equation – either sin kx, or cos kx. The above boundary condition for fixed ends allows *only* the sin kx-type solutions, because sin 0 = 0, and sin $kL = 0$ *if and only if* $kL = n\pi$, n = 1, 2, 3, 4, … Denoting $k_n = n\pi/L$, we see that the allowed standing wave solution(s) to the $U(x)$-wave equation are of the form:

$$U_n(x) = A_n \sin(k_n x) = A_n \sin(n\pi x/L)$$

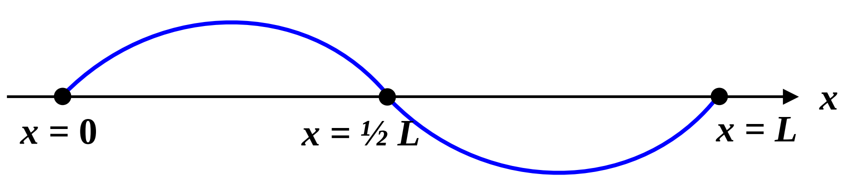
We also see that the boundary condition(s) on the spatial $U(x)$ standing wave solutions to the $U(x)$-wave equation determine the frequencies of the eigen-modes/normal modes of vibration. Since $\lambda = 2\pi/k$, then $\lambda_n = 2\pi/k_n = 2\pi L/n\pi = 2L/n$, n = 1, 2, 3, 4, …Since $f = v/\lambda$, then $f_n = v/\lambda_n$ and thus $f_n = v/\lambda_n = nv/2L$, n = 1, 2, 3, 4, …

The lowest mode of vibration (known as the *fundamental*), is when n = 1; thus this vibrational standing wave eigen-mode is also known as the *first harmonic*, with $f_1 = v/2L$ and $\lambda_1 = 2L$ (*i.e.* $L = ½\,\lambda_1$) and corresponding standing wave solution $U_1(x) = A_1 \sin k_1 x = A_1 \sin \pi x/L$, as shown in the figure below:



Excluding the endpoints, note that the first harmonic/fundamental has no *nodes* in its spatial wave function, $U_1(x)$.

The next highest eigen-mode (n = 2) is known as the *second harmonic* (also known as the *first overtone*) with $f_2 = 2v/2L = v/L$ and $\lambda_2 = 2L/2 = L$ and corresponding standing wave solution $U_2(x) = A_2 \sin k_2 x = A_2 \sin 2\pi x/L$ as shown below. Note that the second harmonic has one *node* in its spatial wavefunction, $U_2(x)$ at $x = L/2$.
The next highest eigen-mode (n = 3) is known as the third harmonic (also known as the *second*



□vertone) with $f_3 = 3v/2L$ and $\lambda_3 = 3L/2$, and corresponding standing wave solution $U_3(x) = A_3 \sin k_3 x = A_3 \sin 3\pi x/L$, which has two nodes in its spatial standing wave function, $U_3(x)$ at $x = L/3$ and $x = 2L/3$; and so on… in general, there are $(n-1)$ nodes for the spatial standing wave function, $U_n(x)$.

We have not yet discussed the allowed solutions to the *temporal* Helmholtz equation, so let us now turn our attention to this. As stated earlier, the physical boundary conditions on the ends of the strings place no *direct* constraints on the allowed solutions, $T(t)$ of the temporal Helmholtz equation. Thus, *both* sin ωt and cos ωt solutions in principle are (or must be) allowed.

Also, since spatial wavelengths and temporal frequencies are intimately related to each other via $f = v/\lambda$, and hence also the eigen-frequencies, $f_n = v/\lambda_n = nv/2L$, $\Rightarrow$ $\omega_n = vk_n = n\pi v/L$, then the allowed solutions to the temporal Helmholtz equation, $T(t) \to T_n(t)$, with both sin $\omega_n t$ and cos $\omega_n t$ type solutions allowed. Note that the eigen-frequencies for standing waves of a string with fixed endpoints are integer-multiples of the lowest mode of vibration, $f_n = nf_1$, n = 1, 2, 3, 4, 5, …. with $f_1 = v/2L$ and $\lambda_n = \lambda_1/n$ with $\lambda_1 = 2L$.

In actuality, the *detailed shape of the vibrating string at some particular instant in time* (usually conveniently taken to be $t = 0$) – known as the "*initial conditions*" (not to be confused with {spatial} boundary conditions!) actually *determines/defines* the *relative* amount(s) of the allowed sin $\omega_n t$ vs. cos $\omega_n t$ type solutions – because this simply specifies, for each eigen-mode what its phase is/how far it is along in its oscillation cycle at time $t = 0$. Temporal phase information is "encrypted" into the allowed eigen-solutions, $T_n(t)$ of the temporal Helmholtz equation as follows:

$$T_n(t) = b_n \sin \omega_n t + c_n \cos \omega_n t$$

With the following constraints/conditions on the coefficients b_n and c_n:

$$-1 \le b_n \le +1$$
$$-1 \le c_n \le +1$$
$$\sqrt{b_n^2 + c_n^2} = 1$$

Of course, one could equivalently write the allowed eigen-solutions of the temporal Helmholtz equation simply as one or the other of the following forms:

$$T_n(t) = \sin\left(\omega_n t + \delta_n\right); \qquad \delta_n = \tan^{-1}\left(c_n/b_n\right) = \cot^{-1}\left(b_n/c_n\right)$$
$$T_n(t) = \cos\left(\omega_n t + \varphi_n\right); \qquad \varphi_n = \tan^{-1}\left(b_n/c_n\right) = \cot^{-1}\left(c_n/b_n\right)$$
$$\delta_n = \varphi_n + \tfrac{\pi}{2}$$

These relations can be obtained directly from the above $T_n(t) = b_n \sin \omega_n t + a_n \cos \omega_n t$ relation using the trigonometric identities for sin(A+B) and cos(A+B), respectively.

We can also write the allowed eigen-solutions of the temporal Helmholtz equation in yet another, equivalent form, using complex notation:

$$T_n(t) = e^{i(\omega_n t + \varphi_n)}$$

The complete eigen-function solution(s) $\psi_n(x,t) = U_n(x)T_n(t)$ for standing waves on a vibrating string with fixed ends are of the form:

$$\begin{aligned}
\psi_n(x,t) &= U_n(x)T_n(t) \\
&= A_n \sin(k_n x)\left[b_n \sin(\omega_n t) + c_n \cos(\omega_n t)\right] = A_n \sin(n\pi x/L)\left[b_n \sin(n\pi vt/L) + c_n \cos(n\pi vt/L)\right] \\
&= A_n \sin(k_n x)\sin(\omega_n t + \delta_n) = A_n \sin(n\pi x/L)\sin\left[(n\pi vt/L) + \delta_n\right] \\
&= A_n \sin(k_n x)\cos(\omega_n t + \varphi_n) = A_n \sin(n\pi x/L)\cos\left[(n\pi vt/L) + \varphi_n\right] \\
&= A_n \sin(k_n x)e^{i(\omega_n t + \varphi_n)} = A_n \sin(n\pi x/L)e^{i\left[(n\pi vt/L) + \varphi_n\right]}
\end{aligned}$$

with $\lambda_n = 2L/n$ and $f_n = v/\lambda_n = nv/2L$, n = 1, 2, 3, 4, 5, … and eigen-energies $E_n = \frac{1}{4}M\omega_n^2 A_n^2$.

The displacement amplitude coefficients, A_n are also formally specified/determined by the *initial conditions* (*i.e.* harmonic content) of the vibrating string at time t = 0. Physically, this means that the *detailed shape/configuration* of the vibrating string at time t = 0, *e.g.* a triangle, sawtooth or square wave-shape, etc. *completely specifies* – by the method of Fourier analysis (*i.e.* harmonic analysis) – the *exact* harmonic content (*i.e.* allowed f_n values), the harmonic amplitudes (values of A_n) and the phases, φ_n (or δ_n). For example, for a *symmetrical* triangle-type standing wave (which has reflection symmetry about its mid-point), only odd-n coefficients A_n are non-zero. For an *asymmetrical* triangle-type standing wave, which does not have reflection symmetry about its mid-point) both even and odd-n coefficients A_n are non-zero. For a 50% duty-cycle type square wave (which also has reflection symmetry about its mid-point), again only odd-n coefficients A_n are non-zero. For further details of how this is accomplished, see *e.g.* the UIUC P498POM lecture notes on Fourier Analysis, I-IV.

As mentioned at the outset of this section, the above eigen-function solutions $\psi_n(x,t) = U_n(x)T_n(t)$ for standing waves on a vibrating string with (idealized) fixed ends are relevant for a broad selection of stringed instruments, such as the violin, viola, cello, guitar, mandolin, piano, etc.

A2. Transverse Standing Waves on a Vibrating String – Free Ends:

Certain kinds of 1-dimensional systems with *free ends* (Neumann boundary conditions) can also exhibit standing wave solutions. Mathematically, the method of analysis is exactly the same as above, except that for free, rather than fixed ends – *i.e.* Neumann boundary conditions, the value of the spatial derivative (= slope) of $U(x)$ must vanish at the endpoints x = 0 and $x = L$ for any/all time(s) t:

$$\left.\frac{dU(x)}{dx}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{dU(x)}{dx}\right|_{x=L} = 0$$

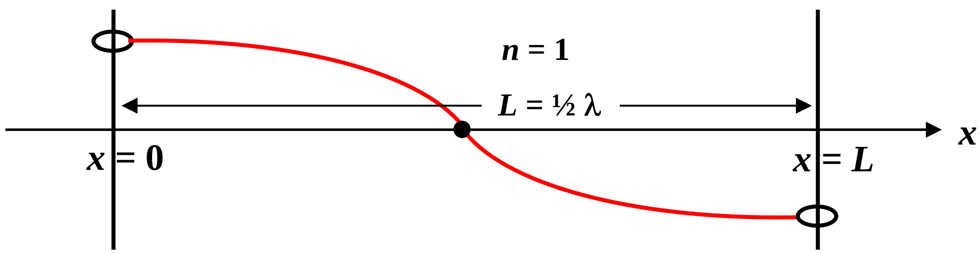
The reader can easily verify that the allowed eigen-solutions $U_n(x)$ for free ends *e.g.* of a vibrating string, or a transversely-vibrating 1-dimensional rod must be of the form cos $k_n x$ (rather than sin $k_n x$, as in the case of fixed ends/Dirichlet boundary conditions). The form of the temporal eigen-solutions, $T_n(t)$ are the same for free ends/Neumann boundary conditions and for fixed ends/Dirichlet boundary conditions.

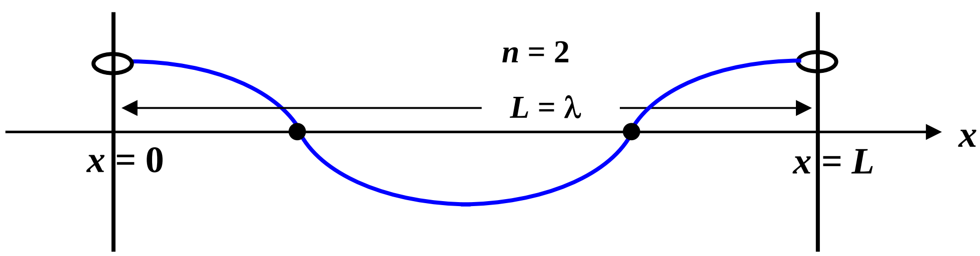
Physically, we can envision an ideal stretched string of length, L with free ends, tension, T and mass per unit length $\mu = M/L$ as having massless, frictionless rings attached to the ends of the string, each sliding frictionlessly on a perpendicular rod that constrain the motion of the ends of the string to be only in the transverse direction, as shown in the figures below.

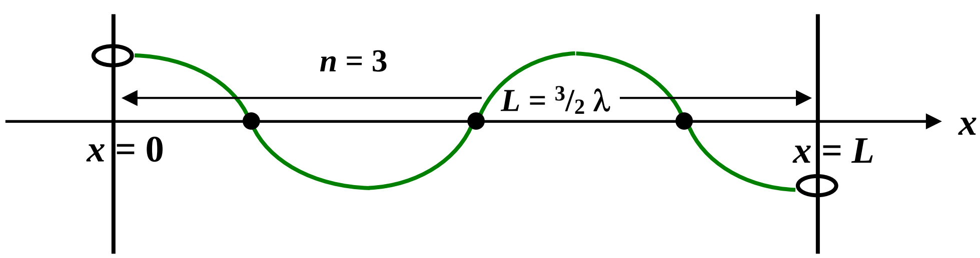
The complete eigen-function solution(s) $\psi_n(x,t) = U_n(x)T_n(t)$ for standing waves on a vibrating string with free ends are of the form:

$$\begin{aligned}
\psi_n(x,t) &= U_n(x)T_n(t) \\
&= A_n \cos(k_n x)\left[b_n \sin(\omega_n t) + c_n \cos(\omega_n t)\right] = A_n \cos(n\pi x/L)\left[b_n \sin(n\pi vt/L) + c_n \cos(n\pi vt/L)\right] \\
&= A_n \cos(k_n x)\sin(\omega_n t + \delta_n) = A_n \cos(n\pi x/L)\sin\left[(n\pi vt/L) + \delta_n\right] \\
&= A_n \cos(k_n x)\cos(\omega_n t + \varphi_n) = A_n \cos(n\pi x/L)\cos\left[(n\pi vt/L) + \varphi_n\right] \\
&= A_n \cos(k_n x)e^{i(\omega_n t + \varphi_n)} = A_n \cos(n\pi x/L)e^{i\left[(n\pi vt/L) + \varphi_n\right]}
\end{aligned}$$

with $\lambda_n = 2L/n$ and $f_n = v/\lambda_n = nv/2L$, $n = 1, 2, 3, 4, 5, \ldots$ and eigen-energies $E_n = \frac{1}{4}M\omega_n^2 A_n^2$.
The first few standing wave eigen-modes for free ends on a string are shown below:







Note the phase relation between relative motion of the displacement amplitude at the left and right ends for odd vs. even n free-end eigen-modes. For odd n, the motion of the ends is 180° out of phase, for even n, the motion of the ends is in-phase.

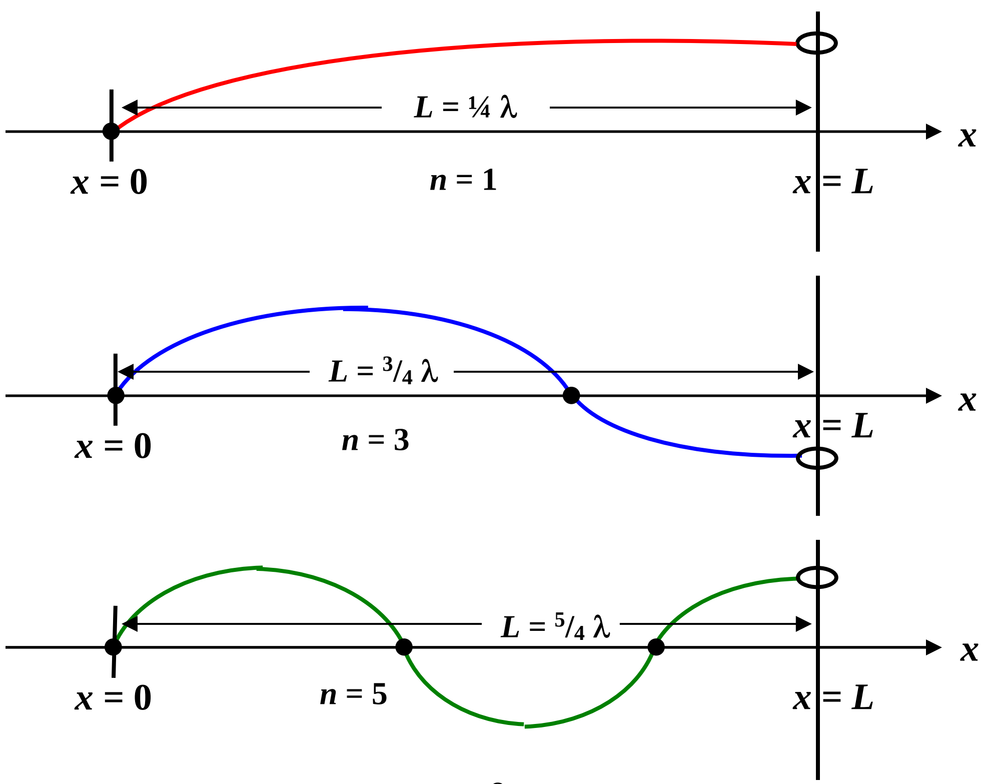
A3. Transverse Standing Waves on a Vibrating String – Fixed End + Free End:

For completeness, we consider the possibility of standing waves on a vibrating 1-dimensional system, such as a stretched string with *mixed* boundary conditions – *i.e.* so-called *Cauchy boundary conditions*, where one end of the string (e.g $x = 0$) is *fixed* (thus $U(x=0) = 0$) and the other end of the string ($x = L$) is *free* ($\partial U(x=L)/\partial x = 0$) (or vice-versa).

For the mixed boundary condition case, the reader can verify that fixed-free end boundary conditions with a *node* at the fixed end ($x = 0$) and an *anti-node* at the free end ($x = L$) requires spatial eigen-solutions of the form $U_n(x) = A_n \sin\left(k_n x\right) = A_n \sin\left(2\pi x/\lambda\right)$ with $\lambda_n = 4L/n$ and $f_n = v/\lambda_n = nv/4L$, with only odd-$n$ integers allowed, *i.e.* $n = 1, 3, 5, 7\ldots$ etc. The complete eigenfunction solutions for this case have the same form as that given in Section A1 above. For free-fixed end conditions ($\partial U(x=0)/\partial x = 0$ and $U(x= L) = 0$), the spatial eigen-function solutions are of the form $U_n(x) = A_n \cos\left(k_n x\right) = A_n \cos\left(2\pi x/\lambda_n\right)$ and thus the complete eigen-function solutions have the same form as that given in Section A2 above, but again only odd-n integers are allowed, *i.e.* $n = 1, 3, 5, 7\ldots$ and with eigen-energies $E_n = \frac{1}{4} M \omega_n^2 A_n^2$.

For either version of mixed fixed-free or free-fixed end conditions, the lowest mode – the fundamental/first harmonic ($n = 1$) has $\lambda_1 = 4L$ and $f_1 = v/4L$; the second harmonic ($n = 3$) has $\lambda_3 = 4L/3$ and $f_3 = 3v/4L$; the third harmonic ($n = 5$) has $\lambda_5 = 4L/5$ and $f_5 = 5v/4L$, and so on.

The spatial waveforms for the first few/lowest eigen-modes for mixed, *fixed-free* end conditions are shown in the figures below. The spatial waveforms for eigen-modes associated with *free-fixed* end conditions are mirror-reflections of the *fixed-free* eigen-mode solutions.



A4. Longitudinal Standing Waves on the Singing Rod – Free Ends:

The singing rod of length, L and diameter, $D << L$ is a one-dimensional vibrating system with eigen-modes that are longitudinal standing waves. The two ends of the rod at $x = 0$ and $x = L$ vibrate longitudinally (*i.e.* along the axis of the rod), thus we have free-free boundary conditions at the rod ends, *i.e.* Neumann boundary conditions, zero slopes at the ends of the rods, $(\partial U(x=0)/\partial x = 0$ and $(\partial U(x=L)/\partial x = 0$. The spatial eigen-solutions for the singing rod are the same as given in Section A2 above, *i.e.* they are of the form $U_n(x) = A_n \cos(k_n x) = A_n \cos(2\pi x/\lambda)$ with $\lambda_n = 2L/n$ and $f_n = v/\lambda_n = nv/2L$, $n = 1, 2, 3, 4, \ldots$ with eigen-energies $E_n = \frac{1}{4} M \omega_n^2 A_n^2$ and longitudinal displacement waveforms for the first few eigen-modes, also as shown in Section A2 above.

For a singing rod made up of an elastic solid, such as a metal, the longitudinal speed of propagation of sound in the solid, v_L is given by $v = \sqrt{Y/\rho}$ where Y (N/m^2) = Young's modulus and ρ (kg/m^3) is the density of the for the material. Physically, Young's modulus, $Y = \sigma/\varepsilon$ = ratio of longitudinal compressive stress (σ = F/A) to longitudinal strain ($\varepsilon = |L_2\text{-}L_1|/L_1$ {dimensionless}, where L_1 is the equilibrium length of the rod, and L_2 is the extended length of the rod).

To make a singing rod "sing" in one of its eigen-modes, one grasps the rod at a displacement node for that mode, *e.g.* with thumb and index finger, and then using the thumb and index finger of the other hand, pull sharply along the axis of the rod, toward the end of the rod, trying to stretch it. This works best coating the pulling thumb & index fingers first with crushed violin rosin, in order to really get a good "pull" on the rod. A typical displacement amplitude for the fundamental, f_1 ~ 1670 Hz on a L ~ 1.5 m, ~ 1" diameter aluminum rod is $\delta(x)$ ~ 1 mm. Its also very loud, with measured sound pressure level of ~ 140 dB at the rod ends! For more information specifically on the singing rod, see the UIUC P498POM lecture notes on the singing rod.

A5. Longitudinal Standing Waves on a Stretched String or a Thin Bar:

Longitudinal modes of vibrations of a stretched string, or a thin bar, while much less common than transverse standing waves in these same systems, do occur in certain circumstances. However, unlike transverse waves on a string or a thin bar, the longitudinal speed of propagation (and hence their eigen-frequencies) do not change with tension (unless the physical properties of the string or thin bar change with tension), since $v_L = \sqrt{Y/\rho}$. For both fixed ends or both free ends, the eigen-solutions $\psi_n(x,t) = U_n(x)T_n(t)$ for longitudinal displacement are as given in Section A1 or A2 above, respectively, both situations have $\lambda_n = 2L/n$ and $f_n = v_L/\lambda_n = nv_L/2L$, $n = 1, 2, 3, 4, 5, \ldots$ and eigen-energies $E_n = \frac{1}{4} M \omega_n^2 A_n^2$. For mixed fixed-free boundary conditions, the eigen-solutions for longitudinal displacement $\psi_n(x,t) = U_n(x)T_n(t)$ are as given in Section A3 above, with $\lambda_n = 4L/n$ and $f_n = v_L/\lambda_n = nv_L/4L$, with only <u>odd-*n*</u> integers allowed, *i.e.* $n = 1, 3, 5, 7\ldots$ and with eigen-energies $E_n = \frac{1}{4} M \omega_n^2 A_n^2$.

A6. Longitudinal Standing Waves in Long, Narrow Organ Pipes:

The standing waves that exist in long, narrow organ pipes (diameter, $D << $ length, L) of various kinds – closed-closed, open-open and closed-open or open-closed end conditions are excited by air flow – hence air pressure supplies the needed energy input, rather than mechanical displacement.

Due to the relation between the air over-pressure amplitude, $\Delta p_n(x,t)$ and the longitudinal displacement amplitude, $\psi_n(x,t)$, namely that $\Delta p_n(x,t) = -B_{air} d\psi_n(x,t)/dx$ where B_{air} is the bulk modulus of air, the mathematical form of longitudinal displacement amplitude eigen-solutions, $\psi_n(x,t) = U_n(x)T_n(t)$ for organ pipes with closed-closed end conditions (longitudinal displacement *nodes* and overpressure *antinodes* at the closed ends of the organ pipe of length, L) are as given above in Section A1, with $\lambda_n = 2L/n$ and $f_n = v/\lambda_n = nv/2L$, n = 1, 2, 3, 4, … and eigen-energies $E_n = \frac{1}{4} M_{air} \omega_n^2 A_n^2$, as shown in the figure below for the first few eigen-modes:

$$f_1 = \frac{c}{2L}, \qquad (8)$$

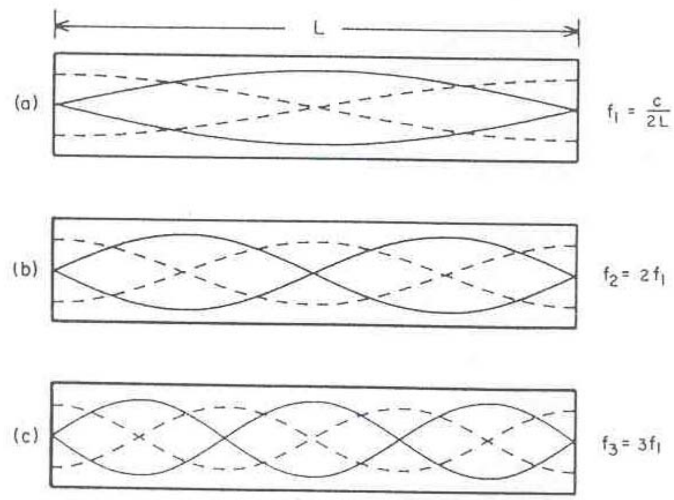


FIG. 7. First three vibration modes of an air column closed at both ends. Solid lines give displacement amplitudes; dashed lines, pressure amplitudes.

For each of the above (and following) organ pipe end-condition cases, the over-pressure amplitude eigen-solutions $\Delta P_n(x,t)$ can be obtained using the above relation between over-pressure amplitude and longitudinal displacement amplitude.

For organ pipes with open-open end conditions (longitudinal displacement *antinodes* and overpressure *nodes* at the open ends of the organ pipe of length, L) the mathematical form of longitudinal displacement eigen-solutions, $\psi_n(x,t) = U_n(x)T_n(t)$ are as given above in Section A2, again with $\lambda_n = 2L/n$ and $f_n = v/\lambda_n = nv/2L$, n = 1, 2, 3, 4, 5, … and eigen-energies $E_n = \frac{1}{4} M_{air} \omega_n^2 A_n^2$, as shown in the figure below for the first few eigen-modes:

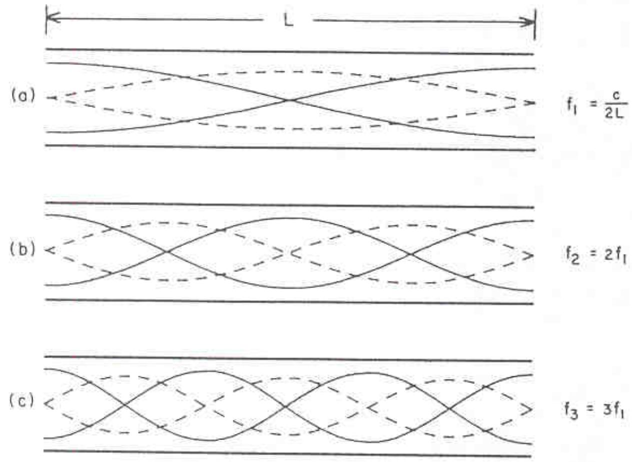


FIG. 8. First three vibration modes of an air column open at both ends. Solid lines give displacement amplitudes; dashed lines, pressure amplitudes.

For organ pipes with open-closed or closed-open end conditions (longitudinal displacement <u>*nodes*</u> (<u>*antinodes*</u>) and overpressure <u>*antinodes*</u> (<u>*nodes*</u>) at the closed (open) ends of the organ pipe of length, L, respectively) the mathematical form of longitudinal displacement eigen-solutions, $\psi_n(x,t) = U_n(x)T_n(t)$ are as given above in Section A3, with $\lambda_n = 4L/n$ and $f_n = v/\lambda_n = nv/4L$, with only <u>odd-*n*</u> integers allowed, $n = 1, 3, 5, 7\ldots$ and eigen-energies $E_n = \frac{1}{4} M_{air} \omega_n^2 A_n^2$ as shown below for the first few eigen-modes of a closed-open organ pipe of length, L:

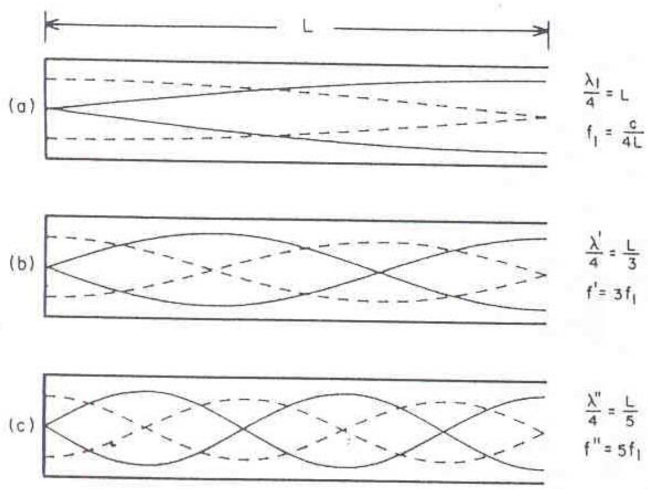


FIG. 9. First three vibration modes of an air column closed at one end and open at the other. Solid lines give displacement amplitudes; dashed lines, pressure amplitudes.

Note that an end correction exists – *e.g.* a last displacement node does not occur <u>*precisely*</u> at the open end of the organ pipe, but is located a distance ~ D outside it. Hence a more accurate wavelength formula is $\lambda_n^{'} \simeq \lambda_n^0 (1 + \alpha D)$, $\alpha \sim O(1)\, m^{-1}$ where the λ_n^0 are as given above.

B. Standing Waves In Two-Dimensional Systems:

We now turn our attention to standing waves associated with two-dimensional systems.

B1. Transverse Standing Waves on a Rectangular Membrane – Fixed Edges:

For a thin, perfectly compliant (*i.e.* flexible) rectangular membrane (*e.g.* an idealized rectangular drum head) of dimensions L_x and L_y, the wave equation in rectangular 2-dimensional coordinates (x,y) for the displacement amplitude, $\psi(x,y,t)$ is given by:

$$\nabla^2\psi(x,y,t)-\frac{1}{v^2}\frac{\partial^2\psi(x,y,t)}{\partial t^2}=0$$

where the longitudinal speed of propagation of transverse waves on a stretched, 2-dimensional membrane is given by $v=\sqrt{T_\ell/\sigma}$; T_ℓ is the membrane surface tension (in *Newtons*/*meter*) and $\sigma \equiv M/A = M/L_xL_y$ is the ***areal*** mass density of the membrane (in kg/m^2). The *L*aplacian operator, ∇^2 in 2-D rectangular coordinates is given by:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

Thus, the 2-dimensional wave equation describing the behavior of waves on a rectangular membrane is given by:

$$\frac{\partial^2\psi(x,y,t)}{\partial x^2}+\frac{\partial^2\psi(x,y,t)}{\partial y^2}-\frac{1}{v^2}\frac{\partial^2\psi(x,y,t)}{\partial t^2}=0$$

Again, we can (trivially) rewrite this as:

$$\frac{\partial^2\psi(x,y,t)}{\partial x^2}+\frac{\partial^2\psi(x,y,t)}{\partial y^2}=\frac{1}{v^2}\frac{\partial^2\psi(x,y,t)}{\partial t^2}$$

Notice again that the LHS (RHS) contains only spatial-dependent (time-dependent) functions, respectively. Thus, we again can use the technique of separation of variables, with $\psi(x,y,t)=U(x,y)T(t)$ where $U(x,y)$ contains only spatially x- and y-dependent terms and $T(t)$ contains only the time-dependent term.

Again, we have the relation $v = f\lambda = (\omega/2\pi)(2\pi/k) = \omega/k$; thus $vk = \omega$. We again obtain a separation constant of $-k^2$, and, after some simple algebraic manipulations, obtain the following two linear, homogeneous differential equations:

$$\frac{\partial^2 U(x,y)}{\partial x^2}+\frac{\partial^2 U(x,y)}{\partial y^2}+k^2U(x,y)=0$$

$$\frac{d^2T(t)}{dt^2}+\omega^2T(t)=0$$

We can again use the separation of variables technique on the above Helmholtz equation, with a product solution of the form $U(x,y) = X(x)Y(y)$. Inserting this into the above Helmholtz equation and carrying out the (partial) differentiations, dividing by $U(x,y) = X(x)Y(y)$ and carrying out a simple algebraic manipulation, we obtain the following equation:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k^2$$

Since the first (second) term on the LHS depends only on *x* (*y*), respectively, in order for this relation to be satisfied for all possible values of (*x*,*y*), each of these two terms ***must*** be equal to a constant, which we call $-k_x^2$ and $-k_y^2$, respectively. Thus we obtain the following relation, known as the so-called ***characteristic equation***:

$$k^2 = k_x^2 + k_y^2$$

with:

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0$$
$$\frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

With fixed-end/Dirichlet boundary conditions on both the *X*(*x*)- and *Y*(*y*)-solutions of:

$$X(x=0) = X(x=L_x) = 0$$
$$Y(y=0) = Y(y=L_y) = 0$$

we again obtain spatial eigen-mode solutions for 2-D transverse standing waves of the form:

$$X_m(x) \sim \sin(k_m x) = \sin(m\pi x/L_x)$$
$$Y_n(y) \sim \sin(k_n y) = \sin(n\pi y/L_y)$$
$$U_{m,n}(x,y) = X_m(x)Y_n(y) = A_{m,n}\sin(k_m x)\sin(k_n y) = A_{m,n}\sin(m\pi x/L_x)\sin(n\pi y/L_y)$$

where *m* and *n* are both integers, *i.e.* *m*, *n* = 1, 2, 3, 4, 5, … Note that solutions with *m* = 0 or *n* = 0 are not allowed, because then $U_{m,n}(x,y) = X_m(x)Y_n(y) = 0$ everywhere, which are not (propagating) transverse standing wave solutions.

The characteristic equation for transverse standing wave eigen-mode solutions for a 2-D rectangular membrane and eigen-wavelengths become:

$$k_{m,n}^2 = k_m^2 + k_n^2 = \left(\pi m/L_x\right)^2 + \left(\pi n/L_y\right)^2; \quad \lambda_{m,n} = 2\pi/k_{m,n} = 2\Big/\sqrt{\left(m/L_x\right)^2 + \left(n/L_y\right)^2}$$
$$k_m = \pi m/L_x; \qquad m = 1, 2, 3, 4, 5, \ldots$$
$$k_n = \pi n/L_y; \qquad n = 1, 2, 3, 4, 5, \ldots$$

Since $\omega = vk$, with $v = \sqrt{T_\ell / \sigma}$ the angular eigen-frequencies are $\omega_{m,n} = vk_{m,n} = v\sqrt{k_m^2 + k_n^2}$ and thus eigen-frequencies $f_{m,n}$ and eigen-wavelengths $\lambda_{m,n}$ are:

$$f_{m,n} = \omega_{m,n} / 2\pi = vk_{m,n} / 2\pi = v / \lambda_{m,n} = \tfrac{1}{2} v \sqrt{\left(m / L_x\right)^2 + \left(n / L_y\right)^2}$$
$$\lambda_{m,n} = 2\pi / k_{m,n} = 2 \Big/ \sqrt{\left(m / L_x\right)^2 + \left(n / L_y\right)^2}$$
$$m, n = 1, 2, 3, 4, 5, ...$$

The eigen-mode solutions of the associated temporal wave equation for two-dimensional standing waves on a rectangular membrane are of the following equivalent form(s):

$$T_{m,n}(t) = b_{m,n} \sin \omega_{m,n} t + c_{m,n} \cos \omega_{m,n} t$$
$$-1 \le b_{m,n} \le +1 \qquad -1 \le c_{m,n} \le +1 \qquad \sqrt{b_{m,n}^2 + c_{m,n}^2} = 1$$
$$T_{m,n}(t) = \sin\left(\omega_{m,n} t + \delta_{m,n}\right); \qquad \delta_{m,n} = \tan^{-1}\left(c_{m,n} / b_{m,n}\right) = \cot^{-1}\left(b_{m,n} / c_{m,n}\right)$$
$$T_{m,n}(t) = \cos\left(\omega_{m,n} t + \varphi_{m,n}\right); \qquad \varphi_{m,n} = \tan^{-1}\left(b_{m,n} / c_{m,n}\right) = \cot^{-1}\left(c_{m,n} / b_{m,n}\right)$$
$$\delta_{m,n} = \varphi_{m,n} + \tfrac{\pi}{2}$$
$$T_{m,n}(t) = e^{i\left(\omega_{m,n} t + \varphi_{m,n}\right)}$$

The complete eigen-mode solutions for two-dimensional transverse standing waves on a rectangular membrane of dimensions with fixed edges are thus given *e.g.* by:

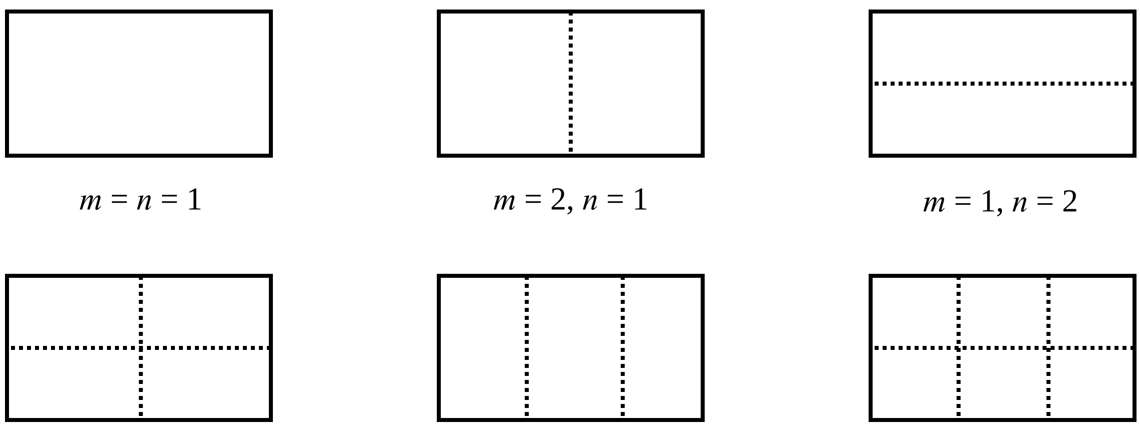
$$\psi_{m,n}(x, y, t) = U_{m,n}(x, y) T_{m,n}(t) = X_m(x) Y_n(y) T_{m,n}(t)$$
$$\psi_{m,n}(x, y, t) = A_{m,n} \sin(k_m x) \sin(k_n y) e^{i\left(\omega_{m,n} t + \varphi_{m,n}\right)}$$
$$\psi_{m,n}(x, y, t) = A_{m,n} \sin(m\pi x / L_x) \sin(n\pi y / L_y) e^{i\left(\omega_{m,n} t + \varphi_{m,n}\right)}$$

With eigen-frequencies, eigen-wavelengths and eigen-energies and eigen-energies of:

$$f_{m,n} = \omega_{m,n} / 2\pi = vk_{m,n} / 2\pi = v / \lambda_{m,n} = \tfrac{1}{2} v \sqrt{\left(m / L_x\right)^2 + \left(n / L_y\right)^2}$$
$$\lambda_{m,n} = 2\pi / k_{m,n} = 2 \Big/ \sqrt{\left(m / L_x\right)^2 + \left(n / L_y\right)^2}$$
$$E_{m,n} = \tfrac{1}{4} M \omega_{m,n}^2 A_{m,n}^2 = \pi^2 M f_{m,n}^2 A_{m,n}^2 = \tfrac{1}{4} \pi^2 v^2 M A_{m,n}^2 \left[\left(m / L_x\right)^2 + \left(n / L_y\right)^2\right]$$
$$m, n = 1, 2, 3, 4, 5, ...$$

Note that the allowed standing wave eigen-solutions all have transverse displacement ***nodes*** along the *x*- and *y*-edges of the rectangular membrane. The lowest frequency standing-wave mode occurs when $m = n = 1$, with $\lambda_{1,1} = 2\sqrt{1/L_x^2 + 1/L_y^2}$ and $f_{1,1} = 2v/\sqrt{1/L_x^2 + 1/L_y^2}$ and $E_{1,1} = 4\pi^2 v^2 M A_{1,1}^2 / \left(1/L_x^2 + 1/L_y^2\right)$. The exact sequence of which *m*,*n* eigen-values have progressively higher frequencies depends on the detailed geometry of the rectangular plate, whether $L_x \geq L_y$ or $L_x \leq L_y$, or $L_x \gg L_y$ or $L_x \ll L_y$.

Some of the lower-order eigen-modes of vibration for transverse standing waves on a rectangular membrane are shown (with dashed nodal lines) in the figure below, for $L_x > L_y$:



$m = n = 1$ $\quad$ $m = 2, n = 1$ $\quad$ $m = 1, n = 2$

$m = n = 2$ $\quad$ $m = 3, n = 1$ $\quad$ $m = 3, n = 2$

For the special case of a *square* membrane, when $L_x = L_y = L$ the square membrane has *exactly* the same number of eigen-modes as that associated with a rectangular membrane $L_x \neq L_y$, however the square membrane now has so-called *2-fold* *degeneracies* associated with it – *i.e.* distinct eigen-states, $\psi_{m,n}(x,y,t)$ with $m \neq n$ (*e.g.* $\psi_{1,2}(x,y,t)$ and $\psi_{2,1}(x,y,t)$), but which have the *same* eigen-frequencies, eigen-wavelengths and eigen-energies:

$$f_{m,n} = f_{n,m} = v/\lambda_{m,n} = \sqrt{m^2+n^2}\, v/[2L]$$

$$\lambda_{m,n} = \lambda_{n,m} = 2L/\sqrt{m^2+n^2}$$

$$E_{m,n} = E_{n,m} = \tfrac{1}{4L}\pi^2\left(m^2+n^2\right)v^2 M A_{m,n}^2$$

The 2-fold degeneracies arise because of the 2 spatial degrees of freedom (*x* & *y*) .AND. the rotational symmetry of the square membrane – *i.e.* a square is invariant under 90° rotations.

B2. Transverse Standing Waves on a Circular Membrane – Fixed Edge:

We now consider standing waves on 2-dimensional system with circular symmetry – that of a thin, perfectly compliant (*i.e.* flexible) circular membrane (*e.g.* an idealized circular drum head) of radius R, the wave equation in cylindrical 2-dimensional coordinates ($x,y \Rightarrow r,\varphi$) for the displacement amplitude, $\psi(r,\varphi,t)$ is given by:

$$\nabla^2\psi(r,\varphi,t) - \frac{1}{v^2}\frac{\partial^2\psi(r,\varphi,t)}{\partial t^2} = 0$$

Where the longitudinal speed of propagation of transverse waves on a stretched, 2-dimensional circular membrane is (also) given by $v = \sqrt{T_\ell / \sigma}$ where T_ℓ is the membrane surface tension (in *Newtons*/*m*) and $\sigma \equiv M/A = M/\pi R^2$ is the ***areal*** mass density of the membrane (in kg/m^2). With $x = r\cos\varphi$, $y = r\sin\varphi$, $d^2r = rdrd\varphi$, the *L*aplacian operator, ∇^2 in cylindrical 2-D coordinates is given by:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

Thus, the 2-dimensional wave equation describing the behavior of waves on a cylindrical membrane is given by:

$$\frac{\partial^2\psi(r,\varphi,t)}{\partial r^2} + \frac{1}{r}\frac{\partial\psi(r,\varphi,t)}{\partial r} + \frac{1}{r^2}\frac{\partial\psi(r,\varphi,t)}{\partial\varphi^2} - \frac{1}{v^2}\frac{\partial^2\psi(r,\varphi,t)}{\partial t^2} = 0$$

Again, we can (trivially) rewrite this as:

$$\frac{\partial^2\psi(r,\varphi,t)}{\partial r^2} + \frac{1}{r}\frac{\partial\psi(r,\varphi,t)}{\partial r} + \frac{1}{r^2}\frac{\partial\psi(r,\varphi,t)}{\partial\varphi^2} = \frac{1}{v^2}\frac{\partial^2\psi(r,\varphi,t)}{\partial t^2}$$

Notice again that the LHS (RHS) contains only spatial-dependent (time-dependent) functions, respectively. Thus, we again can use the technique of separation of variables, with $\psi(r,\varphi,t) = U(r,\varphi)T(t)$ where $U(r,\varphi)$ contains only spatially r- and φ-dependent terms and $T(t)$ contains only the time-dependent term.

Again, we have the relation $v = f\lambda = (\omega/2\pi)(2\pi/k) = \omega/k$; thus $vk = \omega$. We again obtain a separation constant of $-k^2$, and, after some simple algebraic manipulations, obtain the following two linear, homogeneous differential equations:

$$\frac{\partial^2 U(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U(r,\varphi)}{\partial\varphi^2} + k^2 U(r,\varphi) = 0$$

$$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0$$

We can again use the separation of variables technique on the above spatial equation, with a product solution of the form $U(r,\varphi) = R(r)\Phi(\varphi)$. Inserting this into the above spatial equation and carrying out the (partial) differentiations, dividing by $U(r,\varphi) = R(r)\Phi(\varphi)$ and carrying out a simple algebraic manipulation, we obtain the following equation:

$$\frac{1}{R(r)}\left[r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}+k^2r^2R(r)\right]=-\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2}$$

The LHS (RHS) of this equation depends only on r (φ), respectively. Again, this can only be true for all possible values of (r,φ), if both LHS and RHS are equal to a (dimensionless) constant. We know that the $\Phi(\varphi)$-solutions must be periodic/singled valued (*i.e.* $\Phi(\varphi=0) = \Phi(\varphi=2\pi)$) or more generally, $\Phi(\varphi=\varphi_o) = \Phi(\varphi=\varphi_o+2m\pi)$), $m = 0, \pm1, \pm2, \pm3, \pm4, \pm5,\ldots$. Hence we will choose this separation constant to be m^2.

The mathematical form of the standing-wave $\Phi_m(\varphi)$ eigen-solutions we seek for modal vibrations on a circular membrane must satisfy:

$$\frac{d^2\Phi_m(\varphi)}{d\varphi^2}+m^2\Phi_m(\varphi)=0$$

Thus, the $\Phi_m(\varphi)$ eigen-solutions are (one of) the following two equivalent form(s):

$$\Phi_m(\varphi)=\alpha_m\cos m\varphi+\beta_m\sin m\varphi$$

$$-1\le\alpha_m\le+1\quad -1\le\beta_m\le+1\quad \sqrt{\alpha_m^2+\beta_m^2}=1$$

$$\Phi_m(\varphi)=e^{i(m\varphi+\delta_m)};\qquad \delta_m=\tan^{-1}\left(\beta_m/\alpha_m\right)$$

$$m=0,\pm1,\pm2,\pm3,\ldots.$$

The radial equation is the well-known Bessel's equation:

$$\frac{d^2R(r)}{dr^2}+\frac{1}{r}\frac{dR(r)}{dr}+\left(k^2-\frac{m^2}{r^2}\right)R(r)=0$$

The most general solution for Bessel's equation, with m = integer (which *is* the case we have here) is of the form:

$$R_m(r)=A_mJ_m(kr)+B_mY_m(kr)$$

$J_m(x)$ { $Y_m(x)$ } are the ordinary Bessel functions of order, m of the 1st {2nd} kind, respectively.

The $J_m(x)$ are finite at x = 0 and are usually expressed as a power series expansion in x:

$$J_m(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(m+r+1)}\left(\frac{x}{2}\right)^{m+2r}$$

Note that for m = integer, $J_{-m}(x)=(-1)^mJ_m(x)$. The $Y_m(x)$ can be expressed in various ways:

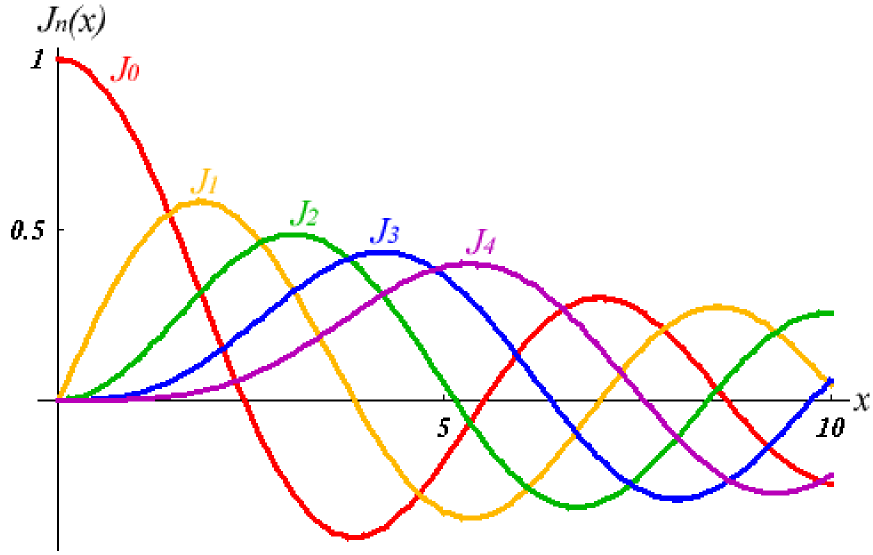
$$Y_m(x)=\frac{J_m(x)\cos(m\pi)-J_{-m}(x)}{\sin(m\pi)}$$

For m = integer, these can also be written as:

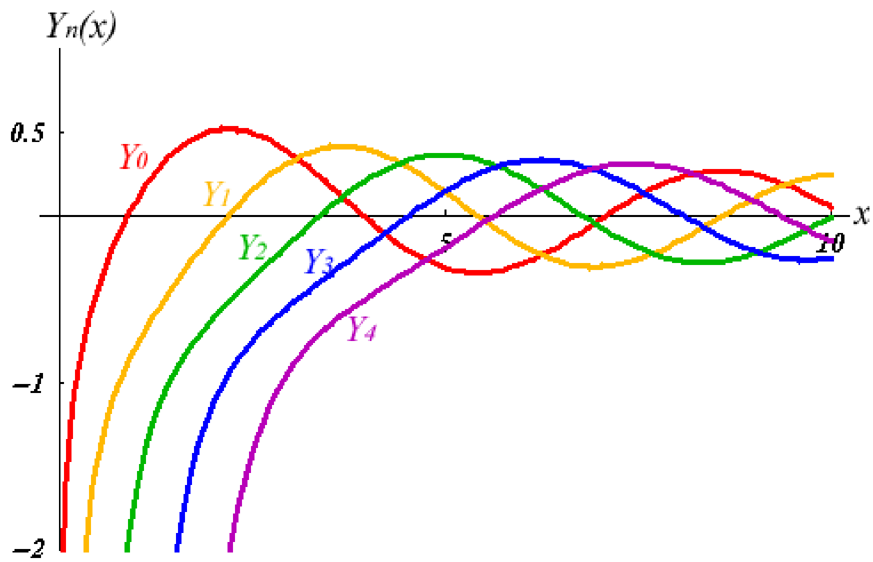
$$Y_m(x)=\frac{1}{\pi}\left[\frac{\partial J_m(x)}{\partial m}-(-1)^m\frac{\partial J_{-m}(x)}{\partial m}\right]$$

The $Y_m(x)$ are ***singular*** (they become (***negative***) infinite) at $x = 0$. However, because we used cylindrical coordinates for our circular membrane, the origin ($r = 0$) ***is*** included in this problem. Physically, we do **NOT** allow infinite amplitude displacements $R(r) \to \infty$ for ***any*** value of r, since an implicit initial assumption was ***small*** amplitude oscillations! Thus ***all*** of the B_m coefficients for the $Y_m(x)$ ***must*** be $B_m = 0$ for physically allowed eigen-mode solutions of the 2-D circular membrane.

Plots of the first few $J_m(x)$ vs. x are shown in the figure below:



Plots of the first few $Y_m(x)$ vs. x are shown in the figure below:



The radial boundary condition for transverse standing waves on a circular membrane with fixed edge (*i.e.* zero transverse displacement) at $r = R$ is $R_m(r=R) = 0$, *i.e.* $J_m(kR) = 0$. Since $r = R > 0$, this means we seek the ***zeroes*** of $J_m(kR)$, *i.e.* $J_m(x=kR) = 0$. Because of the complexity of the form of the $J_m(x)$, the zeroes of $J_m(x)$ (and the $Y_m(x)$) are non-analytic in nature, rather, they are tabulated in many mathematical books, or they can be determined either by graphical and/or computational numeric techniques. We summarize the first few zeroes of the low-order $J_m(x)$ in the table below:

			n=1	n=2	n=3	⇐ zero #
m=0:	$J_0(x)$=0:	$x \approx$	2.40,	5.52,	8.65,	….
m=1:	$J_1(x)$=0:	$x \approx$	3.83,	7.02,	10.17,	….
m=2:	$J_2(x)$=0:	$x \approx$	5.14,	8.42,	11.62,	….

Since $x = kR$, then $k = x/R$ and noting that again, for this 2-dimensional standing wave eigen-value problem we have ***two*** indices, m and n to denote the eigen-wavenumbers $k_{m,n} = x_{m,n}/R$, and the eigen-frequencies $\omega_{m,n} = vk_{m,n}$ $\quad f_{m,n} = v/\lambda_{m,n}$ with $v = \sqrt{T_\ell/\sigma}$. The eigen-energies, $E_{m,n} = \frac{1}{4}M\omega_{m,n}^2 A_{m,n}^2$ and eigen-functions $\psi_{m,n}(r,\varphi,t) = R_{m,n}(r)\Phi_m(\varphi)T_{m,n}(t)$. The eigen-mode solutions of the associated temporal wave equation for two-dimensional standing waves on a circular membrane are of the following equivalent form(s):

$$T_{m,n}(t) = b_{m,n}\sin\omega_{m,n}t + c_{m,n}\cos\omega_{m,n}t$$

$$-1 \le b_{m,n} \le +1 \qquad -1 \le c_{m,n} \le +1 \qquad \sqrt{b_{m,n}^2 + c_{m,n}^2} = 1$$

$$T_{m,n}(t) = \sin\left(\omega_{m,n}t + \delta_{m,n}\right) = \cos\left(\omega_{m,n}t + \varphi_{m,n}\right) \qquad \delta_{m,n} = \varphi_{m,n} + \tfrac{\pi}{2}$$

$$T_{m,n}(t) = e^{i\left(\omega_{m,n}t + \varphi_{m,n}\right)}$$

The complete eigen-mode solutions for two-dimensional standing waves on a circular membrane of radius, R with fixed edges are thus given *e.g.* by:

$$\psi_{m,n}(r,\varphi,t) = R_{m,n}(r)\Phi_m(\varphi)T_{m,n}(t)$$

$$\psi_{m,n}(r,\varphi,t) = A_{m,n}J_m(k_{m,n}R)e^{i\left(m\varphi+\delta_{m,n}\right)}e^{i\left(\omega_{m,n}t+\varphi_{m,n}\right)}$$

$$\psi_{m,n}(r,\varphi,t) = A_{m,n}J_m(k_{m,n}R)\left[\alpha_m\cos m\varphi + \beta_m\sin m\varphi\right]\left[b_{m,n}\cos\omega_{m,n}t + c_{m,n}\sin\omega_{m,n}t\right]$$

with eigen-frequencies, eigen-wavelengths and eigen-energies and eigen-energies of:

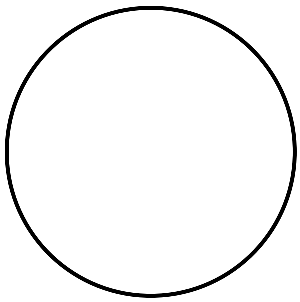
$$f_{m,n} = \omega_{m,n}/2\pi = vk_{m,n}/2\pi = v/\lambda_{m,n} \qquad \lambda_{m,n} = 2\pi/k_{m,n} = 2\pi R/x_{m,n} \qquad E_{m,n} = \tfrac{1}{4}M\omega_{m,n}^2 A_{m,n}^2$$

$$m = 0,1,2,3,4,5,.... \qquad n = 1,2,3,4,5,....$$

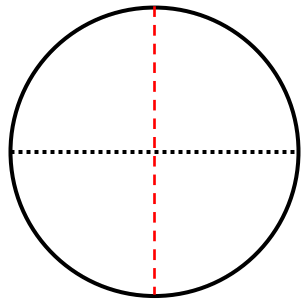
The lowest modes of transverse standing waves on a circular membrane are listed below:

m=0, n=1: $k_{0,1} \approx 2.40/R$ $\quad \omega_{0,1} \approx 2.40v/R$ $\quad \psi_{0,1}(r,\varphi,t) = A_{0,1}\,J_0(k_{0,1}R)\,T_{0,1}(t)$
m=1, n=1: $k_{1,1} \approx 3.83/R$ $\quad \omega_{1,1} \approx 3.83v/R$ $\quad \psi_{1,1}(r,\varphi,t) = A_{1,1}\,J_1(k_{1,1}R)\,[\alpha_1\cos\varphi + \beta_1\sin\varphi]T_{1,1}(t)$
m=2, n=1: $k_{2,1} \approx 5.14/R$ $\quad \omega_{2,1} \approx 5.14v/R$ $\quad \psi_{2,1}(r,\varphi,t) = A_{2,1}\,J_2(k_{2,1}R)\,[\alpha_2\cos2\varphi+\beta_2\sin2\varphi]T_{2,1}(t)$
m=0, n=2: $k_{0,2} \approx 5.52/R$ $\quad \omega_{0,2} \approx 5.52v/R$ $\quad \psi_{0,2}(r,\varphi,t) = A_{0,2}\,J_0(k_{0,2}R)\,T_{0,2}(t)$

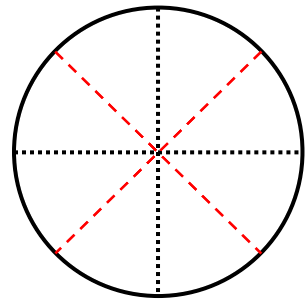
Some of the lower-order eigen-modes of vibration for transverse standing waves on a circular membrane (with dashed nodal lines) are shown in the figure below:



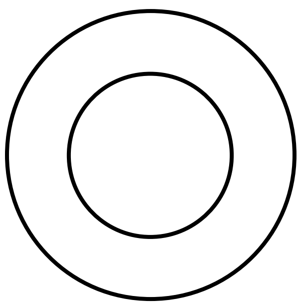
$m=0,\ n=1$
$m+n=1;\ J_0(k_{0,1}r)$



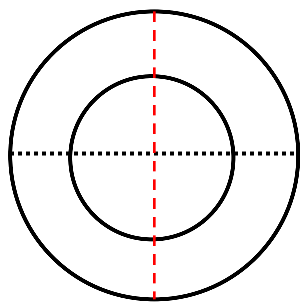
$m=1,\ n=1$
$m+n=2;\ J_1(k_{1,1}r)e^{i\varphi}$
(2-fold degenerate)



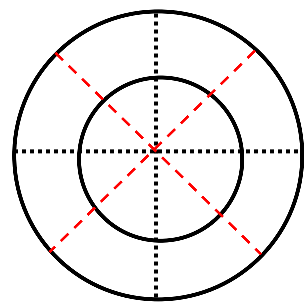
$m=2,\ n=1$
$m+n=3;\ J_2(k_{2,1}r)e^{2i\varphi}$
(2-fold degenerate)



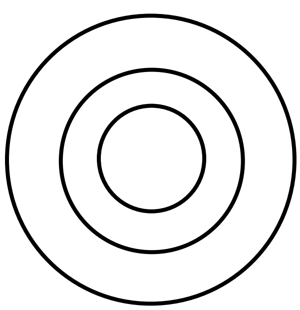
$m=0,\ n=2$
$m+n=2;\ J_0(k_{0,2}r)$



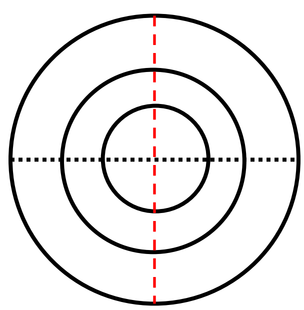
$m=1,\ n=2$
$m+n=3;\ J_1(k_{1,2}r)e^{i\varphi}$
(2-fold degenerate)



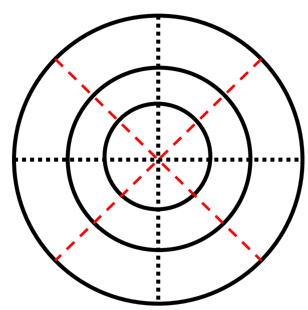
$m=2,\ n=2$
$m+n=4;\ J_2(k_{2,2}r)e^{2i\varphi}$
(2-fold degenerate)



$m=0,\ n=3$
$m+n=3;\ J_0(k_{0,3}r)$



$m=1,\ n=3$
$m+n=4;\ J_1(k_{1,3}r)e^{i\varphi}$
(2-fold degenerate)



$m=2,\ n=3$
$m+n=5;\ J_2(k_{2,3}r)\ e^{2i\varphi}$
(2-fold degenerate)

The 2-fold degeneracies for $m > 0$ again arise because of the 2 spatial degrees of freedom (x and y, or r and φ) .AND. the rotational symmetry of the circular membrane – it is invariant under arbitrary rotations.

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